Reading Seminar
Spring 2020
Introduction to Classical Mirror Symmetry

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Feb 3, 2020

Amazing Predictions

Let $n_d = \#$ rational curves of degree d on the quintic 3-fold $V \subseteq \mathbb{P}^4$.
In 1991, Candelas, de la Ossa, Green and Parkes predicted that:

$$\begin{aligned} n_1 &= 2875 \\ n_2 &= 609250 \\ n_3 &= 317206375 \\ n_{10} &= 70428\ 81649\ 78454\ 68611\ 34882\ 49750 \end{aligned}$$

and formulas for n_d for all d (to be give later).

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- Sheldon Katz proved $n_2 = 609250$ in 1986.
- In 1991, the predictions for $n \geq 3$ were mind-blowing.

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The Basic Idea

Enumerative geometry on quintic 3-fold V (hard)

$\longleftrightarrow$

A superconformal field theory (SCFT) on V

$\longleftrightarrow$

A SCFT on the quintic mirror V°

$\longleftrightarrow$

Hodge theory on V° (easy)

A SCFT is a String Theory

- String theory is 10 dimensional: 4 dimensions for the space-time of general relativity and 6 dimensions for the quantum theory.
- The 6-dimensional quantum piece is a compact manifold V, size $\hbar$.
- Physics $\Rightarrow$ V is complex with trivial canonical bundle and $b_1 = 0$.
- Thus V is a Calabi-Yau 3-fold.

Mirror Symmetry

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A and *B* Models

The SCFT on V has twisted versions called the A-model and the B-model that depend on two types of parameters:

- A-model parameters are Kähler moduli that encode the metric on V.
- B-model parameters are complex moduli that encode the complex structure of V.

The number of parameters of each type is determined by the Hodge numbers of V:

$$h^{pq}(V) = \dim H^q(V, \Omega_V^p).$$

Specifically:

- $h^{11}(V) =$ the number of Kähler moduli parameters.
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The Hodge Diamond

The Hodge numbers $h^{pq} = h^{pq}(V)$ of a smooth projective 3-fold V are often represented in the Hodge diamond shown on the left:

$$
\begin{array}{ccccccc}
 & & & h^{33} & & & \\
 & & h^{32} & & h^{23} & & \\
 & h^{31} & & h^{22} & & h^{13} & \\
h^{30} & & h^{21} & & h^{12} & & h^{03} \\
 & h^{20} & & h^{11} & & h^{02} & \\
 & & h^{10} & & h^{01} & & \\
 & & & h^{00} & & &
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{11} & & 0 & \\
1 & & h^{21} & & h^{21} & & 1 \\
 & 0 & & h^{11} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

The Hodge diamond of a Calabi-Yau 3-fold is shown on the right. Note that this Hodge diamond is completely determined by

$$
\begin{aligned}
h^{11} &= h^{11}(V) = \#\,\text{Kähler parameters} \\
h^{21} &= h^{21}(V) = \#\,\text{complex parameters.}
\end{aligned}
$$

Mirror Symmetry

In mirror symmetry, a given a family of Calabi-Yau 3-folds V has a mirror family of Calabi-Yau 3-folds V° such that the corresponding SCFTs are the same via an isomorphism that does two things:

- Interchanges the A- and B-models.
- Interchanges Kähler and complex moduli.

In particular, V and V° should satisfy

$$h^{11}(V^\circ) = h^{21}(V) \text{ and } h^{21}(V^\circ) = h^{11}(V).$$

It follows that

> The Hodge diamond of V is the mirror image of the Hodge diamond of V° about the 45° line through the center of the diamond.

This is the origin of the name "mirror symmetry".

Picture for the Quintic 3-Fold V and its Mirror V°

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 101 & & 101 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & & \\
 & & & & & & \\
 & & & V & & &
\end{array}
\qquad
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 & & & 1 & & & \\
 & & & & & & \\
 & & & V^\circ & & &
\end{array}
$$

The Quintic Mirror V°

- In $\mathbb{P}^4$, start with

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + \psi\, x_1 x_2 x_3 x_4 x_5 = 0$$

where $\psi \neq 0$ and $\psi^5 \neq -5^5$.

- Take the quotient under the action of

$$G = \{ (\zeta^{a_1}, \dots, \zeta^{a_5}) \in \mathbb{Z}_5^5 \mid \sum_i a_i \equiv 0 \bmod 5 \} / \mathbb{Z}_5,$$

where $\zeta = e^{2\pi i/5}$. Note that $|G| = 125$.

- Finally, V° is a crepant resolution of singularities.

Complex Moduli of V°

Because of the action by roots of unity on the original equation,

$$x = \psi^{-5}$$

parametrizes the complex moduli of V°.

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3-Point Correlation Functions

Experiments based on the Standard Model indicate the existence of 3 generations of particles, with interactions described by 3-point functions.

A-Model 3-Point Function on the Quintic 3-Fold V

Let q be the Kähler moduli parameter on V. Then the hyperplane class $H \in H^{1,1}(V)$ gives the three point function

$$\langle H, H, H\rangle = \int_V H \wedge H \wedge H + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1-q^d} = 5 + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1-q^d},$$

The infinite sum represents world sheet non-perturbative corrections.

B-Model 3-Point Function on the Quintic Mirror V°

Let Ω be holomorphic 3-form on V° and set $\theta = x\frac{d}{dx}$, where x is the complex moduli parameter. Then the Yukawa coupling is

$$Y = \langle \theta, \theta, \theta\rangle = \int_V \Omega \wedge \nabla_\theta \nabla_\theta \nabla_\theta(\Omega),$$

where ∇_θ is the Gauss-Manin connection. No corrections!

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A Mirror Theorem

Mirror symmetry gives a SCFT isomorphism that maps the A-model of V to the B-model of V°. This induces an equality of their 3-point functions via the mirror map $q = q(x)$. After we do two things:

- Compute the mirror map $q = q(x)$ explicitly, and
- Compute the Yukawa coupling $Y = \langle \theta, \theta, \theta \rangle$ explicitly,

we will get the following theorem:

Mirror Theorem for Quintic 3-Fold

For the mirror map

$$q(x) = -x \exp\left(\tfrac{5}{y_0(x)} \sum_{n=1}^{\infty} \tfrac{(5n)!}{(n!)^5} \left[\sum_{j=n+1}^{5n} \tfrac{1}{j}\right] (-1)^n x^n\right),$$

where $y_0(x) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} (-1)^n x^n$, we have

$$5 + \sum_{d=1}^{\infty} n_d \, d^3 \, \frac{q^d}{1-q^d} = \frac{5}{(1+5^5 x)} \frac{1}{y_0(x)^2} \left(\frac{q}{x} \frac{dx}{dq}\right)^3.$$

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The Picard-Fuchs Equation

Since V° has only one complex moduli parameter x, the periods $y = \int_\gamma \Omega$ of the 3-form Ω satisfy the Picard-Fuchs equation, which by standard methods is computed to be

$$\begin{aligned} 0 &= \Big(x\frac{d}{dx}\Big)^4 y + \frac{2\cdot 5^5 x}{1+5^5x}\Big(x\frac{d}{dx}\Big)^3 y + \frac{7\cdot 5^4 x}{1+5^5x}\Big(x\frac{d}{dx}\Big)^2 y \\ &\quad + \frac{2\cdot 5^4 x}{1+5^5x}\Big(x\frac{d}{dx}\Big) y + \frac{24\cdot 5x}{1+5^5x} y. \end{aligned}$$

This enables us to compute the Gauss-Manin connection ∇_θ, $\theta = x\frac{d}{dx}$, with the result that $Y = \int_V \Omega \wedge \nabla_\theta\nabla_\theta\nabla_\theta(\Omega)$ satisfies

$$\Big(x\frac{d}{dx}\Big) Y = \frac{-5^5 x}{1+5^5x} Y.$$

This implies $Y = \dfrac{c}{1+5^5x}$ for some constant c to be determined.

The Mirror Map

The Picard-Fuchs equation is hypergeometric in nature, which means that one can write down some explicit solutions, including

$$\begin{aligned} y_0(x) &= \sum_{n=0}^{\infty} \tfrac{(5n)!}{(n!)^5} (-1)^n x^n \\ y_1(x) &= y_0(x)\log(-x) + 5\sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5}\Big[\sum_{j=n+1}^{5n} \frac{1}{j}\Big] (-1)^n x^n. \end{aligned}$$

Here, y_0 is the only holomorphic solution (up to constant multiple), while y_1 has a logarithmic singularity at $x = 0$. Then:

The Mirror Map

$$q(x) = e^{y_1(x)/y_0(x)} = -x \exp\Big(\frac{5}{y_0(x)} \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5}\Big[\sum_{j=n+1}^{5n} \frac{1}{j}\Big](-1)^n x^n\Big)$$

Remaining Topics

There is a more to say – mirror symmetry is a rich topic.

In the remainder of the talk, I will focus on four things:

- Gromov-Witten invariants and rigorous n_d
- Givental's version of the Mirror Theorem
- Toric varieties and reflexive polytopes
- Experimental evidence for string theory and mirror symmetry

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Gromov-Witten Invariants and Rigorous n_d

The definition of n_d as the number of rational curves of degree d on V is problematic. The finiteness is the Clemens Conjecture, which currently is known only for $d \leq 10$.

The modern approach defines the Gromov-Witten invariants N_d of the quintic 3-fold V using moduli of stable maps and virtual fundamental classes (due to Kontsevich). One then defines n_d via

$$N_d = \sum_{k|d} n_{\frac{d}{k}} k^{-3}.$$

What is Known

- For $n \leq 9$, n_d is the number of rational curves of degree d on V.
- For $n = 10$, the number of degree 10 rational curves on V is

$$\underbrace{70428\,81649\,78454\,68611\,34882\,49750}_{n_{10}} - 6 \times 17,601,000.$$

Mirror symmetry assumes $\mathbb{P}^4$ has a generic complex structure. It isn't!

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Givental Mirror Theorem

If $H \in H^2(\mathbb{P}^4)$ is the hyperplane class, we get two cohomology-valued functions:

- $J_V = e^{tH} 5H \Big(1 + \sum_d N_d\, e^{dt}\, \frac{d}{5}\, H^2 - 2\, N_d\, e^{dt}\, \frac{1}{5}\, H^3\Big)$
- $I_V = e^{tH} 5H \sum_d e^{dt} \frac{\prod_{m=1}^{5d}(5H+m)}{\prod_{m=1}^{d}(H+m)^5}$

Givental's Mirror Theorem

J_V equals a multiple of I_V after a suitable change of variables.

No quintic mirror! No Picard-Fuchs equation!

Givental Mirror Theorem

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- $J_V = e^{tH} 5H\Big(1 + \sum_d N_d\, e^{dt}\, \dfrac{d}{5}\, H^2 - 2\, N_d\, e^{dt}\, \dfrac{1}{5}\, H^3\Big)$
- $I_V = e^{tH} 5H \sum_d e^{dt} \dfrac{\prod_{m=1}^{5d}(5H+m)}{\prod_{m=1}^{d}(H+m)^5}$

Givental's Mirror Theorem

J_V equals a multiple of I_V after a suitable change of variables.

No quintic mirror! No Picard-Fuchs equation!

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Missing Mirrors and Reflexive Polytopes

In the early days of mirror symmetry, physicists used weighted projective spaces to create lots of Calabi-Yau 3-folds V for their theories. Very often, these 3-folds had a mirror V°, such as the case of the quintic 3-fold.

But sometimes the mirror seemed to be missing. Where were the missing mirrors?

- In January 1993, Witten posted Phases of $N = 2$ theories in two dimensions, which used toric varieties to construct gauged linear sigma models.
- In October 1993, Batyrev posted Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, which showed that reflexive polytopes give mirror pairs. This provided the framework needed to find the missing mirrors.

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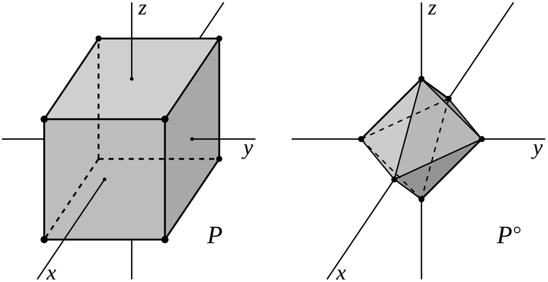
Reflexive Polytopes

A lattice polytope $P \subseteq \mathbb{R}^n$ is reflexive if it contains the origin as an interior point and its dual

$$P^\circ = \{u \in \mathbb{R}^n \mid u \cdot m \geq -1 \text{ for all } m \in P\}$$

is again a lattice polytope. Then P° is reflexive, so that reflexive polytope come in pairs.



Calabi-Yau Threefolds

A lattice polytope P gives the toric variety X_P with an ample divisor D_P. When P is reflexive, $D_P = -K_P =$ anticanonical divisor of X_P.

Hence X_P is Gorenstein Fano. It is then easy to show that a general member $\widehat{V}$ of the linear system $|-K_P|$ is Calabi-Yau (usually singular).

Since P° is reflexive, X_{P° is Gorenstein Fano and the general member $\widehat{V}^\circ \in |-K_{P^\circ}|$ is Calabi-Yau. Thus a reflexive polytope gives a pair $\widehat{V}, \widehat{V}^\circ$ of Calabi-Yau varieties (possibly singular).

In the 3-dimensional case, we have the following theorem of Batyrev.

Theorem (Baytrev 1993)

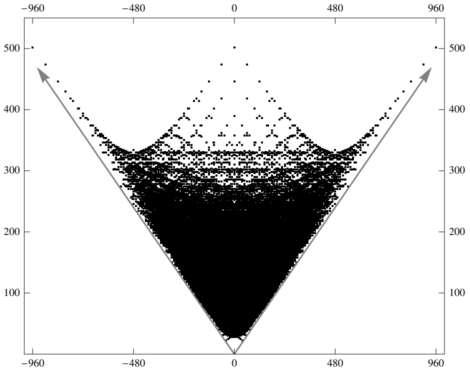
If P is a 4-dimensional reflexive polytope, then normal fans of P and P° have refinements Σ and Σ° such that general members $V \in |-K_\Sigma|$ and $V^\circ \in |-K_{\Sigma^\circ}|$ are smooth Calabi-Yau 3-folds that satisfy

$$h^{11}(V^\circ) = h^{21}(V) \text{ and } h^{21}(V^\circ) = h^{11}(V).$$

Missing Mirrors

In 1995, Candelas, de la Ossa and Katz used 4-dimensional reflexive polytopes to supply the missing mirrors. There are 473,800,776 4-dimensional reflexive polytopes, which gives a lot of mirror pairs.

Plot $\chi = 2(h^{11} - h^{21})$ (horizontal) versus $h^{11} + h^{21}$ (vertical) for all known mirror pairs (including those constructed by non-toric methods):



$\chi =$ topological Euler characteristic

$=$ central charge of the Virasoro algebra

$= \pm 2 \times$ number of fermion generations

Conclusion

- String theory is still conjectural in physics. A proper experiment would require a particle accelerator the size of the solar system.
- Instead, physicists test string theory against the reality of mathematics. Their theories make mathematical predictions, and by proving these predictions to be correct, mathematicians provide experimental confirmation of string theory.
- String theory and mirror symmetry have evolved considerably since the ideas described in this lecture. You will learn about some these new developments in the reading seminar!

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