The Enriques classification of complex algebraic surfaces

David Cox
02/11/2011
notes by Anna Kazanova

To talk about classification of smooth surfaces in one hour is impossible, so we will talk about classification of curves and surfaces.

Smooth projective curves

We denote the smooth projective curve by C. We will use E for elliptic curves.

Key Invariant is a genus g. We have

$$g = \dim \mathcal{L}(K_C) = l(K_C) = \frac{1}{2} b_r(C),$$

where K_C is a canonical class divisor.

Divisor D gives $\deg(D)$, $\mathcal{L}(D)$, $l(D) = h^0(D)$.

Euler characteristic $\chi(D) = h^0(D) - h^1(D)$.

Riemann-Roch: $\chi(D) = \deg(D) + \chi(O)$, where $\chi(O) = 1 - g$.

Key Results: $\deg K_C = 2g - 2$, $l(K_C) = g$, D is ample iff $\deg D \geq 0$, D is very ample iff $\deg D \geq 2g + 1$.

Kodaira Dimension: $\kappa(C) = \sup_n \dim \phi_{|nK_C|}(C)$.

So if $g = 0$, then $\deg K_C = -2 < 0$, so $\mathcal{L}(nK_C) = \{0\}$. Then $\kappa(C) = \dim\{0\} = -\infty$.

If $g = 1$, then C is an elliptic curve, so $K_C \sim 0$, so $\mathcal{L}(nK_C) = \mathbb{C}$. Then $\kappa(C) = 0$.

If $g = 2$, then $\deg K_C > 0$, so K_C is ample. Then $\kappa(C) = 1 = \dim C$. This case corresponds to general type.

Geography and Geology

- $\kappa = -\infty$, $g = 0$, $\mathbb{P}^1$.

- $\kappa = 0$, $g = 1$, elliptic curve, classified by $j(C) \in \mathbb{C} = M_1$

- $\kappa = 1$, $g \geq 2$, general type, topology determined by g, moduli space M_g - irreducible algebraic variety of dimension $3g - 3$, $M_g \subset \bar{M}_g$. $\bar{M}_g$ has interesting "strata"at the boundary.

  Sometimes besides the curve of genus g we throw in some marked points. $M_{g,n} \subseteq \bar{M}_{g,n}$ is interesting even for $g = 0$.

More on K_X Focus only on curves of general type, i.e. $g \geq 2$. Then K_C is ample, but for which n is nK_C very ample?

First $n = 3$, $\deg(nK_C) \geq 6g - 6 \geq 2g + 1$ (true if $g \geq 7/4$).

Second, $n = 2$, $g \geq 3$, $\deg(2K_C) = 4g - 4 \geq 2g + 1$ ($g \geq 5/2$).

Third:

Theorem 1.1. *Let $g \geq 2$. Then either K_C is very ample or the 2:1 map $\phi_{K_C} : C \to \mathbb{P}^1 \subseteq \mathbb{P}^{g-1}$ is hyperelliptic.*

Smooth Projective Surfaces

Restrict to minimal surfaces, i.e. no -1–curves. We have some numerical invariants from RR.

- Euler Characteristic: $\chi(D) = h^0(D) - h^1(D) + h^2(D)$, where $h^2(D) = h^0(K_D)$.

- RR: $\chi(D) = \chi(0) + 1/2D \cdot (D - K_S)$

- Noether Formula: $\chi(0) = 1/12(K_S^2 + e)$, where e is the topological Euler Charactristics, $e = \sum_{i=0}^{4}(-1)^i b_i(S)$.

We will use: $c_1^2 = K_S^2$, $c_2 = e$, $q = h^1(0)$ is called the **irregularity**; $p_g = l(K_S) = h^0(K_S) = h^2(0)$ is called the **geometric genus**.

So we can get from the RR: $\chi(0) = 1 - g + p_g$.

Hodge Decomposition: $H^1(S, \mathbb{C}) = H^{1,0}(S) \oplus H^{0,1}(S)$, but $H^{0,1} = H^1(0)$ and $H^{1,0} = \bar{H}^{0,1}$. Therefore $b_1 = 2q$, or $q = 1/2b_1$.

Enriques – Kodaira Classification

Shafarevich: Algebraic Surfaces

- $\kappa = -\infty$. They all have $p_g = 0$, $q = g$;
  - $\mathbb{P}^2$, $c_1^2 = 9$;
  - Ruled Surfaces $S \to C$, all fibers are copies of $\mathbb{P}^1$, say C has genus g then $c_1^2 = 8(1-g)$,
    - $g = 0$, Hirzebruch Surface $\mathbb{F}_n$, $n \geq 2$;
    - $g \geq 1$, $S = \mathbb{P}(\mathcal{E})$, where E is a rank 2 vector bundle on $\mathbb{C}$.
- $\kappa = 0$, they all have $c_1^2 = 0$
  - Abelian Surfaces, (Ex. $E_1 \times E_2$), $c_2 = 0$, $p_g = 1$, $q = 2$, $K_S \sim 0$.
  - K3 Surface (Ex: smooth quartic in $\mathbb{P}^3$) $c_2 = 24$, $p_g = 1$, $q = 0$, $K_S \sim 0$.
  - Enriques Surfaces, $c_2 = 12$, $p_g = 0$, $q = 0$, $K_S \not\sim 0$, $2K_S \sim 0$. (Ex: K3/(fixed point free convolution), so we have a 2:1 covering map K3 $\to$ Enriques, $\pi_1 = \mathbb{Z}/2$.)
  - Bielliptic Surface $c_2 = 0$, $p_g = 0$, $q = 1$, $nK_S \sim 0$, $n \in \{2, 3, 4, 6\}$. (Ex: G acts on E_1 by translations, E_1/G is still elliptic, G acts on E_2 by automorphisms $E_2/G \simeq \mathbb{P}^1$. Then the surface $(E_1 \times E_2)/G$ is bielliptic.)

    Complication: Abelian and K3 can have complex versions, need not be algebraic. So complex moduli $\neq$ algebraic moduli.
- $\kappa = 1$, then $c_1^2 = 0$ and we have a fibration $f : S \to C$ to a smooth projective curves C of genus g, most fibers (with finitely many exceptions) are elliptic curves, this is called an **elliptic surface**. (Warning: not al elliptic surfaces have $\kappa = 1$.) Furthermore, for a suitable n a map $\phi_{|nK_S|} S \to \mathbb{P}^n$ factors through C like $S \to C \to \mathbb{P}^n$ and the diagram commutes.

Example 1.1. Take $p_g \geq 2$, let $n = 1 + p_g$, work in $\bar{S} \subseteq \mathbb{P}(1, 1, 2n, 3n)$ with variables x, y, z, w. The equation is given by $w^2 = z^3 + P(x, y)z + Q(x, y)$, where $P(X, Y)$ is homogeneous of degree $4n$ and $Q(x, y)$ is homogeneous of degree $6n$.

- $\kappa = 2$ then $c_1^2 \geq 0$, this is the rest!

  $\phi_{|nK_S|} =?$

  Theorem 1.2. *S is a surface of general type. Then $\phi_{|nK_S|} : S \to \bar{S} \subseteq \mathbb{P}^N$ is defined everywhere and birational if $n \geq 5$ or $n = 4$ and $c_1^2 \geq 2$ or $n = 3$, $c_1^2 \geq 6$. Furthemore, $\bar{S}$ is normal, its singularities (if any) are rational double points, and $S \to \bar{S}$ is a minimal resolution of singularities, and $\bar{S}$ has a singularity for every -2 curve in S.*