FUNCTORS OF ARTIN RINGS(1)

BY
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0. **Introduction.** In the investigation of functors on the category of preschemes, one is led, by Grothendieck [3], to consider the following situation. Let Λ be a complete noetherian local ring, μ its maximal ideal, and $k=\Lambda/\mu$ the residue field. (In most applications Λ is k itself, or a ring of Witt vectors.) Let $\boldsymbol{C}$ be the category of Artin local Λ-algebras with residue field k. A covariant functor F from $\boldsymbol{C}$ to *Sets* is called *pro-representable* if it has the form

$$F(A) \cong \operatorname{Hom}_{\text{local }\Lambda\text{-alg.}}(R, A), \qquad A \in \boldsymbol{C},$$

where R is a *complete* local Λ-algebra such that $R/\boldsymbol{m}^n$ is in $\boldsymbol{C}$, all n. ($\boldsymbol{m}$ is the maximal ideal in R.)

In many cases of interest, F is not pro-representable, but at least one may find an R and a morphism $\operatorname{Hom}(R, \cdot) \to F$ of functors such that $\operatorname{Hom}(R, A) \to F(A)$ is surjective for all A in $\boldsymbol{C}$. If R is chosen suitably "minimal" then R is called a "hull" of F; R is then unique up to noncanonical isomorphism. Theorem 2.11, §2, gives a criterion for F to have a hull, and also a simple criterion for pro-representability which avoids the use of Grothendieck's techniques of nonflat descent [3], in some cases. Grothendieck's program is carried out by Levelt in [4]. §3 contains a few geometric applications of these results.

To avoid awkward terminology, I have used the word "pro-representable" in a more restrictive sense than Grothendieck [3] has. He considers the category of Λ-algebras of finite length and allows R to be a projective limit of such rings.

The methods of this paper are a simple extension of those used by David Mumford in a proof (unpublished) of the existence of formal moduli for polarized Abelian varieties. I am indebted to Mumford and to John Tate for many valuable suggestions.

1. **The category C_Λ.** Let Λ be a complete noetherian local ring, with maximal ideal μ and residue field $k=\Lambda/\mu$. We define $\boldsymbol{C}=\boldsymbol{C}_\Lambda$ to be the category of Artinian local Λ-algebras having residue field k. (That is, the "structure morphism" $\Lambda \to A$ of such a ring A induces a trivial extension of residue fields.) Morphisms in $\boldsymbol{C}$ are local homomorphisms of Λ-algebras.

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Let $\hat{C}=\hat{C}_\Lambda$ be the category of complete noetherian local Λ-algebras A for which A/m^n is in C, all n. Notice that C is a full subcategory of $\hat{C}$.

If $p: A \to B$, $q: C \to B$ are morphisms in C, let $A \times_B C$ denote the ring (in C) consisting of all pairs (a, c) with $a \in A$, $c \in C$, for which $pa=qc$, with coordinatewise multiplication and addition.

For any A in $\hat{C}$, we denote by $t^*_{A/\Lambda}$, or just t^*_A, the "Zariski cotangent space" of A over Λ:

$$t^*_A = m/(m^2 + \mu A) \tag{1.0}$$

where m is the maximal ideal of A. A simple calculation shows that the dual vector space, denoted by t_A, may be identified with $\mathrm{Der}_\Lambda(A, k)$, the space of Λ linear derivations of A into k.

LEMMA 1.1. *A morphism $B \to A$ in $\hat{C}$ is surjective if and only if the induced map from t^*_B to t^*_A is surjective.*

Proof. First of all, any A in $\hat{C}$ is generated, as Λ module, by the image of Λ in A and the maximal ideal m of A. (For A and Λ have the same residue field k.) Thus the induced map from μ/μ^2 to $\mu A/(m^2 \cap \mu A)$ is a surjection. If $B \to A$ is a morphism in $\hat{C}$, then denoting the maximal ideal of B by n, we get a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mu A/(\mu A \cap m^2) \arrow[r] & m/m^2 \arrow[r] & t^*_A \arrow[r] & 0 \\
0 \arrow[r] & \mu B/(\mu B \cap n^2) \arrow[r] \arrow[u] & n/n^2 \arrow[r] \arrow[u] & t^*_B \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

in which the left-hand arrow is a surjection. If the right-hand arrow is also a surjection, then the middle arrow is a surjection, so that the induced map on the graded rings is a surjection. From this it follows that $B \to A$ is a surjection [1, §2, No. 8, Theorem 1].

Conversely, if $B \to A$ is a surjection, then the induced map on cotangent spaces is obviously surjective.

Let $p: B \to A$ be a *surjection* in C.

DEFINITION 1.2. p is a *small extension* if kernel p is a nonzero principal ideal (t) such that $mt=(0)$, where m is the maximal ideal of B.

DEFINITION 1.3. p is *essential* if for any morphism $q: C \to B$ in C such that pq is surjective, it follows that q is surjective.

From Lemma 1.1 we obtain easily

LEMMA 1.4. *Let $p: B \to A$ be a surjection in C. Then*

(i) *p is essential if and only if the induced map $p_*: t^*_B \to t^*_A$ is an isomorphism.*

(ii) *If p is a small extension, then p is not essential if and only if p has a section $s: A \to B$, with $ps=1_A$.*

Proof. (i) If p_* is an isomorphism, then by Lemma 1.1, p is essential. Conversely let $\bar{t}_1, \ldots, \bar{t}_r$ be a basis of t_A^*, and lift the $\bar{t}_i$ back to elements t_i in B. Set

$$C = \Lambda[t_1, \ldots, t_r] \subseteq B.$$

Then p induces a surjection from C to A, so if p is essential, $C = B$. But then $\dim_k t_B^* \leqq r = \dim_k t_A^*$, so $t_B^* \cong t_A^*$.

(ii) If p has a section s, then s is not surjective, so p is not essential. If p is not essential, then the subring C constructed above is a proper subring of B, and hence is isomorphic to A, since length (B) = length $(A)+1$. The isomorphism $C \cong A$ yields the section.

2. **Functors on C.** We shall consider only *covariant* functors F, from $\mathbf{C}$ to *Sets*, such that $F(k)$ *contains just one element*. By a *couple* for F we mean a pair (A, ξ) where $A \in \mathbf{C}$ and $\xi \in F(A)$. A *morphism of couples* $u: (A, \xi) \to (A', \xi')$ is a morphism $u: A \to A'$ in $\mathbf{C}$ such that $F(u)(\xi) = \xi'$. If we extend F to $\hat{\mathbf{C}}$ by the formula $\hat{F}(A) = \text{proj Lim } F(A/\mathbf{m}^n)$ we may speak analogously of *pro-couples* and morphisms of pro-couples.

For any ring R in $\hat{\mathbf{C}}$, we set $h_R(A) = \text{Hom}(R, A)$ to define a functor h_R on $\mathbf{C}$. Now if F is any functor on $\mathbf{C}$, and R is in $\hat{\mathbf{C}}$, we have a canonical isomorphism

$$\hat{F}(R) \xrightarrow{\sim} \text{Hom}(h_R, F).$$

Namely, let $\xi = \text{proj Lim } \xi_n$ be in $\hat{F}(R)$. Then each $u: R \to A$ factors through $u_n: R/\mathbf{m}^n \to A$ for some n, and we assign to $u \in h_R(A)$ the element $F(u_n)(\xi_n)$ of $F(A)$. This sets up the isomorphism. We therefore say that a pro-couple (R, ξ) for F *pro-represents* F if the morphism $h_R \to F$ induced by ξ is an isomorphism.

(2.1) *Relation to global functors.* Let G be a *contravariant* functor on the category of preschemes over Spec Λ, and pick a fixed $e \in G(\text{Spec } k)$. For A in $\mathbf{C}$, let $F(A) \subseteq G(\text{Spec } A)$ be the set of those $\xi \in G(\text{Spec } A)$ such that $G(i)(\xi) = e$ where i is the inclusion of Spec k in Spec A. If G is represented by a prescheme X, then e determines a k-rational point $x \in X$, and it is then clear that $F(A)$ is isomorphic to $\text{Hom}_\Lambda(\mathfrak{O}_{X,x}, A)$. Thus the completion of $\mathfrak{O}_{X,x}$ pro-represents F.

Unfortunately, many interesting functors, for example some "formal moduli" functors (§3.7), are not pro-representable. However, one can still look for a "universal object" in some sense, for example in the sense of Definition 2.7 below.

DEFINITION 2.2. A morphism $F \to G$ of functors is *smooth* if for any *surjection* $B \to A$ in $\mathbf{C}$, the morphism

$$(*) \qquad F(B) \to F(A) \times_{G(A)} G(B)$$

is surjective.

Part (i) of the *sorités* below will perhaps motivate this definition.

REMARKS. (2.3) It is enough to check surjectivity in $(*)$ for small extensions $B \to A$.

(2.4) If $F \to G$ is smooth, then $\hat{F} \to \hat{G}$ is *surjective*, in the sense that $\hat{F}(A) \to \hat{G}(A)$ is surjective for all A in $\hat{C}$ (consider the successive quotients $A/\boldsymbol{m}^n$, $n=1, 2, \ldots$).

PROPOSITION 2.5. (i) *Let $R \to S$ be a morphism in $\hat{C}$. Then $h_S \to h_R$ is smooth if and only if S is a power series ring over R.*

(ii) *If $F \to G$ and $G \to H$ are smooth morphisms of functors, then the composition $F \to H$ is smooth.*

(iii) *If $u\colon F \to G$ and $v\colon G \to H$ are morphisms of functors such that u is surjective and vu is smooth, then v is smooth.*

(iv) *If $F \to G$ and $H \to G$ are morphisms of functors such that $F \to G$ is smooth, then $F \times_G H \to H$ is smooth.*

Proof. (i) This is more or less well known (see [3, Theorem 3.1]), but we give a proof for the sake of completeness. Suppose $h_S \to h_R$ is smooth. Let $\boldsymbol{r}$ (resp. $\boldsymbol{s}$) be the maximal ideal in R (resp. S), and pick $x_1, \ldots, x_n$ in S which induce a basis of $t^*_{S/R} = \boldsymbol{s}/(\boldsymbol{s}^2 + \boldsymbol{r}S)$. If we set $T = R[[X_1, \ldots, X_n]]$ and denote the maximal ideal of T by $\boldsymbol{t}$, we get a morphism $u_1\colon S \to T/(\boldsymbol{t}^2 + \boldsymbol{r}T)$ of local R algebras, obtained by mapping x_i on the residue class of X_i. By smoothness u_1 lifts to $u_2\colon S \to T/\boldsymbol{t}^2$, thence to $u_3\colon S \to T/\boldsymbol{t}^3, \ldots$ etc. Thus we get a $u\colon S \to T$ which induces an isomorphism of $t^*_{S/R}$ with $t^*_{T/R}$ (by choice of u_1) so that u is a surjection (1.1). Furthermore, if we choose $y_i \in S$ such that $uy_i = X_i$, we can set $vX_i = y_i$ and produce a local morphism $v\colon T \to S$ of R algebras such that $uv = 1_T$; in particular v is an injection. Clearly v induces a bijection on the cotangent spaces, so v is also a surjection (1.1). Hence v is an isomorphism of $T = R[[X_1, \ldots, X_n]]$ with S.

Conversely, if S is a power series ring over R, then it is obvious that $h_S \to h_R$ is smooth.

The proofs of (ii), (iii), (iv) are completely formal and are left to the reader.

(2.6) NOTATION. Let $k[\varepsilon]$, where $\varepsilon^2 = 0$, denote the ring of dual numbers over k. For any functor F, the set $F(k[\varepsilon])$ is called the *tangent space* to F, and is denoted by t_F. It is easy to see that if $F = h_R$, then there is a canonical isomorphism $t_F \cong t_R$:

$$t_R \cong \operatorname{Hom}_\Lambda(R, k[\varepsilon]).$$

Usually t_F will have an intrinsic vector space structure (Lemma 2.10).

DEFINITION 2.7. A pro-couple (R, ξ) for a functor F is called a *pro-representable hull* of F, or just a *hull* of F, if the induced map $h_R \to F$ is *smooth* (2.2), and if in addition the induced map $t_R \to t_F$ of tangent spaces is a bijection.

(2.8) Notice that if (R, ξ) pro-represents F then (R, ξ) is a hull of F. In this case (R, ξ) is unique up to canonical isomorphism. In general we have only noncanonical isomorphism:

PROPOSITION 2.9. *Let (R, ξ) and (R', ξ') be hulls of F. Then there exists an isomorphism $u\colon R \to R'$ such that $F(u)(\xi) = \xi'$.*

Proof. By (2.4) we have morphisms $u\colon (R, \xi) \to (R', \xi')$ and $u'\colon (R', \xi') \to (R, \xi)$, both inducing an isomorphism on tangent spaces, by the definition of hull. Thus

$u'u$ say, induces an isomorphism on t_R^*, so that $u'u$ is a surjective endomorphism of R, by Lemma 1.1. But an easy argument, which we leave to the reader, shows that a surjective endomorphism of any noetherian ring is an isomorphism. Thus $u'u$ and uu' are isomorphisms and we are done.

REMARK 2.10. Let (R, ξ) be a hull of F. Then R is a power series ring over Λ if and only if F transforms surjections $B \to A$ in $\mathbf{C}$ into surjections $F(B) \to F(A)$. In fact the stated condition on F is equivalent to the *smoothness* of the natural morphism $F \to h_\Lambda$. By applying (2.6), (ii) and (iii) to the diagram

$$\begin{tikzcd} h_R \arrow[rr] \arrow[dr] & & h_\Lambda \\ & F \arrow[ur] & \end{tikzcd}$$

we conclude that $h_R \to h_\Lambda$ is smooth if and only if $F \to h_\Lambda$ is. Now use 2.5 (i).

LEMMA 2.10. *Suppose F is a functor such that*

$$F(k[V] \times_k k[W]) \xrightarrow{\sim} F(k[V]) \times F(k[W])$$

for vector spaces V and W over k, where $k[V]$ denotes the ring $k \oplus V$ of $\mathbf{C}$ in which V is a square zero ideal. Then $F(k[V])$, and in particular $t_F = F(k[\varepsilon])$, has a canonical vector space structure, such that $F(k[V]) \cong t_F \otimes V$.

Proof. $k[V]$ is in fact a "vector space object" in the category $\hat{\mathbf{C}}$ (in which k is the final object), for we have a canonical isomorphism

$$\mathrm{Hom}(A, k[V]) \cong \mathrm{Der}_\Lambda(A, V), \qquad A \in \hat{\mathbf{C}}.$$

The addition map $k[V] \times_k k[V] \to k[V]$ is given by $(x, 0) \mapsto x$, $(0, x) \mapsto x$ $(x \in V)$, and scalar multiplication by $a \in k$ is given by the endomorphism $x \mapsto ax$ $(x \in V)$ of $k[V]$. Thus if F commutes with the necessary products, $F(k[V])$ gets a vector space structure. Finally, we identify V with $\mathrm{Hom}(k[\varepsilon], k[V])$ to get a map

$$t_F \otimes V \to F(k[V])$$

which is an isomorphism since $k[V]$ is isomorphic to the product of $r = \dim_k V$ copies of $k[\varepsilon]$.

THEOREM 2.11. *Let F be a functor from $\mathbf{C}$ to Sets such that $F(k) = (e)$ ($=$one point). Let $A' \to A$ and $A'' \to A$ be morphisms in $\mathbf{C}$, and consider the map*

$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A''). \tag{2.12}$$

Then

(1) *F has a hull if and only if F has properties* (H_1), (H_2), (H_3) *below*:

(H_1) (2.12) *is a surjection whenever $A'' \to A$ is a small extension* (1.2).

(H_2) (2.12) *is a bijection when $A = k$, $A'' = k[\varepsilon]$.*

(H_3) $\dim_k(t_F) < \infty$.

(2) *F is pro-representable if and only if F has the additional property* (H_4):

$$(H_4) \qquad F(A' \times_A A') \xrightarrow{\sim} F(A') \times_{F(A)} F(A')$$

for any small extension $A' \to A$.

Notice that if F is isomorphic to some h_R, then (2.12) is an isomorphism for any morphisms $A' \to A$, $A'' \to A$; that is, the four conditions are trivially necessary for pro-representability.

REMARKS. (2.13) (H_2) implies that t_F is a vector space by Lemma 2.10. In fact, by induction on $\dim_k W$ we conclude from (H_2) that (2.12) is an isomorphism when $A=k$, $A''=k[W]$; in particular the hypotheses of 2.10 are satisfied.

(2.14) By induction on length A''-length A it follows from (H_1) that (2.12) is a surjection for *any* surjection $A'' \to A$.

(2.15) Condition (H_4) may be usefully viewed as follows. For each A in $\boldsymbol{C}$, and each ideal I in A such that $\boldsymbol{m}_A \cdot I=(0)$, we have an isomorphism

$$(2.16) \qquad A \times_{A/I} A \xrightarrow{\sim} A \times_k k[I],$$

induced by the map $(x, y) \mapsto (x, x_0+y-x)$, where x and y are in A and x_0 is the k residue of x. Now, given a small extension $p: A' \to A$ with kernel I, we get by (H_2) and (2.16) a map

$$(2.17) \qquad F(A') \times (t_F \otimes I) \to F(A') \times_{F(A)} F(A')$$

which is easily seen to determine, for each $\eta \in F(A)$, a group action of $t_F \otimes I$ on the subset $F(p)^{-1}(\eta)$ of $F(A')$ (provided that subset is not empty). (H_1) implies that this action is "transitive," while (H_4) is precisely the condition that this action makes $F(p)^{-1}(\eta)$ a (formally) principal homogeneous space under $t_F \otimes I$. Thus, in the presence of conditions (H_1), (H_2), (H_3), it is the existence of "fixed points" of $t_F \otimes I$ acting on $F(p)^{-1}(\eta)$ which obstruct the pro-representability of F. In many applications, where the elements of $F(A)$ are isomorphism classes of geometric objects, the existence of such a fixed point $\eta' \in F(p)^{-1}(\eta)$ is equivalent to the existence of an automorphism of an object y in the class of η which cannot be extended to an automorphism of any (or some) object y' in the class of η'.

Proof of 2.11. (1) Suppose F satisfies conditions (H_1), (H_2), (H_3). Let $t_1, \ldots, t_r$ be a dual basis of t_F, put $S=\Lambda[[T_1, \ldots, T_r]]$, and let $\boldsymbol{n}$ be the maximal ideal of S. R will be constructed as the projective limit of successive quotients of S. To begin, let $R_2=S/(\boldsymbol{n}^2+\boldsymbol{\mu}S) \cong k[\varepsilon] \times_k \cdots \times_k k[\varepsilon]$ (r times). By (H_2) there exists $\xi_2 \in F(R_2)$ which induces a bijection between t_{R_2} ($\cong \operatorname{Hom}(R_2, k[\varepsilon])$) and t_F. Suppose we have found (R_q, ξ_q), where $R_q=S/J_q$. We seek an ideal J_{q+1} in S, minimal among those ideals J in S satisfying the conditions (a) $\boldsymbol{n}J_q \subseteq J \subseteq J_q$, (b) ξ_q lifts to S/J. Since the set $\mathscr{S}$ of such ideals corresponds to a certain collection of vector subspaces of $J_q/(\boldsymbol{n}J_q)$, it suffices to show that $\mathscr{S}$ is stable under pairwise intersection. But if

J and K are in $\mathscr{S}$, we may enlarge J, say, so that $J+K=J_q$, without changing the intersection $J \cap K$. Then

$$S/J \times_{S/J_q} S/K \cong S/(J \cap K)$$

so that by (H_1) (see (2.14)) we may conclude that $J \cap K$ is in $\mathscr{S}$. Let J_{q+1} be the intersection of the members of $\mathscr{S}$, put $R_{q+1}=S/J_{q+1}$, and pick any $\xi_{q+1} \in F(R_{q+1})$ which projects onto $\xi_q \in F(R_q)$.

Now let J be the intersection of all the J_q's ($q=2, 3, \ldots$) and let $R=S/J$. Since $\mathfrak{n}^q \subseteq J_q$, the J_q/J form a base for the topology in R, so that $R=\text{proj Lim } S/J_q$, and it is legitimate to set $\xi=\text{proj Lim } \xi_q \in \hat{F}(R)$. Notice that $t_F \cong t_R$, by choice of R_2.

We claim now that $h_R \to F$ is smooth. Let $p\colon (A', \eta') \to (A, \eta)$ be a morphism of couples, where p is a small extension, $A=A'/I$, and let $u\colon (R, \xi) \to (A, \eta)$ be a given morphism. We have to lift u to a morphism $(R, \xi) \to (A', \eta')$. For this it suffices to find a $u'\colon R \to A'$ such that $pu'=u$. In fact, we have a transitive action of $t_F \otimes I$ on $F(p)^{-1}(\eta)$ (resp. $h_R(p)^{-1}(\eta)$) by (2.15); thus, given such a u', there exists $\sigma \in t_F \otimes I$ such that $[F(u')(\xi)]^\sigma=\eta'$, so that $v'=(u')^\sigma$ will satisfy $F(v')(\xi)=\eta'$, $pv'=u$.

Now u factors as $(R, \xi) \to (R_q, \xi_q) \to (A, \eta)$ for some q. Thus it suffices to complete the diagram

$$\begin{tikzcd} R_{q+1} \arrow[r, dashed] \arrow[d] & A' \arrow[d, "p"] \\ R_q \arrow[r] & A \end{tikzcd}$$

or equivalently, the diagram

$$\begin{tikzcd} \Lambda[[T_1,\ldots,T_r]]=S \arrow[r, "w"] \arrow[d] & R_q \times_A A' \arrow[d, "pr_1"] \\ R_{q+1} \arrow[r] \arrow[ur, dashed, "v"] & R_q \end{tikzcd}$$

where w has been chosen so as to make the square commute. If the small extension pr_1 has a section, then v obviously exists. Otherwise, by 1.4(ii), pr_1 is essential, so w is a surjection. By (H_1), applied to the projections of $R_q \times_A A'$ on its factors, $\xi_q \in F(R_q)$ lifts back to $R_q \times_A A'$, so $\ker w \supseteq J_{q+1}$, by choice of J_{q+1}. Thus w factors through $S/J_{q+1}=R_{q+1}$, and v exists. This completes the proof that (R, ξ) is a hull of F.

Conversely, suppose that a pro-couple (R, ξ) is a hull of F. To verify (H_1), let $p'\colon (A', \eta') \to (A, \eta)$ and $p''\colon (A'', \eta'') \to (A, \eta)$ be morphisms of couples, where p''

is a surjection. Since $h_R \to F$ is surjective, there exists a $u' \colon (R, \xi) \to (A', \eta')$, and hence by smoothness applied to p'', there exists $u'' \colon (R, \xi) \to (A'', \eta'')$ rendering the following diagram commutative:

$$\begin{tikzcd}
& (A' \times_A A'', \zeta) \arrow[dr] \arrow[dd] & \\
(R, \xi) \arrow[ur, "u' \times u''"] \arrow[rr, "u''"] \arrow[dr, "u'"'] & & (A'', \eta'') \arrow[dd, "p''"] \\
& (A', \eta') \arrow[dr, "p'"'] & \\
& & (A, \eta)
\end{tikzcd}$$

Therefore $\zeta = F(u' \times u'')(\xi)$ projects onto η' and η'', so that (H_1) is satisfied.

Now suppose $(A, \eta) = (k, e)$, and $A'' = k[\varepsilon]$. If ζ_1 and ζ_2 in $F(A' \times_k k[\varepsilon])$ have the same projections η' and η'' on the factors, then choosing u' as above we get morphisms

$$u' \times u_i \colon (R, \xi) \to (A' \times_k k[\varepsilon], \zeta_i), \qquad i = 1, 2,$$

by smoothness applied to the projection of $A' \times_k k[\varepsilon]$ on A'. Since $t_F \cong t_R$ we have $u_1 = u_2$, so that $\zeta_1 = \zeta_2$, which proves (H_2). The isomorphism $t_R \cong t_F$ also proves (H_3).

(2) Suppose now that F satisfies conditions (H_1) through (H_4). By part (1) we know that F has a hull (R, ξ). We shall prove that $h_R(A) \xrightarrow{\sim} F(A)$ by induction on length A. Consider a small extension $p \colon A' \to A = A'/I$, and assume that $h_R(A) \xrightarrow{\sim} F(A)$. For each $\eta \in F(A)$, $h_R(p)^{-1}(\eta)$ and $F(p)^{-1}(\eta)$ are both formally principal homogeneous spaces under $t_F \otimes I$ (2.15); since $h_R(A')$ maps *onto* $F(A')$, we have $h_R(A') \xrightarrow{\sim} F(A')$, which proves the induction step.

The necessity of the four conditions has already been noted.

3. **Examples.**

(3.1) *The Picard functor.* If X is a prescheme, we define $\operatorname{Pic}(X) = H^1(X, \mathfrak{O}_X^*)$, the group of isomorphism classes of invertible (i.e., locally free of rank one) sheaves on X. Recall that the group of automorphisms of an invertible sheaf is canonically isomorphic to $H^0(X, \mathfrak{O}_X^*)$.

Now suppose X is a prescheme over Spec Λ. We let X_A abbreviate $X \times_{\operatorname{Spec} \Lambda} \operatorname{Spec} A$ for A in $\boldsymbol{C}$, and set $X_0 = X_k$. If η (resp. L) is an element of $\operatorname{Pic}(X_A)$ (resp. an invertible sheaf on X_A) and $A \to B$ is a morphism in $\boldsymbol{C}$, let $\eta \otimes_A B$ (resp. $L \otimes_A B$) denote the induced element of $\operatorname{Pic}(X_B)$ (resp. induced invertible sheaf on X_B). Let ξ_0 be an element of $\operatorname{Pic}(X_0)$ *fixed once and for all* in this discussion, and let

$\boldsymbol{P}(A)$ be the subset of Pic (X_A) consisting of those η such that $\eta \otimes_A k = \xi_0$. We claim that $\boldsymbol{P}$ is pro-representable under suitable conditions, namely:

PROPOSITION 3.2. *Assume*
(i) *X is flat over Λ,*
(ii) *$A \xrightarrow{\sim} H^0(X_A, \mathfrak{O}_{X_A})$ for each $A \in \boldsymbol{C}$,*
(iii) $\dim_k H^1(X_0, \mathfrak{O}_{X_0}) < \infty$.
Then $\boldsymbol{P}$ is pro-representable by a pro-couple (R, ξ); furthermore $t_R \cong H^1(X_0, \mathfrak{O}_{X_0})$.

Notice that condition (ii) is equivalent to the condition $k \xrightarrow{\sim} H^0(X_0, \mathfrak{O}_{X_0})$, in view of (i). In fact, by flatness, the functor $M \mapsto T(M) = H^0(X, \mathfrak{O}_X \otimes M)$ of Λ modules is left exact. A standard five lemma type of argument then shows that the natural map $M \to T(M)$ is an isomorphism for all M of finite length.

For the proof of 3.2 we need two simple lemmas on flatness.

LEMMA 3.3. *Let A be a ring, J a nilpotent ideal in A, and $u\colon M \to N$ a homomorphism of A modules, with N flat over A. If $\bar{u}\colon M/JM \to N/JN$ is an isomorphism, then u is an isomorphism.*

Proof. Let $K = \operatorname{coker} u$ and tensor the exact sequence

$$M \to N \to K \to 0$$

with A/J. Then we find $K/JK = 0$, which implies $K = 0$, since J is nilpotent. Thus, if $K' = \ker u$, we get an exact sequence

$$0 \to K'/JK' \to M/JM \to N/JN \to 0$$

by the flatness of N. Hence $K' = 0$, so that u is an isomorphism.

LEMMA 3.4. *Consider a commutative diagram*

$$\begin{tikzcd}
N \arrow[rr, "p''"] \arrow[dr] \arrow[dd, dash, "p'"'] & & M'' \arrow[dr, "u''"] \arrow[dd, dash] & \\
& M' \arrow[rr, "u'"'] \arrow[dd, dash] & & M \arrow[dd, dash] \\
B \arrow[rr] \arrow[dr] & & A'' \arrow[dr] & \\
& A' \arrow[rr] & & A
\end{tikzcd}$$

of compatible ring and module homomorphisms, where $B = A' \times_A A''$, $N = M' \times_M M''$ and M' (resp. M'') is a flat A' (resp. A'') module. Suppose
(i) *$A''/J \xrightarrow{\sim} A$, where J is a nilpotent ideal in A'',*
(ii) *u' (resp. u'') induces $M' \otimes_{A'} A \xrightarrow{\sim} M$ (resp. $M'' \otimes_{A''} A \xrightarrow{\sim} M$).*

Then N is flat over B and p' (resp. p'') induces $N \otimes_B A' \xrightarrow{\sim} M'$ (resp. $N \otimes_B A'' \xrightarrow{\sim} M''$).

Proof. We shall consider only the case where M' is actually a *free* A' module. (This case actually suffices for our purposes, since a simple application of Lemma 3.3 shows that a flat module over an Artin local ring is free.) Choose a basis $(x_i')_{i \in I}$ for M'. Then by (ii) we find that M is the free module on generators $u'(x_i')$. Choosing $x_i'' \in M''$ such that $u''(x_i'') = u'(x_i')$, we get a map $\sum A''x_i'' \to M''$ of A'' modules, whose reduction modulo the ideal J is an isomorphism. Therefore M'' is free on generators x_i'' (Lemma 3.3) and it follows easily that $N = M' \times_M M''$ is free on generators $x_i' \times x_i''$, and that the projections on the factors induce isomorphisms

$$N \otimes_B A' \xrightarrow{\sim} M', \quad N \otimes_B A'' \xrightarrow{\sim} M''$$

as desired. (A similar argument for the case of general M' is given in [4, §1, Proposition 2].)

COROLLARY 3.6. *With the notations as above, let L be a B module which may be inserted in a commutative diagram*

$$\begin{tikzcd} & L \arrow[dl, "q'"'] \arrow[dr, "q''"] & \\ M' \arrow[dr, "u'"'] & & M'' \arrow[dl, "u''"] \\ & M & \end{tikzcd}$$

where q' induces $L \otimes_B A' \xrightarrow{\sim} M'$. Then the canonical morphism $q' \times q'' : L \to N = M' \times_M M''$ is an isomorphism.

Proof. Apply Lemma 3.3 to the morphism $u = q' \times q''$.

REMARK. Lemma 3.4 is false, in general, if neither $A'' \to A$ nor $A' \to A$ is assumed surjective. For example, let A' be a sublocal ring of the local ring A, and map $A_1 = A''$ into A by inclusion. Let a be a *unit* of A such that the ideal $(aA') \cap A'$ of A' is not flat ($=$free) over A'. (In C_Λ one could take $A = k[t]/(t^3)$, $A' = k[t^2]$, $a = 1 + t$.) Let $M' = M'' = A'$, $M = A$, $u' =$ inclusion, $u'' =$ multiplication by a^{-1}. Then $B \cong A'$, while $N \cong (aA') \cap A'$ is not flat over B.

Proof of Proposition 3.2. Let $u' : (A', \eta') \to (A, \eta)$, $u'' : (A'', \eta'') \to (A, \eta)$ be morphisms of couples, where u'' is a surjection. Let L', L, L'' be corresponding invertible sheaves on $X' = X_{A'}$, $Y = X_A$, and $X'' = X_{A''}$. Then we have morphisms $p' : L' \to L$, $p'' : L'' \to L$ (of sheaves on the topological space $|X_0|$, compatible with $\mathfrak{O}_{X'} \to \mathfrak{O}_Y$, $\mathfrak{O}_{X''} \to \mathfrak{O}_Y$) which induce isomorphisms $L' \otimes_{A'} A \xrightarrow{\sim} L$, $L'' \otimes_{A''} A \xrightarrow{\sim} L$.

Let $B=A' \times_A A''$, and let $Z=X_B$. Then we have a commutative diagram

$$\begin{tikzcd} & \mathfrak{O}_Z \arrow[dl] \arrow[dr] & \\ \mathfrak{O}_{X'} \arrow[dr] & & \mathfrak{O}_{X''} \arrow[dl] \\ & \mathfrak{O}_Y & \end{tikzcd}$$

of sheaves on $|X_0|$; thus by Corollary 3.6 there is a canonical isomorphism $\mathfrak{O}_Z \xrightarrow{\sim} \mathfrak{O}_{X'} \times_{\mathfrak{O}_Y} \mathfrak{O}_{X''}$, where $\mathfrak{O}_{X'} \times_{\mathfrak{O}_Y} \mathfrak{O}_{X''}$ is the sheaf of B-algebras whose sections over an open U in $|X_0|$ are given by

$$\mathfrak{O}_{X'} \times_{\mathfrak{O}_Y} \mathfrak{O}_{X''}(U) = \mathfrak{O}_{X'}(U) \times_{\mathfrak{O}_{Y}(U)} \mathfrak{O}_{X''}(U).$$

Hence $N=L' \times_L L''$ is a sheaf on Z, obviously invertible, and the projections of N on L' and L'' induce isomorphisms $N \otimes_B A' \xrightarrow{\sim} L'$, $N \otimes_B A'' \xrightarrow{\sim} L''$ by Lemma 3.4.

If M is another invertible sheaf on Z for which there exist isomorphisms

$$M \otimes A' \xrightarrow{\sim} L', \quad M \otimes A'' \xrightarrow{\sim} L'',$$

we have morphisms $q'\colon M \to L'$, $q''\colon M \to L''$ which induce these isomorphisms, and thus a commutative diagram

$$\begin{tikzcd} & & M \arrow[dll, "q'"'] \arrow[drr, "q''"] & & \\ L' \arrow[dr, "u'"'] & & & & L'' \arrow[dl, "u''"] \\ & L \arrow[rr, "\theta"'] & & L & \end{tikzcd}$$

Here θ is the automorphism of L given by the composition

$$L \xrightarrow{\sim} L' \otimes_{A'} A \xrightarrow{\sim} M \otimes_B A \xrightarrow{\sim} L'' \otimes_{A'} A \xrightarrow{\sim} L.$$

By hypothesis (ii) of 3.2, θ is multiplication by some unit $a \in A$. Lifting a back to a'' in A'', we can change q'' to $a''q''$; thus we may assume that $u'q'=u''q''$. It follows from Corollary 3.6 that $M \xrightarrow{\sim} N$. We have therefore proved that

$$\boldsymbol{P}(A' \times_A A'') \xrightarrow{\sim} \boldsymbol{P}(A') \times_{\boldsymbol{P}(A)} \boldsymbol{P}(A'')$$

for *any* surjection $A'' \to A$ in $\boldsymbol{C}$.

Finally, letting $Y = X_{k[\varepsilon]}$, we have $\mathfrak{O}_Y = \mathfrak{O}_{X_0} \oplus \varepsilon\mathfrak{O}_{X_0}$, so there is a split exact sequence

$$0 \longrightarrow \mathfrak{O}_{X_0} \xrightarrow{\exp} \mathfrak{O}_Y^* \longrightarrow \mathfrak{O}_{X_0}^* \longrightarrow 1$$

where exp maps the (additive) sheaf $\mathfrak{O}_{X_0}$ into $\mathfrak{O}_Y^*$ by $\exp(f) = 1 + \varepsilon f$. Hence

$$F(k[\varepsilon]) \cong \ker\{H^1(X_0, \mathfrak{O}_Y^*) \to H^1(X_0, \mathfrak{O}_{X_0}^*)\} \cong H^1(X_0, \mathfrak{O}_{X_0})$$

which has finite dimension, by assumption. This completes the proof of Proposition 3.2.

(3.7) *Formal moduli.* Let X be a fixed prescheme over k, and $A \in \boldsymbol{C}$. By an (infinitesimal) *deformation* of X/k to A we mean a *product* diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \to & \operatorname{Spec} A \end{array} \qquad X \xrightarrow{\sim} Y \times_{\operatorname{Spec} A} \operatorname{Spec} k$$

where Y is flat over Spec A and i is (necessarily) a closed immersion. We will suppress the i and refer to Y as a deformation, if no confusion is possible. If Y' is another deformation to A then Y and Y' are *isomorphic* if there exists a morphism $f\colon Y \to Y'$ over A which induces the identity on the closed fibre X. (f must then be an isomorphism of preschemes, by Lemma 3.3.) Given the deformation Y over A and a morphism $A \to B$ in $\boldsymbol{C}$, one has evidently an induced deformation $Y \otimes_A B$ over B; and if Z is a deformation over B, one can define the notion of morphism $Z \to Y$ of deformations. (Notice that there is a one-to-one correspondence between such morphisms and the isomorphisms $Z \xrightarrow{\sim} Y \otimes_A B$ which they induce.

Define the deformation functor $\boldsymbol{D} = \boldsymbol{D}_{X/k}$ by setting

$$\boldsymbol{D}(A) = \text{Set of isomorphism classes of deformations of } X/k \text{ to } A.$$

We shall find that, in general, $\boldsymbol{D}$ is not pro-representable, but that with rather weak finiteness restrictions on X, $\boldsymbol{D}$ will have a hull.

Suppose that $(A', \eta') \to (A, \eta)$ and $(A'', \eta'') \to (A, \eta)$ are morphisms of couples, where $A'' \to A$ is a surjection. Letting X', Y, X'' denote deformations in the class of η', η, η'' respectively, we have a diagram

$$X' \xleftarrow{u'} Y \xrightarrow{u''} X''$$

of deformations. Therefore we can construct, as in the proof of 3.2 the sheaf $\mathfrak{O}_{X'} \times_{\mathfrak{O}_Y} \mathfrak{O}_{X''}$ of $A' \times_A A''$ algebras, and $(|X|, \mathfrak{O}_{X'} \times_{\mathfrak{O}_Y} \mathfrak{O}_{X''})$ defines a prescheme Z flat over $A' \times_A A''$. (The fact that Z is actually a prescheme consists of straightforward checking; in fact it is the *sum* of X' and X'' in the category of preschemes *under* Y, *homeomorphic* to Y. Z is flat over $A' \times_A A''$ by Lemma 3.4.) Furthermore the closed immersions $X \to Y \to Z$ give Z a structure of deformation of X/k to $A' \times_A A''$ such that

$$\begin{tikzcd} & Z & \\ X' \arrow[ur, hook, "p'"] & & X'' \arrow[ul, "p''"'] \\ & Y \arrow[ul, "u'"] \arrow[ur, hook, "u''"'] & \end{tikzcd}$$

is a commutative diagram of deformations. In particular this shows that

$$\boldsymbol{D}(A' \times_A A') \to \boldsymbol{D}(A') \times_{\boldsymbol{D}(A)} \boldsymbol{D}(A')$$

is surjective, for every surjection $A'' \to A$. That is, condition (H_1) of 2.11 is satisfied.

Suppose now that W is another deformation over B, inducing the deformations

$$\begin{tikzcd} & & W & & \\ X' \arrow[urr, "q'"] & & & & X'' \arrow[ull, "q''"'] \\ & Y \arrow[ul, "u'"] & & Y \arrow[ll, "\theta"] \arrow[ur, "u''"'] & \end{tikzcd}$$

X' and X''. Then there is a commutative diagram of deformations, where θ is the composition

$$Y \xrightarrow{\sim} X' \otimes_{A'} A \xrightarrow{\sim} W \otimes_B A \longrightarrow X'' \otimes_{A''} A \xrightarrow{\sim} Y.$$

If θ can be lifted to an automorphism θ' of X', such that $\theta' u' = u' \theta$, then we can replace q' with $q'\theta'$; then we would have an isomorphism $W \xrightarrow{\sim} Z$ by Corollary 3.6. Now if $A = k$ (so that $Y = X$, $\theta = \mathrm{id}$) θ' certainly exists, so condition (H_2) is satisfied.

To consider the condition (H_4), let $p\colon (A', \eta') \to (A, \eta)$ be a morphism of couples, where p is a small extension. For each morphism $B \to A$, let $\boldsymbol{D}_\eta(B)$ denote as usual the set of $\zeta \in \boldsymbol{D}(B)$ such that $\zeta \otimes_B A = \eta$. Pick a deformation Y' in the class of η'; then

LEMMA 3.8. *The following are equivalent*

(i) $\boldsymbol{D}_\eta(A' \times_A A') \xrightarrow{\sim} \boldsymbol{D}_\eta(A') \times \boldsymbol{D}_\eta(A')$,

(ii) *Every automorphism of the deformation* $Y = Y' \otimes_{A'} A$ *is induced by an automorphism of the deformation* Y'.

Proof. (i) $\Rightarrow$ (ii). Let $u\colon Y \to Y'$ be the induced morphism of deformations.

If θ is an automorphism of Y, then one can construct deformations Z, W over $A' \times_A A'$ to yield "sum diagrams"

$$\begin{tikzcd} & Z & \\ Y' \arrow[ur] & & Y' \arrow[ul] \\ & Y \arrow[ul, "u\theta"] \arrow[ur, "u"'] & \end{tikzcd} \qquad \begin{tikzcd} & W & \\ Y' \arrow[ur] & & Y \arrow[ul] \\ & Y \arrow[ul, "u"] \arrow[ur, "u"'] & \end{tikzcd}$$

of deformations. Since Z and W have isomorphic projections on both factors, there is an isomorphism $\rho: Z \xrightarrow{\sim} W$. ρ induces automorphisms θ_1 and θ_2 of Y', and an automorphism ϕ of Y such that

$$\theta_1 u\theta = u\phi, \quad \theta_2 u = u\phi.$$

Therefore $u\theta = \theta_1^{-1}\theta_2 u$ and $\theta_1^{-1}\theta_2$ induces θ.

(ii) $\Rightarrow$ (i). In a similar manner, it follows from (ii) that $t_F \otimes I$ ($I = \ker p$) acts freely on η' (i.e., $(\eta')^\sigma = \eta'$ implies $\sigma = 0$). Since the action of $t_F \otimes I$ on $\boldsymbol{D}_\eta(A')$ is transitive, it follows that $\boldsymbol{D}_\eta(A')$ is a principal homogeneous space under $t_F \otimes I$, which is equivalent to (i).

It should be remarked that the obstruction to lifting θ lies in $t_F \otimes I$ and is often nonzero (see e.g., [4, §4]).

Finally it remains to consider the finiteness condition (H_3). If X is smooth over k (in ancient terminology *absolutely simple*), then Grothendieck has shown in S.G.A. III, Theorem 6.3, that

$$t_{\boldsymbol{D}} \cong H^1(X, \Theta)$$

where Θ is the tangent sheaf of X over k. Thus $t_{\boldsymbol{D}}$ has finite dimension if X is smooth and proper over k. In general, it is shown in [4] that for any scheme X locally of finite type over k, there is an exact sequence

$$0 \to H^1(X, T^0) \to t_{\boldsymbol{D}} \to H^0(X, T^1) \to H^2(X, T^0) \tag{3.9}$$

where T^0 is the sheaf of derivations of $\mathfrak{O}_X$, and T^1 is a (coherent) sheaf isomorphic to the sheaf of germs of deformations of X/k to $k[\varepsilon]$. If X is smooth over k, then $T^0 = \Theta$, $T^1 = 0$. Thus, in summary

PROPOSITION 3.10. *If X is either*

(a) *proper over k or*

(b) *affine with only isolated singularities,*

then $\boldsymbol{D}$ has a hull (R, ξ). (R, ξ) pro-represents $\boldsymbol{D}$ if and only if for each small extension $A' \to A$, and each deformation Y' of X/k to A', every automorphism of the deformation $Y' \otimes_{A'} A$ is induced by an automorphism of Y'.

(3.11) *The automorphism functor.* One can formalize the obstructions to pro-representing $\boldsymbol{D}$ as follows. Let X be a prescheme *proper* over k, and let (R, ξ) be a hull of the deformation functor $\boldsymbol{D}$. ξ is represented by a formal prescheme $\mathfrak{X} = \text{inj Lim } X_n$ over R, where X_n is a deformation of X/k to $R/\boldsymbol{m}^n$. For each morphism $R \to A$ in $\boldsymbol{C}_\Lambda$, we get a deformation $\mathfrak{X}_A = \mathfrak{X} \times_{\text{Spec } R} \text{Spec } A$ of X/k to A. We can therefore define a group functor $\boldsymbol{A}$ on the category $\boldsymbol{C}_R$ of Artin local R-algebras:

$$\boldsymbol{A}: A \mapsto \text{group of automorphisms of the deformation } \mathfrak{X}_A.$$

If $A' \to A$ and $A'' \to A$ are morphisms in $\boldsymbol{C}_R$ with $A'' \to A$ a surjection, and if we put $B = A' \times_A A''$ then we have a canonical isomorphism, respecting the structures as deformations:

$$\mathfrak{O}_{\mathfrak{X}_B} \cong \mathfrak{O}_{X_{A'}} \times_{\mathfrak{O}_{X_A}} \mathfrak{O}_{X_{A''}}$$

by Corollary 3.6. It follows easily that (2.12) is an isomorphism, so that (H_1), (H_2) and (H_4) of Theorem 2.11 are satisfied. Finally the computations of Grothendieck in S.G.A. III, §6, show that the tangent space of $\boldsymbol{A}$ is given by

$$t_{\boldsymbol{A}/R} \cong H^0(X_0, T^0)$$

where T^0 is, again, the (coherent) sheaf of derivations of $\mathfrak{O}_X$ over k. Thus $t_{\boldsymbol{A}}$ has finite dimension, and we find:

PROPOSITION 3.12. *If X is proper over k, the functor $\boldsymbol{A}$ is pro-represented by a complete local R algebra, S, which is a group object in the category dual to $\hat{\boldsymbol{C}}_R$ (i.e., S is a formal Lie group over R). The deformation functor $\boldsymbol{D}$ is pro-representable (by R) if and only if S is a power series ring over R.*

The last statement follows from Lemma 3.8 and the smoothness criterion of Remark 2.10.

In a future paper I will discuss the deformation functor in more detail, with particular attention to the contribution of singular points on X.

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