Reconstruction problem in mirror symmetry, II

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Recall from last time: we consider a family $X^\times/\Delta^\times$ of smooth Calabi–Yau n-folds X_t over the punctured disc $\Delta^\times$, such that the monodromy is maximally unipotent. We have a metric g_t on X_t, the unique Ricci-flat Kähler metric in a given Kähler class. We rescale g_t so that the diameter of (X_t, g_t) is independent of t. Then we expect that the sequence (X_t, g_t) of metric spaces converges (in the Gromov–Hausdorff sense) to a metric space (B, g) where B is homeomorphic to S^n. Moreover, there is a dense open subset $B^0 \subset B$ with complement of real codimension ≥ 2 such that B^0 carries an integral affine structure and the metric g is Monge-Ampère. Roughly speaking, the space B is the base of the SYZ fibration of X_t (the fibres of the SYZ fibration shrink to zero volume as $t \to 0$, leaving the base as the GH limit).

We ask: can we reconstruct the nearby smooth fibre X_t from (B, g) with its integral affine structure?

There are two approaches we would like to discuss. The first is due to Kontsevich–Soibelman [KS06]. They construct the generic fibre as a nonarchimedean analytic space (in the sense of Berkovich) over the power series field $\mathbb{C}((t))$. The second is due to Gross–Siebert (see [GS08] for an introduction and [GS07] for the proofs). They define a degenerate Calabi–Yau X_0 associated to a singular integral affine manifold B and inductively construct an infinitesimal deformation X_n of X_0 over the Artinian ring $A_n = \mathbb{C}[t]/(t^{n+1})$ for each $n \geq 0$. General theory shows that the X_n/A_n are induced by an actual deformation X/Δ over the disc. Finally one checks that the general fibre of X/Δ is a smooth Calabi–Yau.

We study the Gross–Siebert method first as it is algebraic in nature.

1 Motivating example

Let X/Δ be a family of smooth elliptic curves X_t degenerating to a singular curve X_0 which is a cycle of smooth rational curves (copies of $\mathbb{P}^1_{\mathbb{C}}$). Recall that the Ricci-flat metric g_t on $X_t = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau(t)$ is just the flat metric induced by the standard metric on $\mathbb{C}$, suitably scaled. So it is possible to relate the Gromov–Hausdorff limit of the (X_t, g_t) and the special fibre X_0 by an elementary calculation — see [G09, p. 20–21]. We find that a small neighbourhood of a vanishing cycle in X_t corresponds to an interval in the GH limit S^1. The complement of these neighbourhoods in X_t is a union of cylinders, and each cylinder shrinks to a point in the GH limit (please draw a picture and read the treatment in [G09]). So we obtain a subdivision of the GH limit S^1 into a union of intervals, with each interval corresponding to a 0-stratum (a node) of X_0 and each vertex corresponding to a component of X_0. In other words, the GH limit can be identified with the *dual complex* of X_0.

We expect that a similar picture holds in arbitrary dimension (this was first proposed by Kontsevich).

2 Gross–Siebert construction

We consider a *Gorenstein toroidal degeneration* X/Δ of Calabi–Yau manifolds. That is, for $t \neq 0$, X_t is a smooth Calabi–Yau, and X_0 is a singular variety satisfying the following properties:

(1) Irreducible components of X_0 are toric varieties, meeting transversely in codimension 1 along their toric boundaries.

(2) At any 0-dimensional stratum $p \in X_0$, the total space X of the family is locally analytically isomorphic to a neighbourhood of the origin in a Gorenstein affine toric variety V, with the special fibre X_0 corresponding to the toric boundary. Explicitly, we have a polytope $\sigma_p \subset \mathbb{R}^n$; let C_p be the cone over $\sigma_p \subset \mathbb{R}^n \times 1$ in $\mathbb{R}^{n+1}$. Then V is the affine toric variety with coordinate ring

$$\mathbb{C}[C_p^* \cap \mathbb{Z}^{n+1}]$$

and the toric map $V \to \mathbb{A}^1$ (corresponding to the projection $X \to \Delta$) is given by the map of cones $C_p \to \mathbb{R}_{\geq 0}$ obtained by projection onto the last coordinate. In other words, the coordinate t on $\mathbb{A}^1$ is given by the vector $(0, \dots, 0, 1)$ in the coordinate ring of V.

(3) There is a closed subvariety $Z \subset X_0$ of complex codimension ≥ 2, not containing any toric strata, such that at each point of $X_0 \setminus Z$ the deformation is locally toric as in (2).

Now we produce a sphere $B = S^n$ and an integral affine structure on a dense open subset $B^0 \subset B$ with complement of real codimension ≥ 2. We first construct B as a topological manifold. For each 0-stratum $p \in X_0$ we have the polytope σ_p from (2) above. If p and q are connected by a 1-stratum of X_0 (a copy of $\mathbb{P}^1_{\mathbb{C}}$) we glue the polytopes σ_p along the corresponding faces. This gives B. As we mentioned last time, if X_t is simply connected and has full $SU(n)$ holonomy, we expect that B is a copy of S^n.

We now define the integral affine structure on a dense open subset $B^0 \subset B$ of B. We take the integral affine structure on the interior of each polytope σ_p to be the integral affine structure induced by the inclusion $\sigma_p \subset \mathbb{R}^n$. A vertex v of B corresponds to an irreducible component X_0^v of X_0. Recall that X_0^v is a toric variety by assumption, and the intersection of the singular locus of X_0 with X_0^v is its toric boundary. Let $\Sigma \subset \mathbb{R}^n$ be the fan of X_0^v. Then a neighbourhood of $v \in B$ is naturally identified with a neighbourhood of $0 \in |\Sigma| = \mathbb{R}^n$, such that the two subdivisions match up, and the identification is integral affine linear on each cell. So, this gives a chart for an integral affine structure on B in a neighbourhood of v, compatible with the integral affine structure on the interiors of the polytopes.

However, we do *not* get an integral affine structure on the whole of B this way, because the two integral affine structures along an edge $e = vw$ of B defined by v and w are not compatible. However, we can define a discriminant locus $\Delta \subset B$ and an integral affine structure on $B^0 := B \setminus \Delta$. We will explain one way to do this in the case of dimension $n = 2$. For each edge e of B choose a point δ_e in the interior of e and let Δ be the set of points δ_e. Now for a vertex $v \in B$ let U_v be the open set given by the union of all the interiors of cells of B containing v, minus the segments $[\delta_e, w]$ for $e = vw$ an edge incident to v (please draw a picture!). Then we can define the integral affine structure on the neighbourhood U_v of v as before and these charts define an integral affine structure on $B^0 = B \setminus \Delta$. Moreover, the monodromy of the integral affine structure about a point δ_e will be given by the matrix

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

where $k \in \mathbb{Z}_{\geq 0}$, and the invariant direction is along the edge e. In terms of the degeneration X/Δ, this corresponds (typically) to k distinct singularities of the total space X of type $(xy + zt = 0)$ along the 1-stratum Γ of X_0

corresponding to e. Note: it may be more realistic to perturb the singularity above into k distinct singularities along the edge e each with monodromy

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(the focus–focus singularity).

Example 2.1. Quartic K3. Here $X_t \subset \mathbb{P}^3$ degenerates to the union of the 4 coordinate planes. The total space X of the degeneration has 24 singular points, this is our set $Z \subset X_0$ contained in the interior of the 1-strata (with 4 points along each edge). Our $B = S^2$ is the dual tetrahedron, each face σ_p is a triangle (because X is locally of the form $(xyz = t)$ near $p \in X_0$). We have 4 focus–focus singularities of the affine structure along each edge.

Example 2.2. The K3 surface of degree 6. We have $X_t = Q \cap R \subset \mathbb{P}^4$, where Q is a quadric hypersurface and R is a cubic hypersurface, and X_t degenerates to the union X_0 of 6 coordinate planes by degenerating Q to $x_0x_1 = 0$ and R to $x_2x_3x_4 = 0$ (where $x_0, \ldots, x_4$ are homogeneous coordinates on $\mathbb{P}^4$). Here

$$X_0 = \mathbb{P}^2_{x_0,x_2,x_3} \cup \mathbb{P}^2_{x_0,x_3,x_4} \cup \mathbb{P}^2_{x_0,x_4,x_2} \cup \mathbb{P}^2_{x_2,x_3,x_1} \cup \mathbb{P}^2_{x_3,x_4,x_1} \cup \mathbb{P}^2_{x_4,x_2,x_1}$$

and we can draw X_0 as the boundary of the union of two pyramids with vertices corresponding to the coordinate points $e_0, \ldots, e_4$ (e_2, e_3, e_4 being the vertices of the common base of the two pyramids). The dual complex is a triangular prism. The integral affine structure has 3 focus–focus singularities along each edge of the triangular faces and 2 along each of the remaining edges.

The reconstruction problem á la Gross–Siebert is then the following. We begin with an integral affine manifold with singularities B together with a polyhedral subdivision as above. Then B corresponds to a degenerate Calabi–Yau X_0, together with some additional data (a *log structure*) which specifies, everywhere locally on X_0, the local form of the deformation of X_0 we want to construct (compare the definition of σ_p above). We also choose an ample line bundle on X_0 (that is, a holomorphic line bundle L on X_0 such that the global sections of $L^{\otimes N}$ define an embedding of X_0 in projective space for $N \gg 0$). This corresponds to a multivalued function ϕ on B which is piecewise integral affine linear with respect to the polyhedral subdivision of B. Then we construct a deformation X/Δ of X_0 over the disc with general fibre a smooth projective Calabi–Yau manifold X_t.

Note: Although the line bundle L is used in the construction, it is expected that the actual result is independent of the choice of L.

References

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