Wall crossing for stable objects in Calabi Yau 3-fold categories

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1 Reineke [R03], [R08]

1.1 Representations of quivers

Let k be a field. Let Q be a finite quiver (directed graph) without oriented cycles. Then the path algebra $A = kQ$ of Q is a finite dimensional k-algebra. It has k-basis the set of directed paths in Q and multiplication given by concatenation of paths. Let $V = \{1, \ldots, n\}$ be the set of vertices and E the set of edges.

We assume that the ground field k is a finite field. Write $q := |k|$.

A representation of Q is the data of a (finite dimensional) vector space V_i for each vertex $i \in V$ and a linear map $\theta_e \colon V_i \to V_j$ for each edge $e = \vec{ij} \in E$. This is the same thing as a (finitely generated) A-module. The dimension vector of a representation is $d = (d_1, \ldots, d_n)$, $d_i = \dim V_i$.

Let $R_d = \prod_{\vec{ij} \in E} \operatorname{Hom}(k^{d_i}, k^{d_j})$ denote the space of representations of Q with dimension vector d together with a choice of basis of each V_i. Let $G_d = \prod_{i \in V} \mathrm{GL}_{d_i}$ be the group acting on R_d by change of basis. So the quotient R_d/G_d is the set of isomorphism classes of representations of Q with dimension d (the coarse moduli space). One can also consider the moduli stack $[R_d/G_d]$ (the stack quotient).

1.2 Hall Algebra

The Hall algebra $H(Q) = \prod_d H_d(Q)$ is defined as follows. Let $H_d(Q)$ be the set of $\mathbb{Q}$-valued G_d-invariant functions f on R_d (or equivalently the set of $\mathbb{Q}$-valued functions $\bar{f}$ on R_d/G_d). Define the convolution product

$$f * g(X) = q^{\langle e,d \rangle/2} \sum_{Y \subset X} f(Y) g(X/Y)$$

for $f \in H_d$, $g \in H_e$, where

$$\langle e, d\rangle = \chi(V_e, V_d) = \dim \operatorname{Hom}(V_e, V_d) - \dim \operatorname{Ext}^1(V_e, V_d)$$

where V_e, V_d are representations with dimensions e, d. Note that the category of representations of Q has homological dimension 1 ($\operatorname{Ext}^i = 0$ for $i > 0$). Explicitly

$$\langle e, d\rangle = \sum_{i \in V} e_i d_i - \sum_{\vec{ij} \in E} e_i d_j.$$

1.3 Integration map

Let

$$R_q = \prod_d \mathbb{Q} x_d$$

be the $\mathbb{Q}$-algebra with topological basis x_d for $d \in \mathbb{Z}^n_{\geq 0}$ a dimension vector and multiplication rule

$$x_d \cdot x_e = q^{-\langle e,d\rangle/2} x_{d+e}.$$

One checks that R_q is associative with unit $x_0 = 1$. The algebra R_q is the quotient of the noncommutative formal power series ring

$$\mathbb{Q}\langle\langle x_1, \ldots, x_n\rangle\rangle$$

by the relations

$$x_i x_j = q^{\{e_i,e_j\}/2} x_j x_i$$

where

$$\{d, e\} := \langle d, e\rangle - \langle e, d\rangle,$$

and $x_i = x_{e_i}$ where $e_i \in \mathbb{Z}^n$ is the ith standard basis vector.

We define the *integration map*

$$I \colon H(Q) \to R_q, \quad I(f) = \sum_d \left(\frac{1}{|G_d|} \sum_{X \in R_d} f(X) \right) \cdot x_d.$$

Lemma 1.1. *[R08, 6.1] I is a ring homomorphism.*

Proof. We need to show $I(f * g) = I(f) \cdot I(g)$. We may assume $f = \chi_{\mathcal{O}_M}, g = \chi_{\mathcal{O}_N}$ are the indicator functions of orbits $\mathcal{O}_M, \mathcal{O}_N$ of representations M, N of dimensions d, e. Then

$$I(f) = \frac{|\mathcal{O}_M|}{|G_d|} \cdot x_d = \frac{1}{|\operatorname{Aut} M|} \cdot x_d, \quad I(g) = \frac{1}{|\operatorname{Aut} N|} \cdot x_e.$$

By the definition of the convolution product,

$$f * g = q^{\langle e,d\rangle/2} \sum_X F^X_{M,N} \cdot \chi_{\mathcal{O}_X}$$

where $F^X_{M,N}$ denotes the number of submodules $M' \subset X$ with $M' \simeq M$ and $X/M' \simeq N$. We observe that

$$F^X_{M,N} = \frac{|\operatorname{Ext}^1(N,M)_X| \cdot |\operatorname{Aut} X|}{|\operatorname{Aut} M| \cdot |\operatorname{Aut} N| \cdot |\operatorname{Hom}(N,M)|}$$

where $\operatorname{Ext}^1(N,M)_X$ denotes the number of isomorphism classes of extensions

$$0 \to M \to X' \to N \to 0$$

such that $X' \simeq X$. This follows from the definition of $F^X_{M,N}$ and the fact that the automorphism group of an extension as above is identified with $\operatorname{Hom}(N,M)$ via $\theta \mapsto \theta - \mathrm{id}$. Now

$$\begin{aligned} I(f * g) &= q^{\frac{1}{2}\langle e,d\rangle} \cdot \frac{\sum_X |\operatorname{Ext}^1(N,M)_X|}{|\operatorname{Hom}(N,M)_X|} \cdot \frac{1}{|\operatorname{Aut} M| \cdot |\operatorname{Aut} N|} \cdot x_{d+e} \\ &= q^{\frac{1}{2}\langle e,d\rangle} \cdot q^{-\langle e,d\rangle} \frac{1}{|\operatorname{Aut} M| \cdot |\operatorname{Aut} N|} \cdot x_{d+e} = I(f) \cdot I(g). \end{aligned}$$

□

We have used the fact that the category of representations of Q has homological degree 1. According to Kontsevich and Soibelman, it seems that an analogous construction is possible for a Calabi Yau 3-fold category (a category of homological dimension 3 such that $\operatorname{Ext}^i(M,N) \simeq \operatorname{Ext}^{3-i}(N,M)^*$). However the product on the Hall algebra uses the virtual fundamental class of the moduli space of objects (in Behrend's formulation, we integrate a constructible weight function over the moduli space).

2 Kontsevich–Soibelman [KS08]

Let $\mathcal{C}$ be a triangulated category over a field k.

Let $\mathcal{M}$ be the moduli stack of objects in $\mathcal{C}$ (an algebraic stack in the sense of Artin). WARNING: Actually we need to restrict attention to objects E of $\mathcal{C}$ satisfying $\operatorname{Ext}^i(E,E) = 0$ for $i < 0$, see [L06]. This will be true in the cases we need (E will be contained in an abelian category $\mathcal{A}$ which is the heart of a t-structure on $\mathcal{C}$).

2.1 Hall algebra

We define the *Hall algebra* $H(\mathcal{C})$ as follows. Elements are morphisms

$$f\colon \mathcal{X} \to \mathcal{M}$$

where $\mathcal{X}$ is a stack and f is a morphism (*not* assumed to be representable). Roughly speaking, f is a family of objects of $\mathcal{C}$ parametrized by $\mathcal{X}$. Let

$$\begin{array}{ccc} \mathcal{M}^{(2)} & \longrightarrow & \mathcal{M} \\ \downarrow & & \\ \mathcal{M} \times \mathcal{M} & & \end{array}$$

be the universal extension of objects of $\mathcal{M}$, where

$$\begin{array}{ccc} (A_1 \to B \to A_2) & \longmapsto & B \\ \downarrow & & \\ (A_1, A_2) & & \end{array}$$

for each distinguished triangle

$$A_1 \to B \to A_2 \to A_1[1]$$

in $\mathcal{C}$. Given $f\colon \mathcal{X} \to \mathcal{M}$ and $g\colon \mathcal{Y} \to \mathcal{M}$, form the fiber product diagram

$$\begin{array}{ccccc} \mathcal{Z} & \longrightarrow & \mathcal{M}^{(2)} & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow & & \\ \mathcal{X} \times \mathcal{Y} & \xrightarrow{f \times g} & \mathcal{M} \times \mathcal{M} & & \end{array}$$

Then the product $f * g$ is the composition $\mathcal{Z} \to \mathcal{M}$ of the arrows along the top row.

One shows that $H(\mathcal{C})$ is an associative ring with unit. (The unit is the map from a point to $\mathcal{M}$ with image the zero object of $\mathcal{C}$.)

Example 2.1. Let $f\colon \mathcal{X} \to \mathcal{M}$, $g\colon \mathcal{Y} \to \mathcal{M}$ be the maps from a point with images the objects E, F of $\mathcal{C}$. Then $\mathcal{Z}$ is the fiber of $\mathcal{M}^{(2)} \to \mathcal{M} \times \mathcal{M}$ over (E, F), which is the quotient stack

$$[\operatorname{Ext}^1(F, E) / \operatorname{Hom}(F, E)]$$

where $\operatorname{Hom}(F, E)$ acts trivially on $\operatorname{Ext}^1(F, E)$.

2.2 Integration map

Assume that $\mathcal{C}$ is a CY3 category. Let $(Z, \mathcal{C}^{\mathrm{ss}}, \mathrm{Log})$ be a stability condition on $\mathcal{C}$ (see Lecture notes from 3/12/10).

Let $\Gamma = K(\mathcal{C})/\sim$ be the source of the central charge $Z\colon \Gamma \to \mathbb{C}$, a quotient of the K-theory $K(\mathcal{C})$ of $\mathcal{C}$. For example, if $\mathcal{C} = D(X)$ is the bounded derived category of coherent sheaves on a smooth projective Calabi-Yau 3-fold X, then we can take Γ to be the topological K-theory of X. We will assume that Γ is a finitely generated free abelian group. We also assume that the Euler form

$$\chi(E, F) = \sum (-1)^i \dim \mathrm{Ext}^i(E, F)$$

descends to $\mathbb{Z}$-valued bilinear form $\langle\, ,\, \rangle$ on Γ. (In the example above, this follows from the Hirzebruch-Riemann-Roch formula.) Notice that the form $\langle\, ,\, \rangle$ is skew because $\mathcal{C}$ is CY3 ($\mathrm{Ext}^i(E, F) \simeq \mathrm{Ext}^{3-i}(F, E)^*$).

Let $V \subset \mathbb{C} = \mathbb{R}^2$ be a strict sector. That is, $x, y \in V$ implies $x + y \in V$, $x \in V$, $c \in \mathbb{R}_{>0}$ implies $cx \in V$, and V does *not* contain a line. We also insist that $0 \notin V$. Let $\mathcal{C}_V$ denote the full subcategory of $\mathcal{C}$ given by extensions of semistables E such that $Z(E) \in V$ and $\mathrm{Log}\, Z(E) \in \mathrm{Log}\, V$ (where we have chosen a branch of the logarithm over V). Then $\mathcal{C}_V$ is a quasiabelian category, cf. [B07, §4]. By the support property axiom for stability conditions, the classes of all objects in $\mathcal{C}_V$ are contained in a convex cone $C(V, Z) \subset \Gamma \otimes \mathbb{R}$. Let P denote the monoid (semigroup)

$$P := (C(V, \mathbb{Z}) \cap \Gamma) \cup \{0\}.$$

Define the associated completion of the Hall algebra

$$\hat{H}(\mathcal{C}_V) := \prod_{\gamma \in P} H(\mathcal{C}_V \cap \mathrm{cl}^{-1}(\gamma))$$

where cl is the map

$$\mathrm{cl}\colon K(\mathcal{C}) \to \Gamma.$$

Define a $\mathbb{Q}(q)$-algebra

$$R_{V,q} := \prod_{\gamma \in P} \mathbb{Q}(q) \cdot x_\gamma$$

with multiplication

$$x_\gamma \cdot x_\mu = q^{\frac{1}{2}\langle \gamma, \mu \rangle} \cdot x_{\gamma+\mu}$$

and unit $x_0 = 1$. There is an integration map

$$I\colon \hat{H}(\mathcal{C}_V) \to R_{V,q}.$$

Let E be an object of $\mathcal{C}_V$, and also denote by E the map from a point to $\mathcal{M}$ with image E. If $\mathrm{Ext}^2(E,E)=0$, then

$$I(E) = q^{-(\dim \mathrm{Ext}^1(E,E) - \dim \mathrm{Hom}(E,E))/2} \cdot x_{[E]}.$$

In general, we need to correct this formula using a Behrend weight function. If we have a continuous family $f\colon X \to \mathcal{M}$ of objects E such that $\mathrm{Ext}^*(E,E)$, $[E]$ and the weight function are constant, then

$$I(f) = I(E) \cdot P_X(q)$$

where $P_X(q)$ is the *Serre polynomial*

$$P_X(q) = \sum_i (-1)^i \sum_w \dim H_c^{i,w}(X) q^{w/2}.$$

Here $H_c^{i,w}$ denotes the weight w part of the degree i cohomology with compact supports. For example $P_{\mathbb{A}^n}(q) = q^n$. If X is defined over $\mathbb{Z}$ then $P_X(q)$ is related to the number of $\mathbb{F}_q$-points of X.

Conjecture 2.2. [KS08] I is a ring homomorphism.

Some versions of this conjecture have been proven in [KS08].

Consider the element $A_V \in \hat{H}(\mathcal{C}_V)$ corresponding to the open substack $\mathcal{M}_V \subset \mathcal{M}$ parametrizing objects of $\mathcal{C}_V$. Write $A_{V,q} = I(A_V)$.

Now suppose that $V = V_1 \sqcup V_2$ is the disjoint union of two strict sectors V_1, V_2 in clockwise order. Then the Harder-Narasimhan property for the stability condition implies the *factorization property*

$$A_V = A_{V_1} \cdot A_{V_2},$$

see [KS08, p. 88], and so

$$A_{V,q} = A_{V_1,q} \cdot A_{V_2,q}.$$

The element $A_{V,q} \in R_{V,q}$ does not change as we continuously vary the stability condition unless the central charge of a semistable object enters or leaves the sector V, see [KS08, §2.3]. This leads to the wall crossing formula.

3 Examples

3.1

Let $\mathcal{C}$ be a CY3 category generated by a single *spherical object* E (that is, $\mathrm{Ext}^*(E,E) \simeq H^*(S^3,k)$). We have $\Gamma = \mathbb{Z}\cdot[E]$ and $\langle\,,\,\rangle = 0$. Define a stability condition by $Z(E) = z \in \mathbb{C}\setminus\{0\}$,

$$\mathcal{C}^{\mathrm{ss}} = \mathrm{Ob}(\mathcal{C}) = \{0, E, E^{\oplus 2}, \ldots\}$$

and $\operatorname{Arg} z \in [0, 2\pi)$. Let V be a strict sector containing the ray $\mathbb{R}_{>0}z$. Then

$$A_V = \sum_{n\geq 0} [E^{\oplus n} / \operatorname{Aut}(E^{\oplus n})] = \sum_{n \geq 0} [E^{\oplus n} / \operatorname{GL}(n)]$$

so

$$A_{V,q} = \sum q^{\frac{1}{2}n^2} \cdot \frac{1}{P_{\operatorname{GL}(n)}(q)} \cdot x^n = \sum_{n\geq 0} \frac{q^{\frac{1}{2}n^2}}{(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})} \cdot x^n,$$

where $x = x_{[E]}$. That is, $A_{V,q}$ is the *quantum dilogarithm.* We will denote this function $E(x) = E(q^{\frac{1}{2}}, x)$.

3.2

Let $\mathcal{C}$ be a CY3 category generated by 2 spherical objects E_1, E_2 such that $\dim \mathrm{Ext}^1(E_2,E_1) = 1$ and $\mathrm{Ext}^i(E_2,E_1) = 0$ for $i \neq 1$. We have $\Gamma = \mathbb{Z}[E_1] \oplus \mathbb{Z}[E_2]$ and $\langle E_1, E_2\rangle = 1$. Let E_{12} be the unique (up to isomorphism) nontrivial extension of E_2 by E_1. Then E_{12} is a spherical object. Define a stability condition by $Z(E_i) = z_i \in \mathbb{C}$, $\mathrm{Im}(z_i) > 0$, $\operatorname{Arg} z_i \in (0,\pi)$, with semistables $0, E_1^{\oplus n}, E_2^{\oplus n}$, $n \in \mathbb{N}$ if $\operatorname{Arg}(z_1) > \operatorname{Arg}(z_2)$ and $0, E_1^{\oplus n}, E_2^{\oplus n}, E_{12}^n$, $n \in \mathbb{N}$ if $\operatorname{Arg}(z_1) < \operatorname{Arg}(z_2)$. Let V be a strict sector containing z_1, z_2. The image of the integration map is contained in the subring

$$R_q := \mathbb{Q}(q)\langle\langle x_1, x_2\rangle\rangle/(x_1x_2 - qx_2x_1)$$

where $x_1 = x_{[E_1]}$, $x_2 = x_{[E_2]}$. We have $x_{12} := x_{[E_{12}]} = q^{-\frac{1}{2}}x_1x_2 = q^{\frac{1}{2}}x_2x_1$. The wall crossing formula implies the identity

$$E(x_1)E(x_2) = E(x_2)E(x_{12})E(x_1).$$

This is the so called pentagon identity for the quantum dilogarithm.

References

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