Examples of stability conditions on derived categories

Paul Hacking

3/12/10

1 Bridgeland stability conditions

Let $D = D(X)$ be the bounded derived category of coherent sheaves on a smooth projective variety X. Let $K(D)$ be the numerical K-theory of D (the quotient of the K-theory by the kernel of the Euler form $\chi(E,F) = \sum(-1)^i \dim \operatorname{Ext}^i(E,F)$). This is rationally the same as the topological K-theory or equivalently the even cohomology of X (via the Chern character).

Definition 1.1. A *stability condition* on D is the following data

(a) A homomorphism $Z\colon K(D) \to \mathbb{C}$ (the *central charge*),

(b) A collection $D^{\text{ss}} \subset D$ of nonzero *semistable objects*, such that $Z(E) \neq 0$ for $E \in D^{\text{ss}}$, and

(c) A choice of logarithm $\operatorname{Log}(Z(E)) = \log|Z(E)| + i \operatorname{Arg}(Z(E))$ for each $E \in D^{\text{ss}}$,

satisfying the following conditions:

(1) $\operatorname{Arg}(Z(E[n])) = Arg(Z(E)) + n\pi$ for all $E \in D^{\text{ss}}$ and $n \in \mathbb{Z}$.

(2) If $E_1, E_2 \in D^{\text{ss}}$ and $\operatorname{Arg}(Z(E_1)) > \operatorname{Arg}(Z(E_2))$ then $\operatorname{Hom}_D(E_1, E_2) = 0$.

(3) (Harder-Narasimhan filtration) For each $E \in \operatorname{Ob}(D)$ there is an $n \geq 0$ and a "filtration"

$$0 = E_0 \to E_1 \to \cdots \to E_n = E$$

such that the "quotients" $F_i := \operatorname{Cone}(E_{i-1} \to E_i)$ are semistable and $\operatorname{Arg}(F_1) > \cdots > \operatorname{Arg}(F_n)$.

(4) Local finiteness [B02, Def. 5.7, p. 17](details omitted).

Here is some background:

(1) $E[1]$ denotes the left shift: we have

$$E = \cdots \to E^0 \to E^1 \to \cdots$$

and $E[1]^n := E^{n+1}$ (with differential $d_{E[1]} = -d_E$). Then for example for sheaves $E = E^0$ and $F = F^0$ we have $\mathrm{Hom}_D(E, F[i]) = \mathrm{Ext}^i(E, F)$.

(2) Note that $[E[1]] = [-E] \in K(D)$. This essentially forces the behaviour of Arg under shifts.

(3) D is not an abelian category but a *triangulated category*. The role of short exact sequences is played by so called *distinguished triangles*. These are sequences of complexes

$$E \to F \to G \to E[1]$$

quasi-isomorphic to

$$E \to F \to \mathrm{Cone}(E, F) \to E[1]$$

where the *cone* $\mathrm{Cone}(E, F)$ of $f \colon E \to F$ is defined by

$$\mathrm{Cone}(E, F)^i = E^{i+1} \oplus F^i$$

with differential

$$\begin{pmatrix} d_E^{i+1} & 0 \\ f^{i+1} & d_F^i \end{pmatrix}.$$

See [C05, §2]. This is why we use quotes in 1.1(3) above.

Example 1.2. Let X be a smooth projective curve. Let $D^{\mathrm{ss}} \subset D = D(X)$ be the shifts of nonzero semistable sheaves. Define the central charge Z by

$$Z(E) = -\deg(E) + i \,\mathrm{rank}(E)$$

for $E \in D$. For $E = E^0$ a sheaf define $\mathrm{Log}(Z(E))$ by

$$\mathrm{Arg}(Z(E)) \in (0, \pi]$$

(Then $\mathrm{Arg}(Z(E[n])) \in (n\pi, (n+1)\pi]$ by condition (1) above.) Notice that Arg and the slope $\mu(E) = \deg(E)/\mathrm{rank}(E)$ give the same ordering, so 1.1(2) and (3) follow from the usual results for slope stability in this case.

Note that $Z(E) \neq 0$ for $E \neq 0$ a sheaf. This fails if we try to define a stability condition in the same way for $\dim X > 1$. (In that case, slope stability should be thought of as a degenerate limit of stability conditions.)

If X is not Calabi–Yau (for example a curve of genus $g \neq 1$), the definition of stability condition should probably be modified (I am not sure how exactly).

Theorem 1.3. *[B07, Thm. 1.2] The space* $\mathrm{Stab}(D)$ *of stability conditions on* D *is naturally a complex manifold and the map*

$$\mathrm{Stab}(D) \to \mathrm{Hom}(K(D), \mathbb{C}), \quad (Z, D^{\mathrm{ss}}, \mathrm{Arg}) \mapsto Z$$

is a local homeomorphism.

Note: Actually, Bridgeland's statement is that the map is a local homeomorphism onto a open subset of a linear subspace, but I believe (?) that with the stronger version of local finiteness defined by Kontsevich–Soibelman this subspace is all of $\mathrm{Hom}(K(D), \mathbb{C})$. See [KS08, p. 6–9]

2 Group action

We have a right action of the universal cover $\tilde{\mathrm{GL}}_2^+(\mathbb{R})$ of $\mathrm{GL}_2^+(\mathbb{R})$ on the space $\mathrm{Stab}(D)$ of stability conditions as follows. An element $\tilde{A} \in \tilde{\mathrm{GL}}_2^+(\mathbb{R})$ is given by a pair (A, f) where $A \in GL_2^+(\mathbb{R})$ and $f \colon \mathbb{R} \to \mathbb{R}$ is an increasing function such that A and f agree on $S^1 = (\mathbb{R}^2 \setminus \{0\})/\mathbb{R}_{>0} = \mathbb{R}/2\mathbb{Z}$. Then

$$\tilde{A} \colon (Z, D^{\mathrm{ss}}, \mathrm{Arg}) \mapsto (A^{-1} \circ Z, D^{\mathrm{ss}}, f^{-1} \circ \mathrm{Arg}).$$

Also we have a left action of the group $\mathrm{Aut}(D)$ of autoequivalences of D commuting with the $\tilde{\mathrm{GL}}_2^+(\mathbb{R})$ action, given by

$$\phi \colon (Z, D^{\mathrm{ss}}, \mathrm{Arg}) \mapsto (Z \circ \Phi^{-1}, \Phi(D^{\mathrm{ss}}), \mathrm{Arg}).$$

3 Elliptic curve

Let X be a smooth projective curve of genus 1.

Theorem 3.1. *[B07, Thm. 9.1] The action of* $\tilde{\mathrm{GL}}_2^+(\mathbb{R})$ *on* $\mathrm{Stab}(D(X))$ *is free and transitive.*

So, up to the $\tilde{\mathrm{GL}}_2^+(\mathbb{R})$ action, every stability condition is given by slope stability as in 1.2.

We can now explain the connection with mirror symmetry in this case. Recall that the space $\mathrm{Stab}(D)$ of stability conditions on $D = D(X)$ is expected to contain the Kähler moduli space of X, which is identified with the complex moduli space of the mirror Y. In the case of an elliptic curve X, the mirror Y is also an elliptic curve, and we find

$$\operatorname{Stab}(D)/\operatorname{Aut}(D) = \tilde{\operatorname{GL}}_2^+(\mathbb{R})/\operatorname{SL}(2,\mathbb{Z}) \tag{1}$$

which is a $\mathbb{C}^\times$ bundle over

$$\mathcal{H}/\operatorname{PSL}(2,\mathbb{Z}) = M_{1,1},$$

the moduli space of elliptic curves Y (where $\mathcal{H}$ denotes the upper half plane). The $\mathbb{C}^\times$ bundle is given by a choice of nonzero holomorphic 1-form Ω on the elliptic curves Y. Here we used the description of $\operatorname{Aut}(D)$ due to Mukai [M81] as follows. We have

$$\operatorname{Aut}(D) = \langle \operatorname{Aut}(X), \operatorname{Pic}(X), [1], \Phi \rangle$$

where $L \in \operatorname{Pic}(X)$ acts by $\otimes L$, and Φ is the (original) Fourier–Mukai transform defined by the Poincaré bundle $\mathcal{P}$ on $X \times X$. That is,

$$\mathcal{P} = \mathcal{O}_{X\times X}(\Delta) \otimes p_1^*\mathcal{O}_X(-P_0) \otimes p_2^*\mathcal{O}_X(-P_0)$$

is the universal line bundle on $X \times X = X \times \operatorname{Pic}^0(X)$, suitably normalized, where $\Delta \subset X \times X$ denotes the diagonal and $P_0 \in X$ is the origin, and

$$\Phi \colon D(X) \to D(X), \quad F \mapsto Rp_{2*}(p_1^*F \otimes \mathcal{P}).$$

Mukai showed that Φ is an equivalence and satisfies $\Phi^2 = \iota^* \circ [-1]$, where $\iota \colon X \to X$ is the involution $P \to -P$. The action of $\operatorname{Aut}(D)$ on $\operatorname{Stab}(D)$ is described as follows. $\operatorname{Aut}(X)$ and $\operatorname{Pic}^0(X)$ act trivially. We have $K(D) = \mathbb{Z}^2$, given by rank and degree (so standard basis is $[\mathcal{O}_X]$, $[\mathcal{O}_P]$, where $P \in X$ is a point). Then $[1]$, $\otimes\mathcal{O}(P)$, Φ, act on $K(D) = \mathbb{Z}^2$ via

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The last two are standard generators for $\operatorname{SL}(2,\mathbb{Z})$, and this gives the identification (1) above.

4 Conifold

Let $P \in X$ be a 3-fold ordinary double point singularity and $f \colon Y \to X$, $f^+ \colon Y^+ \to X$ the two small resolutions of X. Thus the exceptional locus of f, f^+ is an smooth rational curve $C \subset Y, C^+ \subset Y^+$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. The birational map $Y \dashrightarrow Y^+$ is called a *flop*. We regard Y, Y^+ as analytic tubular neighbourhoods of C, C^+. They are so called local Calabi–Yau manifolds (the canonical line bundle is trivial). There is an equivalence of derived categories

$$\Phi\colon D(Y) \to D(Y^+)$$

given by

$$F \mapsto Rp_{+*}(p^* F \otimes \mathcal{O}_Z),$$

where $Z = Y \times_X Y^+ \subset Y \times Y^+$.

We consider the full subcategory $D(Y/X) \subset D(Y)$ of complexes whose cohomology sheaves have set-theoretic support on C. We have $K(D(Y/X)) = \mathbb{Z}^2$, generated by $[\mathcal{O}_C]$ and $[\mathcal{O}_y]$, where $y \in C$ is a point. By using the (free) action of $\mathbb{C} \subset \tilde{\mathrm{GL}}_2^+(\mathbb{R})$ (the inverse image of $\mathbb{C}^\times \subset \mathrm{GL}_2^+(\mathbb{R})$) on $\mathrm{Stab}(D(Y/X))$ we may restrict our attention to *normalized* stability conditions satisfying $Z(\mathcal{O}_y) = -1$. (I suppose here we are assuming that $Z(\mathcal{O}_y) \neq 0$).

For $\beta + i\omega \in H^2(Y, \mathbb{C}) = \mathbb{C}$ and $\omega \cdot [C] > 0$, define a normalized stability condition by

$$Z(E) = (\beta + i\omega)\,\mathrm{ch}_2(E) - \mathrm{ch}_3(E),$$

D^{ss} = shifts of semistable sheaves on Y, and $\operatorname{Arg} Z(E) \in (0, \pi]$ for E a sheaf. Let $U(Y/X)$ denote the space of such stability conditions. We define $U(Y^+/X)$ in the same way. Note that, since Y and Y^+ are related by codimension 2 surgery, we have an identification $H^2(Y^+, \mathbb{Z}) = H^2(Y, \mathbb{Z})$, and under this identification $[C^+] = -[C] \in H^2(Y, \mathbb{Z})^*$. So points of $U(Y^+/X)$ correspond to classes $\beta + i\omega \in H^2(Y, \mathbb{C})$ such that $\omega \cdot [C] < 0$.

Write $\mathrm{Stab}(Y/X)$ for the connected component of the space of normalized stability conditions containing $U(Y/X)$.

Theorem 4.1. *[T08, Ex., p. 22] The map*

$$\mathrm{Stab}(Y/X) \to \mathbb{C}, \quad Z \mapsto Z(\mathcal{O}_C)$$

is an covering space over $\mathbb{C} \setminus \mathbb{Z}$. Let Γ be the group of autoequivalences of $D(Y/X)$ preserving $[\mathcal{O}_y] \in K(D(Y/X))$ and the connected component $\mathrm{Stab}(Y/X)$ *of the space of normalized stability conditions. Then we have an exact sequence*

$$\pi_1(\mathbb{C} \setminus \mathbb{Z}) \to \Gamma \to \operatorname{Pic} Y \to 0,$$

Note that $\mathrm{Pic}\, Y \simeq \mathbb{Z}$ *acts on* $\mathbb{C} \setminus \mathbb{Z}$ *by translation. So*

$$\mathrm{Stab}(Y/X)/\Gamma = (\mathbb{C} \setminus \mathbb{Z})/\mathbb{Z} = \mathbb{C}^{\times} \setminus \{1\} = \mathbb{P}^1_z \setminus \{0, 1, \infty\}.$$

The inverse images of the punctured discs $0 < |z| < 1$ *and* $1 < |z| < \infty$ *in* $\mathrm{Stab}(Y/X)$ *are the half planes* $U(Y/X)$ *and* $U(Y^+/X)$*. The inverse image of the punctured equator* $\{|z| = 1\} \setminus \{1\}$ *consists of stability conditions with semistable objects being shifts of perverse sheaves in the sense of [B02].*

Toda proves an analogous result for $Y \to X$ a small resolution of an isolated Gorenstein 3-fold singularity, see [T08, Thm. 1.2] for the precise statement.

One way to describe the perverse sheaves arising above is as follows. The derived category of $C \simeq \mathbb{P}^1$ is generated by $\mathcal{O}_C, \mathcal{O}_C(1)$. Identify Y with an analytic neighbourhood of the zero section in the normal bundle $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$ of C, and let $\pi \colon Y \to C$ be the projection. Let $E = \pi^*\mathcal{O}_C \oplus \pi^*\mathcal{O}_C(1)$. Let $A = \mathrm{End}(E)$. Then A is a noncommutative algebra of global dimension 3 with center $\mathcal{O}_X$, and the map

$$\Phi = \mathrm{RHom}(E, \cdot) \colon D(Y) \to D(\mathrm{Mod}\, A)$$

is an equivalence of categories. It restricts to an equivalence between $D(Y/X)$ and the full subcategory of $D(\mathrm{Mod}\, A)$ of complexes with finite dimensional cohomology. The abelian category of perverse sheaves on Y is identified with $\mathrm{Mod}\, A$ via Φ.

The algebra A can be described explicitly as follows. Let Q be the quiver with two vertices $0, 1$ and arrows $a_1, a_2 \colon 0 \to 1$ and $b_1, b_2 \colon 1 \to 0$. Let W be the *potential* on Q given by

$$W = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1.$$

Then A is the quotient of the path algebra of Q by the ideal of relations generated by the partial derivatives $\frac{\partial W}{\partial a_i}, \frac{\partial W}{\partial b_i}$ of W with respect to the variables a_i, b_i corresponding to the arrows of Q. This is an instance of [KS08, Thm. 9, p. 129], which establishes a correspondence between CY3 categories together with a collection of spherical generators and quivers with potential. The spherical generators in our example are $i_*\mathcal{O}[1], i_*\mathcal{O}(1)$, where $i \colon C \subset Y$ is the inclusion of the zero section.

References

[B02] T. Bridgeland, Flops and derived categories, Invent. Math. 147 (2002), no. 3, 613–632, and arXiv:math/0009053v1 [math.AG] .

[B07] T. Bridgeland, Stability conditions on triangulated categories, Ann. of Math. (2) 166 (2007), no. 2, 317–345, and arXiv:math/0212237v3 [math.AG] .

[C05] A. Caldararu, Derived categories of sheaves: a skimming. Snowbird lectures in algebraic geometry, 43–75, Contemp. Math., 388, AMS 2005, and arXiv:math/0501094v1 [math.AG] .

[KS08] M. Kontsevich and Y. Soibelman, Stability structures, motivic Donaldson-Thomas invariants and cluster transformations, preprint arXiv:0811.2435v1 [math.AG] .

[M81] S. Mukai, Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves. Nagoya Math. J. 81 (1981), 153–175.

[T08] Y. Toda, Stability conditions and crepant small resolutions, Trans. Amer. Math. Soc. 360 (2008), no. 11, 6149–6178, and arXiv:math/0512648v3 [math.AG] .