Reading Seminar Oct 5, 2012 David Cox

Deformation Theory IV

what can be computed in algebraic Geometry alg-geom/9304003

1) Gröbner and Deformations (Bayer-Mumford, 1993)

2) Computing T^1 using Macaulay 2 (Ilten, 2012)

Versal Deformation a package for computing versal deformations and local Hilbert Schemes
math.AG/1107.2416

See also
math.berkeley.edu/~nilten/new-def-calc.

1. GB and Deformations

Use weights to deform polynomials in $\mathbb{C}[x_1, \dots, x_n]$

Let x_i have weight $w_i \geq 0$.

Then $f = \sum a_\alpha x^\alpha \neq 0$ has weight $w(f) = \max w \cdot \alpha$

Assume f has unique term of weight $w(f)$

Define
$$\begin{aligned} F(\underline{x}, t) &= t^{w(f)} f(t^{-w_1}x_1, \dots, t^{-w_n}x_n) \in \mathbb{C}[\underline{x}, t] \\ &= t^{w(f)} \left(\sum a_\alpha t^{-w \cdot \alpha} x^\alpha \right) \\ &= \underbrace{a_{\alpha_0} x^{\alpha_0}}_{\substack{\text{leading } \cancel{\text{term}} \\ \text{(max weight)}}} + \sum_{\alpha \neq \alpha_0} a_\alpha \underbrace{t^{(w(f) - w \cdot \alpha)}}_{\text{pos power of } t} x^\alpha \end{aligned}$$

Given $I = \langle f_1, \dots, f_s \rangle \subseteq \mathbb{C}[\underline{x}]$,

get $X = V(I) \subseteq \mathbb{A}^n$

and $\mathcal{X} = V(F_1, \dots, F_s) \subseteq \mathbb{A}^{n+1}$

$\searrow \mathbb{A}^1$ ← last coord

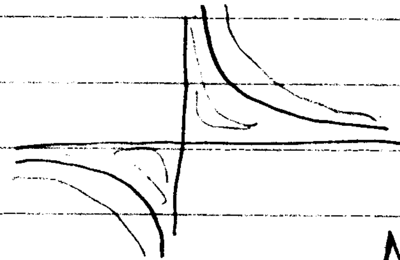
note: 1) for $t \neq 0$, $\mathcal{X}_t \simeq X$ via $x_i \to t^{-w_i} x_i$

2) $\mathcal{X}_0 = V(\text{leading terms of } f_1, \dots, f_s)$
$= \text{union of coordinate subspaces (with mult)}$

Ex 1 $f = xy - 1$

$\omega = (1, 0)$

$F = xy - t$



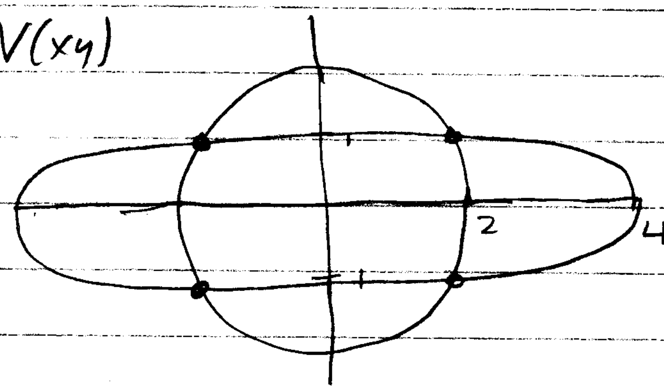
Nice!

$\mathcal{X}_t \underset{t \neq 0}{\simeq} V(f)$ $\quad \mathcal{X}_0 = V(xy)$

Ex 2 $f_1 = x^2 + y^2 - 4$

$f_2 = \frac{x^2}{16} + y^2 - 1$

$\omega = (2, 1)$



$F_1 = x^2 + t^2y^2 - 4t^4$

$F_2 = \frac{x^2}{16} + t^2y^2 - t^4$

Not!

$\mathcal{X}_t \underset{t \neq 0}{\simeq} V(f_1, f_2)$ 4 pts $\quad \mathcal{X}_0 = V(x^2)$ = y-axis with mult 2

Ex 3 $f_1 = x^2 + y^2 - 4$

$f_3 = 15y^2 - 12$ $\quad [= 16f_2 - f_1]$

Same $\omega = (2, 1)$

Same variety as Ex 2!

$F_1 = x^2 + t^2y^2 - 4t^4$

$F_3 = 15y^2 - 12t^2$

Nice!

$\mathcal{X}_t \underset{t \neq 0}{=} V(f_1, f_3) = V(f_1, f_2)$ 4 pts $\quad \mathcal{X}_0 = V(x^2, y^2)$ = origin with mult 4

Key point: Ex 1, 3 are GB $\quad$ Ex 2 isn't!

<u>Theorem</u> Suppose $f_1,\dots,f_s$ are a GB for a monomial order $<$. Then weights $w_i \geq 0$ can be chosen so that $F_i = t^{wt(f)} f_i(t^{-w_1}x_1,\dots,t^{-w_n}x_n) \in \mathbb{C}[x_1,\dots,x_n,t]$ give a *flat* family

$$\begin{tikzcd}
\mathcal{X} = V(F_1,\dots,F_s) \arrow[dr] & \subseteq & \mathbb{A}^{n+1} \arrow[dl, "\text{last coord}"] \\
& \mathbb{A}^1 &
\end{tikzcd}$$

such that $\mathcal{X}_t \simeq V(f_1,\dots,f_s)$ for $t \neq 0$

$\mathcal{X}_0 = V(LT(f_1),\dots,LT(f_s))$ $\quad$ LT = leadin term wrt.

<u>Pf</u> The syzygy module of $LT(f_1),\dots,LT(f_s)$ is generated by finitely many $\mathbb{Z}^n$-homogeneous syzygies. ($\deg(x^\alpha) = \alpha$)
Such a syzygy is of the form $(h_1,\dots,h_s)$ where

1) $\sum h_i LT(f_i) = 0$
2) h_i = const $\cdot$ monomial and $h_i LT(f_i) = c_i x^\alpha$ $\leftarrow$ same α for every i

Thm: f_i GB $\Leftrightarrow$ alway $\mathbb{Z}^n$-homo syzygy $\sum \overbrace{h_i LT(f_i)}^{c_i x^\alpha} = 0$ satisfies

$$\underbrace{\sum h_i f_i \xrightarrow[\text{by } f_1 \dots f_s]{\text{division alg}} 0}$$

[IVA Thm 9 of ch 2, §9]

This means $\sum h_i f_i = \sum A_i f_i$ with $LT(A_i f_i) < x^\alpha$

Cor: f_i GB $\Leftrightarrow$ a basis of $\mathbb{Z}^n$-homo syzygies $\sum \overbrace{h_i LT(f_i)}^{c_i x^\alpha} = 0$ lift to syzygies $\sum (h_i - A_i) f_i = 0$ with $LT(A_i f_i) < x^\alpha$

Now pick weight w_i s.t. $LT(f_i)$ is max wt term of f_i $\forall i$

$wt(A_i f_i) < w \cdot \alpha$ " $\forall i$

Set $g_i = h_i - A_i$. Then $t^{w \cdot \alpha} \sum_i g_i(t_1^{-w_1} x_1, \ldots) f_i(t_1^{-w_1} x_1, \ldots) = 0$

$\Rightarrow$ syz lifts to $\sum \underbrace{G_i} F_i = 0$ in $\mathbb{C}[x_1, \ldots x_n, t]$

Just need to do this for (finite) basis of $\mathbb{Z}^n$-homo syz of $LT(f_i)$. Done!

QED

Ex

from (Bayer-Mumford)

$f_1 = \underline{w^2} - xy$

$f_2 = \underline{wy} - xz$

$f_3 = \underline{wz} - y^2$

generate ideal of twisted cubic $(w, x, y, z) = (u^2v, u^3, uv^2, v^3)$ in $\mathbb{P}^3$

$f_4 = \underline{xz^2} - y^3 = -zf_2 + yf_3$

GB for lex $w > x > y > z$

Syz of w^2, wy, wz, xz^2 gen by

$y(w^2) - w(wy) = 0 \quad \longleftrightarrow \quad (y, -w, 0, 0)$

$z(w^2) - w(wz) = 0 \quad \longleftrightarrow \quad (z, 0, -w, 0)$

$z(wy) - y(wz) = 0 \quad$ etc

$xz(wz) - w(xz^2) = 0$

lift to syz to $f_1 \; f_2 \; f_3 \; f_4$

$yf_1 - wf_2 - xf_3 = 0$

$zf_1 - yf_2 - wf_3 = 0$

$zf_2 - yf_3 + f_4 = 0$

$-y^2 f_2 + xzf_3 - wf_4 = 0$ ∎

Pick weights 16, 4, 1, 0 for w, x, y, z

$$F_1 = t^{32}\left((t^{-16}w)^2 - (t^{-4}x)(t^{-1}y)\right) = \underline{w^2} - t^{27}xy$$
$$F_2 = \underline{wy} - t^{13}xz$$
$$F_3 = \underline{wz} - t^{14}y^2$$
$$F_4 = \underline{xz^2} - ty^3$$

Lifts of syz to $F_1, \ldots, F_4$

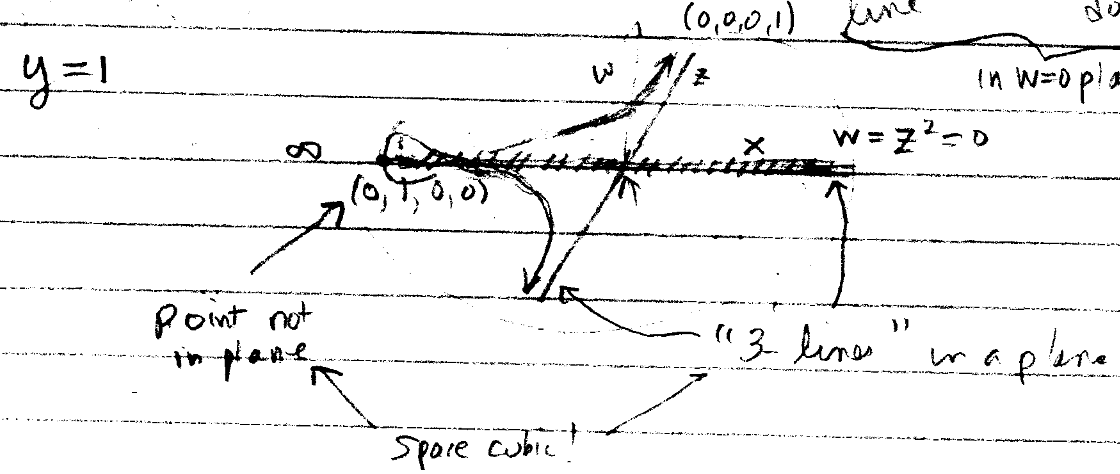
$$yF_1 - wF_2 - t^{13}xF_3 = 0 \qquad \left\{ \begin{array}{l} wt(\overset{1}{y}\overset{32}{f_1}) = wt(\overset{16}{w}\overset{17}{f_2}) = 3 \\ wt(\underset{4}{x}\underset{16}{f_3}) = 20 \end{array} \right.$$
$$zF_1 - t^{14}yF_2 - wF_3 = 0$$
$$zF_2 - yF_3 + t^{13}F_4 = 0$$
$$-ty^2F_2 + xzF_3 - wF_4 = 0$$

Flat deformation of twisted cubic to $V(w^2, wy, wz, xz^2)$

$$\langle w^2, wy, wz, xz^2 \rangle = \underbrace{\langle w, x \rangle}_{\text{line}} \cap \underbrace{\langle w, z^2 \rangle}_{\text{double line}} \cap \underbrace{\langle w^2, y, z \rangle}_{\text{point}}$$

in W=0 plane

not in w=0 plane



2. Computing T^1

Recall from Paul's lecture that for $X = V(I) \subseteq \mathbb{A}^n$,

$$\left\{ \begin{matrix} 1^{st} \text{ order defs} \\ \text{of } X \end{matrix} \right\} \Big/ \text{iso} \cong T^1_X$$

$$= \text{Coker}\left(H^0(T_{\mathbb{A}^n}|_X) \to H^0(N_{X/\mathbb{A}^n}) \right)$$

For $X = V(f)$: 1) $T^1_X = \underbrace{k[x_1, \ldots, x_n]}_{P} / \langle f, \frac{\partial f}{\partial x_i} \rangle$

2) X isolated sing $\Rightarrow \dim_k T^1_X < \infty$

3) In this case, can lift 1^{st} order def to versal deformation over $\mathbb{A}^m$, $m = \dim_k T^1_X$

To compute T^1_X in general for $X = V(I)$, $I \subseteq P = \mathbb{C}[x_1, \ldots, x_n]$

① Define $\varphi : P^n \longrightarrow \overbrace{\text{Hom}_P(I, P/I)}^{N_X}$ by

$$e_i \mapsto \left(f \in I \longmapsto [\tfrac{\partial f}{\partial x_i}] \in P/I \right)$$

② Then $T^1_X = \text{coker}\, \varphi$. Easy to do in Macaulay 2

The VersalDeformation package for Macaulay 2 will compute this.

Ex Consider $X \subseteq \mathbb{P}^4$ param by $(\underset{x_0}{u^4}, \underset{x_1}{u^3v}, \underset{x_2}{u^2v^2}, \underset{x_3}{uv^3}, \underset{x_4}{u^4})$

X is defined by

$$\text{rank}\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix} \leq 1, \quad \text{i.e. } 2\times 2 \text{ minors vanish}$$

i.e.

$$\begin{aligned}
&-x_1^2 + x_0x_2 - t_1x_1 \qquad\qquad t_1^2 = 0 \\
&-x_1x_2 + x_0x_3 \\
&-x_2^2 + x_1x_3 + t_1x_3 \\
&-x_2x_3 + x_0x_4 \\
&-x_2x_3 + x_1x_4 + t_1x_4 \\
&-x_3^2 + x_2x_4
\end{aligned}$$

mult by $-t_1$

The command CT^1 applied to gives output

$$\begin{bmatrix}
x_1 & x_0 & 0 & 0 \\
0 & 0 & 0 & x_0 \\
-x_3 & -x_2 & 0 & x_1 \\
0 & 0 & x_2 & 0 \\
-x_4 & -x_3 & x_3 & 0 \\
0 & 0 & x_4 & -x_3
\end{bmatrix}$$

This says T^1_X has dim 4; each col gives inf def

Can show: 1) cols 1, 2, 4 correspond to

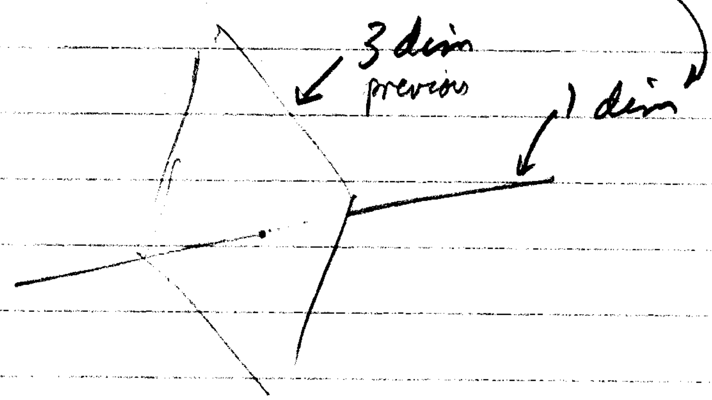
$$\text{rank}\begin{pmatrix} x_0 & x_1+t_1 & x_2 & x_3 \\ x_1 & x_2+t_2 & x_3+t_4 & x_4 \end{pmatrix} \leq 1 \quad \text{3 dim family of inf defs}$$

2) X is defined by

$$\operatorname{rank}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix} \le 1$$

and 3^{rd} col + $(-t_3)2^{nd}$ col is the inf def

$$\operatorname{rank}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2+t_3 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix} \le 1 \quad \leftarrow 1 \text{ dim family}$$



"picture" of versal def

This ignores lifting problems + obstructions to lifting T_X^2.