Linear systems and examples of surfaces of general type

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Let X be an algebraic surface and $k(X)$ its field of rational functions. Given $x \in X$, define the local ring $\mathcal{O}_{X,x} := \{f \in k(X) | f \text{ regular at } x\}$. Given $\mathcal{U} \subset X$, let

$$\mathcal{O}(\mathcal{U}) = \bigcap_{x \in \mathcal{U}} \mathcal{O}_{X,x}$$

be the *structure sheaf*, i.e. the sheaf of regular functions on $\mathcal{U}$. An irreducible curve $C \subset X$ is called a *prime divisor*. In this case, if $\mathcal{U} \subset X$ is affine, then $I(C \cap \mathcal{U}) \subset \mathcal{O}(\mathcal{U})$ and $P = I(C \cap \mathcal{U})$ is a prime ideal.

1 Divisors

Basic Fact. If $x \in X^{\text{sm}}$, then $\mathcal{O}_{X,x}$ is a UFD. In particular, if C is a prime divisor containing x, then $I(C) \cap \mathcal{O}_{X,x} = (\pi)$. There exists an affine chart $x \in \mathcal{U}$ such that $I(C \cap \mathcal{U}) = (\pi) \subset \mathcal{O}(\mathcal{U})$.

Suppose $f \in \mathcal{O}(\mathcal{U})$. Define *the order of vanishing of f along C* to be $\text{ord}_C f = \max_n\{f \in (\pi^n)\}$. Recall from the qualifying exam that $\cap_n(\pi^n) = 0$, so $\text{ord}_C f$ is well-defined.

If $f \in k(X)$, then $f = \frac{p}{q}$ where p, $q \in \mathcal{O}(\mathcal{U})$ and $\text{ord}_C f = \text{ord}_C p - \text{ord}_C q$. We define the *principal divisor* of f to be

$$(f) = \sum_{\text{irreducible curves } C_i} \text{ord}_{C_i}(f) C_i.$$

Note that this is a finite sum, and is well-defined if X has isolated singularities, i.e. $X^{\text{sing}} = \{p_1, ..., p_r\}$. In particular, if X is smooth we can define the canonical divisor $K_X = (\omega)$ where ω is a meromorphic 2-form. If X has isolated singularities, then we can define $K_{X^{\text{sm}}} = (\omega)$, where ω is meromorphic on X^{sm} and then close it at X to get K_X.

Given $f \in k(X)$ such that $(f) \geq 0$, the divisor (f) is called an *effective* divisor. Does this imply that f is regular? Not necessarily. Consider the variety $X =< e_1, e_2 > \cup < e_3, e_4 >\subset \mathbb{A}^4$. Then $X^{\text{sing}} = \{0\}$. Choose f such that

$$f|_{<e_1,e_2>} = 0 \text{ and } f|_{<}e_3, e_4 >= 1.$$

Then (f) is an effective divisor on X, but f is not a regular function on X. To fix this we have the following

Definition 1.1. *X is a* normal *surface if $\mathcal{O}_{X,x}$ is integrally closed in $k(X)$ for all $x \in X$.*

In particular, if X is normal then

1. $X^{\text{sing}} = \{p_1, ..., p_r\}$ and
2. (Hartog's Principal) if $(f) \geq 0$ then f is regular.

Definition 1.2. *A* Cartier *or* locally principal *divisor on X is a covering $X = \bigcup U_i$ together with functions $\{f_i\} \subset k(X)$ such that $(f_i) \subset U_i$ and $\frac{f_i}{f_j} \in \mathcal{O}^*(U_i \cap U_j)$ where $\mathcal{O}^*(U_i \cap U_j)$ is the ring of invertible regular functions on $U_i \cap U_j$.*

A Weil *divisor is a divisor of the form $\sum a_i C_i$ where $a_i \in \mathbb{Z}$ and C_i are prime divisors.*

Note that if X is smooth, then Weil divisors are Cartier divisors. In general we have

$$\text{(Principal Divisors)} \subset \text{(Cartier Divisors)} \subset \text{(Weil Divisors)}$$

We define the *Picard Group* of X to be

$$Pic(X) = \frac{\text{Cartier divisors}}{\text{Principal divisors}}$$

and the *Divisor Class Group* of X to be

$$\text{Cl}(X) = \frac{\text{Weil divisors}}{\text{Principal divisors}}.$$

Example 1.3. *Consider the surface $S = \{z^2 = x^2 + y^2\} \subset \mathbb{A}^2$. Let R be a ruling of S. Then $R \subset S$ is a prime divisor, but is not Cartier. However, the divisor $2R$ is Cartier. In fact, $2R = (f)$ where f is the equation of the tangent plane along R. Thus, $Pic(X) = 0$, but $Cl(X) = \mathbb{Z}/2\mathbb{Z}$.*

Definition 1.4. *A divisor D is called* $\mathbb{Q}$-Cartier *if mD is Cartier for some integer $m > 0$. (In the example above, R is $\mathbb{Q}$-Cartier.) If K_X is Cartier, then X is called* Gorenstein*. If K_X is $\mathbb{Q}$-Cartier, then X is called* $\mathbb{Q}$-Gorenstein.

2 Linear Systems

Suppose X is a projective irreducible surface. Let D be a Cartier divisor. Define

$$\mathcal{L}(D) = \{f \in k(X) | (f) + D \geq 0\}.$$

And let $l(D) = \dim \mathcal{L}(D) < \infty$. The *Linear system* of D is

$$|D| = \{(f) + D | (f) \in \mathcal{L}(D)\}.$$

Choose a linearly independent subset f_0,..., f_r of $\mathcal{L}(D)$. Then there exists a rational map $\phi_D : X \dashrightarrow \mathbb{P}^r$ given by $x \mapsto [f_0(x) : \cdots : f_r(x)]$. If the subset f_0,..., f_r is in fact a basis of $\mathcal{L}(D)$, then the map ϕ_D is called a *complete linear system.* Note that the indeterminacy locus consists of a bunch of points. If the subset f_0,..., f_r is not a basis of $\mathcal{L}(D)$, then the map ϕ_D is called a *complete linear system.*

On the other hand, given any rational map $f : X \dashrightarrow \mathbb{P}^r$, we can write $f = [f_0 : \cdots : f_r]$. Let D be the common denominator of the f_i's. Then by definition, we see that $f_i \in \mathcal{L}(D)$ for all i. Thus, any rational map is given by an incomplete linear system, and the divisors $(f_0) + D$,..., $(f_r) + D$ are pull-backs of coordinate hyperplanes.

Let D be an effective Cartier divisor and consider $|D|$={effective divisors linearly equivalent to D}. (Recall that $D \sim D'$ if $D - D' = (f)$ for some $f \in k(X)$). We can decompose D into $D = F + M$ where F is the "fixed part" (i.e. if $D' \in |D|$ then $D' \geq F$) and M is the "moving divisor." Then $|D| = F + |M|$ and so $\phi_D = \phi_M$.

Let D be a divisor on a surface S. Then $l(D) = h^0(D)$. There are numbers $h^1(D)$ and $h^2(D)$, the dimensions of the cohomology groups. Serre's Duality tells us that $h^2(D) = h^0(K - D)$, where K is the canonical divisor of S, and also that $h^1(D) = h^1(K - D)$.

Definition 2.1. *The* Euler Characteristic *is* $\chi(D) = h^0(D) - h^1(D) + h^2(D)$.

Theorem 2.2 (Riemann-Roch)**.** $\chi(D) = \chi(0) + \frac{D\cdot(D-K)}{2}$ *where here $\cdot$ is the intersection product.*

Theorem 2.3 (Noether's Formula)**.** $\chi_{hol} = \chi(0) = \frac{K^2+e}{12}$ *where e is the topological Euler characteristic.*

3 Surfaces of General Type

Definition 3.1. *A divisor D is called* very ample *if the map $\phi_D : X \hookrightarrow \mathbb{P}^r$ is an embedding. A divisor D is called* ample *if for some $m > 0$, the divisor mD is very ample.*

Some Examples:

1. Let $S \subset \mathbb{P}^3$ be a quintic surface. By the adjunction formula, we have $K_S = H \cap S$, where H is the hyperplane class of S. Thus, $\phi_{K_S} : S \hookrightarrow \mathbb{P}^3$ is an embedding. In particular, K is very ample.

2. Suppose $f : S \to \mathbb{P}^2$ is a 2:1 map ramified along an octic C. Then $K_S = f^*L$ where $L \subset \mathbb{P}^2$ is a line. And ϕ_K, the "canonical map" is the same as f. So in this case, K is not very ample, but since $2K$ is very ample K is ample.

3. Consider a 2:1 map $f : S \to \mathbb{P}^2$, whose branch locus is a degree 10 curve. Then $K_S = f^*(2L)$, and $\phi_{K_S} : S \to \mathbb{P}^5$ is the composition of f and the

Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. Thus, K_S is not very ample, but since $2K$ is very ample, K is ample.

Let X be a smooth projective surface. We now have "pluricanonical maps" $\phi_{nK} : X \dashrightarrow \mathbb{P}^N$. We define the *Kodaira dimension* $\kappa(X) = \sup \dim \phi_{nK}(X)$. If K is ample, then $\kappa(X) = 2$.

Definition 3.2. *X is a* surface of general type *if $\kappa(X) = 2$.*

In particular, the examples above are surfaces of general type.

How can we tell whether or not a divisor is ample?

The idea: If D is ample, then nD is a hyperplane section in some projective embedding. Thus $D \cdot C > 0$ for any irreducible curve C. Consider the Neron-Severi subspace $NS \subset H^2(X, \mathbb{R})$, ie the subspace spanned by algebraic curves.

The *Hodge Index Theorem* states that the interection pairing has signature (+,-,-,...) on NS. The Nori cone, i.e., the cone spanned by algebraic curves, is a subset of NS.

Theorem 3.3 (Kleiman's Criterion)**.** *D is ample if and only if*

$$\overline{NE_1} \backslash \{0\} \subset \{C | D \cdot C > 0\}$$

Suppose S is a surface of general type. If $K \cdot C < 0$ for some algebraic curve C, then S contains a (-1)-curve C', i.e. $C' \backsimeq \mathbb{P}^1$, $C'^2 = -1$, and $K \cdot C' = -1$.

Theorem 3.4 (Castelnuovo's Criterion)**.** *Any (-1)-curve can be contracted, i.e., there exists a map from S to a surface S' such that S' is of general type.*

Definition 3.5. *X is called* minimal *if it has no (-1)-curves.*

So any surface of general type is an iterated blow-up of a minimal surface of general type.

Questions:

1. Classify the numerical invariants (pairs $(\kappa^2, \chi_{\text{top}})$) of minimal surfaces.
2. Fix numerical types and understand the moduli space.

Theorem 3.6 (Gieseker)**.** *Let X be a smooth minimal surface of general type. the $\phi_{5K} : X \to \mathbb{P}^r$ is birational onto its image and $\phi_{5K}(X)$ has duVal singularities. ϕ_{5K} is a "canonical model."*