STABILITY CONDITIONS ON DERIVED CATEGORIES OF CY 3-FOLDS (AFTER KONTSEVICH–SOIBELMAN)

PAUL HACKING

Consider a complex manifold X and holomorphic vector bundles E over X.

Objects of "bounded derived category $D(X)$ of coherent sheaves on X" are finite complexes of vector bundles

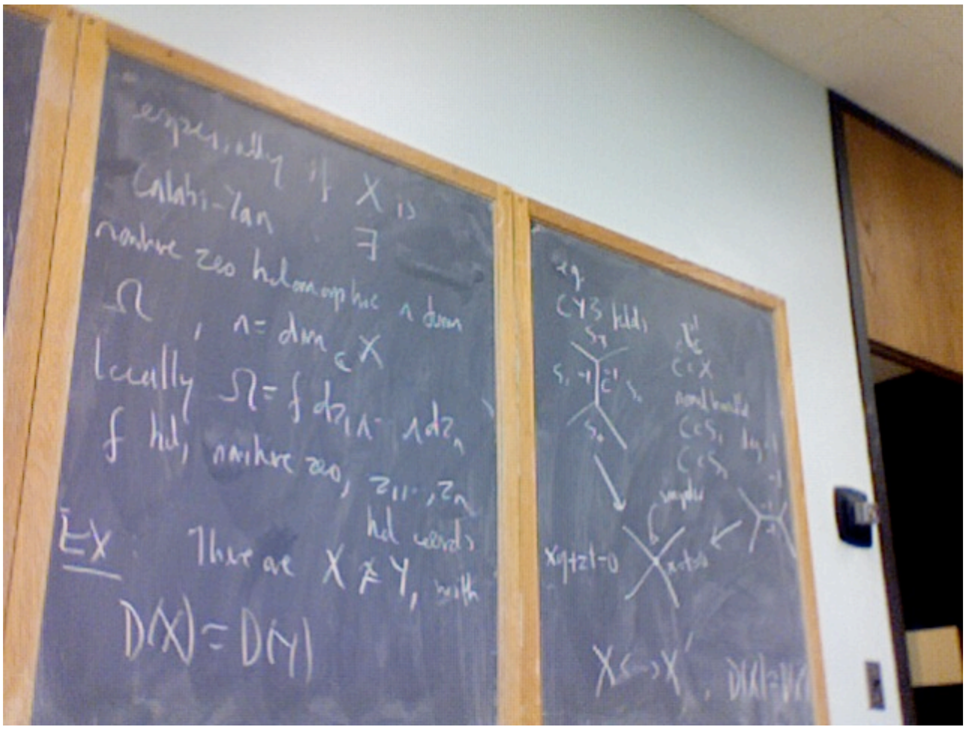
$$E_0 \to^d E_1 \to^d \ldots, \qquad d^2 = 0$$

Morphisms $f : (E, d) \to (F, d)$ are commutative diagrams (more precisely, equivalence classes of such up to homotopy) plus add inverses of quasi-isomorphisms, i.e. maps inducing isomorphism on cohomology.

One reason why we care is because $\mathrm{Aut}(D(X))$ is richer than $\mathrm{Aut}(X)$ in an interesting way, especially if X is a Calabi-Yau, i.e., if there exists a nowhere zero holomorphic n-form Ω, where $n = \dim X$: locally $\Omega = f dz_1 \wedge \ldots \wedge dz_n$, where f is a nowhere zero holomorphic function and $z_1, \ldots, z_n$ are local coordinates.

If X is a curve of genus > 1 (or any variety with an ample canonical class), $D(X)$ determines X. But not so for CY.

For example, suppose we have a curve $C \subset X$ contained in two surfaces, S_1 and S_2. Suppose the normal bundle of C is $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$.



One can contract C to get (locally) $xy + zt = 0$ and then flop to get another CY X'. Then X and X' are not isomorphic but have equivalent derived categories.

This was a pretalk, now a talk.

Big Picture: suppose X and Y are compact mirror CY 3-folds.

Complex geometry of X is related to symplectic geometry of Y. We choose Kähler metric on Y, which gives a Kähler form ω: this is a symplectic form. Theorem of Yau: we can assume that this metric is Ricci-flat.

Kontsevich (94): Homological mirror symmetry proposal. There is an equivalence between the derived category of X and the Fukaya category of Y. Objects of the former are, roughly speaking, vector bundles, and morphisms are $\mathrm{Ext}^i(E,F)$. Objects of the latter are, roughly speaking, Lagrangian submanifolds $L \subset Y$ (i.e. $\omega|_L = 0$ and $\dim L = \frac{1}{2}\dim Y$) and morphisms are "Floer homology"

$$HF^*(L,L') = \oplus_{p\in L\cap L'}\mathbb{C}$$

with a differential given by counts of holomorphic disks $f : \Delta \to Y$, $\partial\Delta \subset L \cup L'$. To define "holomorphic disks" it suffices to choose an almost complex structure $J : TY \to TY$ compatible with the symplectic form ω. The counts do not depend on the choice of J.

Classical mirror symmetry: Kähler moduli of X is related to the complex moduli of Y. For example, periods, i.e. integrals $\int_\gamma \Omega$, where $\gamma \in H_3(Y,\mathbb{Z})$ is a cycle, are related to counts of rational curves $C \simeq \mathbb{P}^1 \to X$.

What are the Kähler moduli?

$$B + i\omega \in U \subset H^2(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^*$$

(U is an open analytic subset). More precisely,

$$U = H^2(X,\mathbb{R}) + iK/H^2(X,\mathbb{Z}) \subset H^2(X,\mathbb{C})/H^2(X,\mathbb{Z}) = H^2(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^*,$$

where $K \subset H^2(X,\mathbb{R})$ is the Kähler cone of X.

What is the meaning of the Kähler moduli in terms of $D = D(X)$?

M. Douglas, T. Bridgeland proposal: They should correspond to stability conditions on $D(X)$

Mumford: X a complex manifold, E a holomorphic vector bundle, we can use the Kähler class $[\omega]$ to define stable vector bundles: E is stable if whenever $F \subset E$,

$$\mu(F) < \mu(E),$$

where the slope μ is defined as follows:

$$\mu(E) = \frac{c_1(E)\cdot[\omega]^{n-1}}{\mathrm{rk}(E)} = \frac{\deg E}{\mathrm{rk}\, E}$$

Stable vector bundles have nice moduli spaces, etc. We want to generalize this to $D(X)$.

Warning: Kähler moduli will be a subspace of $\mathrm{Stab}(X)$ locally given by

$$H^{1,1} \subset \oplus_p H^{p,p}$$

(algebraic cohomology). So:

$$\mathcal{M}_{cpx}(Y) \simeq \mathcal{M}_{\textit{Kähler}}(X) \subset \mathrm{Stab}(X)$$

Donagi–Markman: Consider the $\mathbb{C}^*$ bundle $\{(Y,\Omega)\} \mapsto \{Y\}$:

$$\tilde{\mathcal{M}}_{cpx}(Y) \to \mathcal{M}_{cpx}(Y).$$

Consider the *intermediate Jacobian*

$$J = H^3(Y,\mathbb{C})/(H^{3,0} \oplus H^{2,1} + H^3(Y,\mathbb{Z})) = (H^{1,2} \oplus H^{0,3})/H^3(Y,\mathbb{Z}),$$

a complex torus. We can do it in families which gives a torus bundle

$$J \to \tilde{\mathcal{M}}_{cpx}(Y).$$

0.1. THEOREM (DM 94). *J is a hyperk"ahler manifold.*

Recall that X is hyperkähler if it is a smooth manifold with maps

$$I, J, K : TX \to TX$$

that satisfy the usual quaternionic relations. So X has an S^2-space of complex structures $aI + bJ + cK$, $a^2 + b^2 + c^2 = 1$.

Idea of KS: try to construct J on the mirror side, i.e. over $\mathcal{M}_{\textit{Kähler}}(X)$. In one of cpx structures, $J \to \tilde{\mathcal{M}}_{cpx(Y)}$ will be a C^∞ Lagrangian torus fibration. We can try to construct the corresponding family $J \to \tilde{\mathcal{M}}_{\textit{Kähler}}(Y)$ by scattering diagrams.

Locally $J \to \tilde{\mathcal{M}}_{\textit{Kähler}}(Y)$ is modeled on

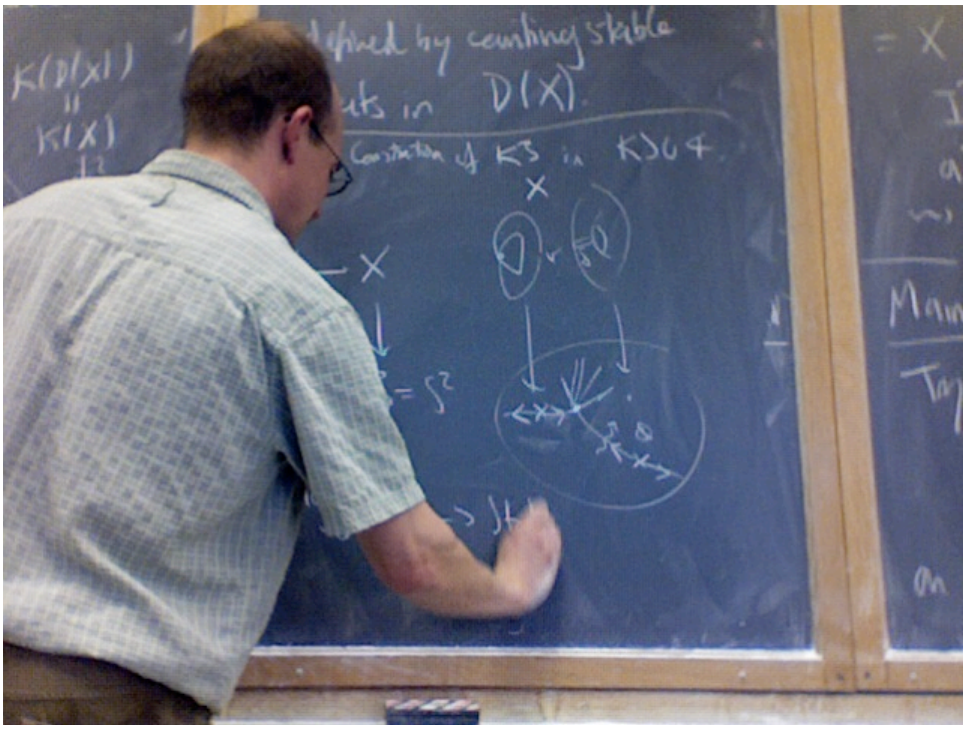
$$(\mathbb{C}^*)^k \to \mathbb{R}^k, \quad (z_i) \mapsto (\log |z_i|)$$

$$(\mathbb{C}^*)^k = K(D)^* \otimes \mathbb{C}^*,\ K(D(X)) = K(X) = H^{ev}(X,\mathbb{Z})$$

(on CY3 all cohomology is algebraic). Gluing of these pieces is encoded in chamber decomposition of $\mathrm{Stab}(X)$ with automorphisms attached to codimension 1 walls defined by counts of stable objects in $D(X)$.

Picture for K3 in Kontsevich-Soibelman as a fibration over S^2 with 24 singular fibers. They introduced scattering diagrams on the base S^2 to encode gluing.

Analogy: S^2 corresponds to the space $\mathrm{Stab}(X)$ of stability conditions. And the K3 X corresponds to Donagi–Markman's J. To be continued ...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