Singularities of Surfaces
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Smooth varieties can degenerate into singular varieties:

Example 1. Consider the family

$$\mathcal{X} = (xy = t) \subset \mathbb{C}^3_{x,y,t} \xrightarrow{p} \mathbb{C}^1_t$$

where t is a deformation parameter. Then the fibers $\mathcal{X}_t$ are smooth where $t \neq 0$, but $\mathcal{X}_0$ is singular. We can think of the fibers as a Riemann surface where we have specified a copy of S^1 that encircles a neck (or passes through a hole) of the surface. Then as t tends to zero the loop constricts to a point.

A. Ordinary Double Point

The simplest singularity on a complex surface is the ordinary double point.

Example 2. Let's take X to be a complex surface and suppose that it has such a singularity at a point P, which we'll take to be 0 on this surface.

$$P = 0 \in X = (f = 0) \subset \mathbb{C}^3_{x,y,z}$$

Recall that X is singular at P if and only if

$$f = f_2 + f_3 + f_4 + \cdots$$

where f_i is homogeneous of degree i. If f_2 is a non degenerate quadratic form on $\mathbb{C}^3$, then we call P an ordinary double point.

After an analytic change of coordinates, we can write

$$X = (x^2 + y^2 + z^2 = 0) \subset \mathbb{C}^3 \quad \text{or} \quad X = (xz = y^2) \subset \mathbb{C}^3$$

With the latter choice of coordinates, we see that X is given by $\mathbb{C}^2$ modulo a group action

$$X = \mathbb{C}^2_{u,v}/(\mathbb{Z}/2\mathbb{Z})$$

where the action is $(u, v) \mapsto (-u, -v)$ and the invariant polynomials are generated by $x = u^2$, $y = uv$, and $z = v^2$. As we saw last semester, this singularity is also known as an A_1 singularity.

Degeneration: The degeneration is given by the family

$$\mathcal{X} = (x^2 + y^2 + z^2 = t) \subset \mathbb{C}^3_{x,y,z} \times \mathbb{C}^1_t$$

The general fibers of this family are smooth quadrics, and we get one special fiber when $t = 0$. The fiber above $t = 0$ is the cone that we saw in the example above.

The *vanishing cycle* on a general fiber is a copy of S^2 that vanishes on the singular fiber. In this example, if $t = r^2$ where $r \in \mathbb{R}$ then

$$S^2 = (x^2 + y^2 + z^2 = r^2) = \mathcal{X}_t \cap \mathbb{R}^3 \subset \mathcal{X}_t \subset \mathbb{C}^3.$$

We write $\delta = [S^2] \in H_2(\mathcal{X}_t, \mathbb{Z})$. Then we get a short exact sequence

$$0 \longrightarrow \mathbb{Z} \underset{1 \mapsto \delta}{\longrightarrow} H_2(\mathcal{X}_t, \mathbb{Z}) \longrightarrow H_2(\mathcal{X}_0, \mathbb{Z}) \longrightarrow 0$$

with the additional properties that $\delta^2 = -2$ and $K_{X_t} \cdot \delta = 0$.

We can also look at the monodromy: take a small loop around the origin in the $\mathbb{C}^1_t$ plane and look at what happens to the fiber as we move t around the loop. The result is diffeomorphism $\mathcal{X}_t \xrightarrow{\sim} \mathcal{X}_t$ that is well defined up to homotopy. This gives a map defined by the Picard–Lefschetz formula

$$T\colon H_2(\mathcal{X}_t,\mathbb{Z}) \to H_2(\mathcal{X}_t,\mathbb{Z})$$
$$\alpha \mapsto \alpha + (\alpha\cdot\delta)\delta$$

and $T^2 = id$. (This has connections to Weyl groups.)

B. Wahl Singularity

Example 3. Let

$$P \in X = (xy = z^n) \subset \mathbb{C}^3_{x,y,z}/(\mathbb{Z}/n\mathbb{Z})$$

where the group action is defined by

$$(x,y,z) \mapsto (\zeta x, \zeta^{-1}y, \zeta^a z)$$

with $\zeta = \exp(2\pi i/n)$ and $(a,n) = 1$.

Recall that if $Y = (xy = z^n) \subset \mathbb{C}^3$, then Y is the quotient singularity $\mathbb{C}^2_{u,v}/(\mathbb{Z}/n\mathbb{Z})$ under the action $(u,v) \mapsto (\zeta u, \zeta^{-1}v)$ with invariants $x = u^n, y = v^n, z = uv$. So

$$X = Y/(\mathbb{Z}/n\mathbb{Z}) = (\mathbb{C}^2_{u,v}/(\mathbb{Z}/n\mathbb{Z}))/(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{C}^2_{u,v}/(\mathbb{Z}/n^2\mathbb{Z})$$

where the action now is $(u,v) \mapsto (\xi u, \xi^{na-1}v)$ with $\xi = \exp(2\pi i/n^2)$.

Degeneration: Take

$$\mathcal{X} = (xy = z^n + t) \subset \mathbb{C}^3(x,y,z)/(\mathbb{Z}/n\mathbb{Z})$$

Note that the fibers are smooth away from $t = 0$ and $\mathcal{X}_0 = X$ has a Wahl singularity. If we assume now that $\mathcal{X}_t$ is a compact 4-manifold and $H_1(\mathcal{X}_t,\mathbb{Z}) = 0$ for $t = 0$, then we get an exact sequence

$$0 \longrightarrow H_2(\mathcal{X}_t,\mathbb{Z}) \longrightarrow H_2(\mathcal{X}_0,\mathbb{Z}) \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

where $H_2(\mathcal{X}_t)$ and $H_2(\mathcal{X}_0)$ are torsion-free. This follows from the Mayer-Vietoris sequence and a local analysis near the singularity.

Now we give global examples of the above types of degeneration.

Example 4. Let us return to the ordinary double point. Consider

$$(X_0^2 + X_1^2 + X_2^2 - tX_3^2 = 0) \subset \mathbb{P}^3 \times \mathbb{C}^1_t$$

The fibers $\mathcal{X}_t$ are smooth quadrics isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ except for $\mathcal{X}_0$ which is a rank-3 quadric. We can also think of $\mathcal{X}_0$ as a cone over a conic $Q \subset \mathbb{P}^2$.

Example 5. (Wahl singularity) Consider the embedding

$$\mathcal{X} = (X_0X_2 = X_1^2 + tY) \subset \mathbb{P}(1,1,1,2) \times \mathbb{C}^1_t$$

where $\mathbb{P}(1,1,1,2)$ is a weighted projective space. When $t \neq 0$, $\mathcal{X}_t \simeq \mathbb{P}^2$. If $t = 0$ then

$$\mathcal{X}_0 = (X_0X_2 = X_1^2) \subset \mathbb{P}(1,1,1,2).$$

Actually,

$$\mathcal{X}_0 = \mathbb{P}(1,1,4) \hookrightarrow \mathbb{P}(1,1,1,2)$$
$$(U_0,U_1,V) \mapsto (U_0^2, U_0U_1, U_1^2, V)$$

The singularity is a Wahl singularity with $n = 2$ and $a = 1$. The surface $\mathbb{P}(1,1,4)$ is isomorphic to the cone over the rational normal curve of degree 4 in $\mathbb{P}^4$. The associated short exact sequence is

$$0 \longrightarrow H_2(\mathcal{X}_t,\mathbb{Z}) \underset{1\mapsto 2}{\longrightarrow} H_2(\mathcal{X}_0,\mathbb{Z}) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

where we can think of $H_2(\mathcal{X}_t, \mathbb{Z})$ as $\mathbb{Z} \cdot$ [line] and $H_2(\mathcal{X}_0, \mathbb{Z})$ as $\mathbb{Z} \cdot$ [ruling]

The ordinary double point and the Wahl singularity are important examples because they correspond to boundary divisors of moduli spaces of surfaces. If we let M be a moduli space of smooth surfaces, in most cases it is not compact. There is a compactification $M \subset \overline{M}$, which is a moduli space of stable surfaces where we allow "mild" singularities. Then typically $M = \overline{M} \setminus D$ where D is a divisor (codimension 1).

Other singularities

ADE Singularities. These are the Du Val singularities that we saw last semester. They all arise as a quotient $\mathbb{C}^2/G$ where G is a finite subgroup of $SU(2)$ and can be realized as hypersurfaces $X = (f = 0) \subset \mathbb{C}^3$.

Elliptic Singularities. Let E be an elliptic curve of genus 1, and let L/E be a line bundle on E of degree n corresponding to the divisor $D = nP$. Then we have the following

$$\begin{array}{ccc} E & \hookrightarrow & \mathbb{P}^{n-1} \\ \uparrow & & \uparrow \rho \\ \rho^{-1}(E) & \hookrightarrow & \mathbb{C}^n \setminus \{0\} \end{array}$$

Example 6. If $n = 3$, then $E \hookrightarrow \mathbb{P}^2$ as a plane cubic and $X = \text{cone}(E) = \overline{\rho^{-1}(E)} \subset \mathbb{C}^3$.

A cone over a variety has a "cylinder resolution". Think of a blowup: $\widetilde{X} \xrightarrow{\pi} X$ where $E \subset \widetilde{X}$ gets sent to $P \in X$ and $\widetilde{X} \setminus E \xrightarrow{\sim} X \setminus P$.

Call $L^* = N_{E/\tilde{X}}$ = normal bundle of $E \subset \widetilde{X}$. Then $N_{E/\tilde{X}} = \mathcal{O}_{\tilde{X}}(E)|_E$ where $\deg N_{E/\tilde{X}} = E \cdot E$ and $\deg N_{E/\tilde{X}} = \deg L^* = -n$. So $E^2 = -n$.

We need $n \geq 3$ to get $E \hookrightarrow \mathbb{P}^{n-1}$.

For the lower degree cases: $(n = 1)$ $\quad E = (Z^2 = Y^3 + E_4 X^4 Y + E_6 X^6) \hookrightarrow \mathbb{P}(1, 2, 3)$
$(n = 2)$ $\quad E \hookrightarrow \mathbb{P}(1, 1, 2)$

We have the following result of Pinkham: If $X = C(E, L)$ is a cone over an elliptic curve, it is smoothable if and only if $n \leq 9$. Here, X is smoothable if $X = \mathcal{X}_0 \subset \mathcal{X}/\mathbb{C}^1_t$ with $\mathcal{X}_t$ smooth for $t \neq 0$.

If $P \in X$ is singular, then the *deformation space* of P, denoted $\text{Def}(P \in X)$ is the parameter space of all possible deformations of the singularity.

Example 7. Let $X = C(E, L)$, then $\dim(\text{Def}(P \in X)) = \begin{cases} 11 - n & \text{if } n \leq 10 \\ 1 & \text{if } n \geq 11 \end{cases}$

Connection with del Pezzo Surfaces

S is a del Pezzo surface if $-K_S$ is ample. Then $S = Bl^r(\mathbb{P}^2)$ where $r \leq 8$, or $S = \mathbb{P}^1 \times \mathbb{P}^1$.

$\phi_{|-K_S|} \colon S \hookrightarrow \mathbb{P}^n$ where $n = K_S^2 = 9 - r$ if $n \geq 3$.
If $n = 2$, then $S \hookrightarrow \mathbb{P}(1, 1, 1, 2)$.
If $n = 1$, then $S \hookrightarrow \mathbb{P}(1, 1, 2, 3)$.

Consider $C(S) \hookrightarrow \mathbb{C}^{n+1}$, the cone over S with singularity P, and compactify: $Y = \overline{C(S)} \subset \mathbb{P}^{n+1}$. Then consider a hyperplane section of Y. If $P \notin H$ then $H \cap Y \cong S$. Else, if $P \in H$, then

$H \cap Y = C(H \cap S) = \overline{C(E)}$. Furthermore, $D = H \cap S \in |-K_S|$ so D is an elliptic curve by the adjunction formula: $0 = (K_S + D)|_D = K_D$, and $D^2 = n$.

Therefore we get a family $\begin{array}{ccc} \mathcal{X} & \supset & X \\ \downarrow & & \downarrow \\ \Delta & \in & 0 \end{array}$

Warning: To make this argument rigorous, some extra work needs to be done to show that $C(S)$ is 'Cohen-Macaulay'.