Geography of Surfaces of General Type

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February 18, 2011

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Let X be a smooth projective variety. To X we associate cycle classes $c_1(X), c_2(X), \ldots, c_n(X)$, where $n = \dim X$. The $c_i(X)$ are defined via the tangent bundle T_X, where $c_i(X)$ is the class of a bunch of codimension i subvarieties, e.g. $c_1(X)$ is a divisor class and $c_n(X)$ is the class of a sum of points. We have $c_1(X) = -K_X$, the negative of the canonical class of X, and over $\mathbb{C}$, $c_n(X) = e$ where e is the topological Euler characteristic, $e = 1 - b_1 + b_2 - \cdots + b_{2n}$ where the b_i are the Betti numbers.

Out of these classes we get numbers. For example, assuming $\dim X = n$, $c_1^n(X) = (-1)^n K_X^n$ and we can consider $c_2(X)c_{n-2}(X)$, etc. Try all combinations and we get the Chern numbers of X. Under deformations these Chern numbers will be preserved.

Now we can talk about moduli=$M_{\text{Chern\#'s}}$. The main geography question is:

$$\text{When is } M_{\text{Chern\#'s}} \neq \emptyset?$$

Example Let $\dim X = 1$ (curves). Then $c_1(X) = -K_X = 2 - 2g$, where g is the genus (number of handles). For $g \geq 0$ let M_g denote the moduli space (parameter space) of curves of genus g. We can see that $M_g \neq \emptyset$ by considering hyperelliptic curves $y^2 = \prod_{i=1}^{2g+2}(x - a_i)$ where $a_i \neq a_j$.

What about surfaces ($\dim X = 2$) where X is minimal? We will use the classification of Enriques. Let κ be the Kodaira dimension of X. Facts we will use:

(1) If $\overline{X}$ is the blowup of X at p then

$$c_1^2(\overline{X}) = c_1^2(X) - 1$$
$$c_2(\overline{X}) = c_2(X) + 1$$

(2) If X is a surface, B is a curve, and $\pi : X \to B$ is a fibration with general fiber F a smooth projective curve, then

$$e(X) = e(B)e(F) + \sum_s (e(F_s) - e(F))$$

where the sum is over $s \in B$ such that the fiber $F_s := \pi^{-1}(s)$ is singular.

(3) If X is a smooth projective surface and G is a finite group acting freely on X, then

$$|G|c_1^2(X/G) = c_1^2(X)$$
$$|G|c_2(X/G) = c_2(X)$$

Also, $p_g = h^0(K_X)$ and $q = h^1(\mathcal{O}_X)$.

We consider cases:

($\kappa = -\infty$) In this case X is birational to $C \times \mathbb{P}^1$ where C is any smooth projective curve. The minimal such X are $\mathbb{P}_C(E)$, where E is a rank 2 vector bundle on C. Then

$$c_1^2(X) = 8(1 - g)$$
$$c_2(X) = 4(1 - g)$$

($\kappa = 0$) Here there are four cases:

(i) X is a $K3$ surface. Then $c_1^2 = 0$ and $e(X) = 24$.

(ii) X is an Enriques surface (in which case we have a $2:1$ map from a $K3$). Here $c_1^2 = 0$ and $e(X) = 12$.
(iii) X is an Abelian surface. Then $c_1^2 = 0$ and $e(X) = 0$.
(iv) X is a bi-elliptic surface, then $c_1^2 = 0$ and $e(X) = 0$.

($\kappa = 1$) Now we have a fibration $\pi : X \to B$ to a smooth projective curve B where a general fiber is a curve of genus 1 (i.e. π is an elliptic fibration). Here K_X is a bunch of fibers and so $K_X^2 = 0 = c_1^2$ since the self intersection of fibers is 0. In this case

$$e(X) = \sum_s e(F_s) \geq 0.$$

For $e(X) = 0$ we can take $E \times C$ for any curve C.

If $e > 0$ there are two cases: Recall Noether's formula:

$$12\chi(\mathcal{O}_X) = c_1^2 + c_2.$$

The cases are

(i) $\chi \geq 3$ (simple). We can construct $X \to B$ as a pullback of a rational elliptic fibration. (Note: If we try the same approach for $\chi < 3$ the result has $\kappa < 1$.)
(ii) $\chi = 1, 2$ (this case is hard). There are Dolgachev surfaces: $X_{2,q} \to \mathbb{P}^1$ an elliptic fibration. It is simply connected and has two multiple fibers of order $2, q$. Here $\chi = 1$, $p_g = 0$.

($\kappa = 2$)

Now we are in the case where X is a surface of general type (and we will assume X is minimal over $\mathbb{C}$). The main question we wish to answer is: When is $M_{K_X^2, e} \neq \emptyset$? We have the following restrictions on K_X^2, e for the existence of X.

$$K_X^2 > 0 \tag{1}$$

$$e > 0 \tag{2}$$

$$K_X^2 + e \equiv 0(12) \tag{3}$$

$$5K_X^2 - e(X) + 36 \geq 12q \geq 0 \tag{4}$$

(From Noether's inequality $2p_g - 4 \leq K_X^2$.)

Theorem (Bogomolov-Miyaoka-Yau (1977) Inequality)**.**

$$K_X^2 \leq 3e(X) \tag{5}$$

These are believed to be the only restrictions.

Comments:

(i) If $Char(\mathbb{K}) > 0$ then (5) and (2) are not necessarily true.
(ii) One can prove that for any complete intersection in $\mathbb{P}^{r+2}$, $c_1^2 < 2c_2$.

Notice that $c_1^2/c_2 \in [-1/5, 3]$. By the 1950's there were examples only with $c_1^2/c_2 < 2$. Then Hirzebruch proved the following. Write

$$\mathbb{H} = \{z \in \mathbb{C} : |z| < 1\}$$
$$\mathbb{B} = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 < 1|\}$$

Suppose a discrete group Γ acts freely and discontinuously on $\mathbb{H}\times\mathbb{H}$. Let $X = (\mathbb{H}\times\mathbb{H})/\Gamma$ denote the quotient (a complex manifold) and assume X is compact. Then X satisfies $c_1^2 = 2c_2$. Similarly for $X = \mathbb{B}/\Gamma$ we have $c_1^2 = 3c_2$ (examples of this type were constructed by Hirzebruch and Borel).

Theorem. *(Hirzebruch's signature formula) The signature σ of X equals $1/3(c_1^2 - 2c_2)$.*

People try to populate the range $2 < c_1^2/c_2 < 3$ (where $\sigma > 0$).

When Yau proved the BMY inequality he also proved that X satisfies $c_1^2 = 3c_2$ if and only if $X = \mathbb{B}/\Gamma$. In particular $\pi_1(X) = \Gamma$ is not trivial.

Sketch of the proof of the inequalities:

Theorem. *If X a minimal surface of general type then $2p_g - 4 \le K_X^2$.*

We can assume $p_g \ge 3$. Let $F : X \dashrightarrow \mathbb{P}^{p_g-1}$ be the rational map defined by the linear system $|K_X|$. There are two cases: either $|K_X|$ is composed to a pencil (that is, the image of F is a curve) or the image of F is a surface. We will assume $|K_X|$ contains a smooth projective curve C. Then we are in the second case. We have an exact sequence

$$0 \to \mathcal{O}_X(K_X - C) \to \mathcal{O}_X(K_X) \to \mathcal{O}_C(K_X|_C) \to 0.$$

The associated left exact sequence of global sections gives $p_g - 1 \le h^0(C, K_X|_C)$. By the adjunction formula, $K_X|_C = K_C - C|_C$ so $C|_C$ is a special divisor on C. We have Clifford's inequality

$$h^0(D) \le (\text{deg}D)/2 + 1,$$

so we obtain $p_g - 1 \le K_X^2/2 + 1$ as required.

Proof of $B-M-Y$ inequality.

This is hard. First an example.

Example: Let $\{L_1, \ldots, L_d\}$ be d lines in $\mathbb{P}^2_{\mathbb{C}}$. Say $\tau_k = \#\{k-\text{uple points }\}$. Then BMY implies that over $\mathbb{C}$, if $\tau_d = \tau_{d-1} = 0$ then

$$2\tau_2 + \tau_3 \ge 3 + d + \sum_{k\ge 4}(k-4)\tau_k.$$

In particular double or triple points exist.

$\mathbb{P}^2_{\mathbb{R}}$ is really a surface with $e(\mathbb{P}^2_{\mathbb{R}}) = 1$. It can be proven that

$$0 \le \left(\sum_{k\ge 3}(k-3)\right) p_k = -3 - \sum_{k\ge 2}(k-3)\tau_k,$$

where p_k is the number of two cells bounded by k one-cells. This number is 0 if and only if we have only triangles (simplicial).

Now back to the proof. This is due to the magic of Miyaoka

Suppose (for a contradiction) that $\alpha := c_2/c_1^2 < 1/3$ and write $\beta = 1/4(1-3\alpha) > 0$. Set key $= S^n\Omega_X^1 \otimes \mathcal{O}_X(-n(\alpha+\beta)K_X)$. Then the Riemann-Roch formula gives an inequality

$$h^0(\text{key}) + h^2(\text{key}) \ge \frac{1}{6\cdot 16}(3\alpha^2 - 22\alpha + 7)K_X^2 n^3 + O(n^2).$$

Hence for $n \gg 0$ we have $h^0(\text{key}) + h^2(\text{key}) > 0$ (note $3\alpha^2 - 22\alpha + 7 > 0$ since $\alpha < 1/3$). This can be used to obtain a contradiction.

The proof of Yau (assuming K_X is ample) shows that there exists a Kähler-Einstein metric on X. Then

$$c_1^2 - 3c_2 = \int_X f d\text{vol}$$

where $f \geq 0$, so $c_1^2 - 3c_2 \geq 0$.

Now we draw a map of the geography for minimal surfaces of general type.