Reconstruction problem in mirror symmetry

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1 Large complex structure limit of Calabi-Yaus

Let X/Δ be a degenerating family of Calabi-Yau n-folds over the disc Δ. So X_t is smooth for $t \neq 0$ and X_0 is singular. We have the monodromy operator

$$T\colon H^n(X_t, \mathbb{Z}) \to H^n(X_t, \mathbb{Z}).$$

After a base change $t \mapsto t^k$ we may assume that X is smooth and $X_0 \subset X$ is a normal crossing divisor, that is, looks locally like a union of coordinate hyperplanes (this is Mumford's semistable reduction theorem). In this case one can show the map T is unipotent, $(T-I)^{n+1} = 0$. We say X/Δ has *maximally unipotent monodromy* if $(T-I)^n \neq 0$. We say X_t is approaching a large complex structure limit as $t \to 0$.

Example 1.1. The elliptic curve ($n = 1$). We consider a family X/Δ of smooth elliptic curves X_t, $t \neq 0$, degenerating to a rational nodal curve (a pinched torus). The monodromy is

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The invariant class is the vanishing cycle.

2 SYZ (philosophical)

Let X be a Calabi–Yau n-fold that is close to a large complex structure limit. Then we expect that there is a fibration $f\colon X \to B$ of X by special Lagrangian real n-tori L. (Recall that $L \subset X$ is *special Lagrangian* if $\omega|_L = 0$ and $\operatorname{Im}\Omega|_L = 0$ where ω is the Kähler form and Ω is the holomorphic n-form.) B is a real manifold of dimension n (away from the locus of singular fibres of f).

Example 2.1. Elliptic curve. $f\colon X = \mathbb{C}/\Lambda \to S^1$ with fibre S^1, given by $z = x + iy \mapsto y$ (here $\Omega = dz$).

Example 2.2. K3 surface. $f\colon X \to S^2$ with general fibre $S^1 \times S^1$, and (typically) 24 singular fibres, each a pinched torus.

Let $B^0 \subset B$ denote the locus of smooth fibres of f. Then f defines an *integral affine structure* on B^0, that is, a distinguished atlas of charts with transition functions of the form $x \mapsto Ax + b$ for $A \in \mathrm{GL}(n,\mathbb{Z})$ and $b \in \mathbb{R}^n$. Indeed the integral affine coordinates can be defined as follows. Let $\gamma_1, \ldots, \gamma_n$ be a basis of $H_1(L,\mathbb{Z})$ for $L = X_b$ a smooth fibre (a real n-torus). For $v \in T_bB$ a tangent vector, let $\tilde{v}$ be a lift of v to a vector field along $L \subset X$ and define 1-forms η_i on B by

$$\eta_j(v) = \int_{\gamma_j} i_{\tilde{v}}\omega = \int_{\gamma_j} \omega(\tilde{v}, \cdot)$$

(the choice of lift $\tilde{v}$ is irrelevant because $\omega|_L = 0$). The forms η_j are closed, so locally we can write $\eta_j = dy_j$. Then $y_1, \ldots, y_n$ are local integral affine coordinates.

There is also a natural metric g on B, the *MacLean metric*, defined by

$$g(v_1, v_2) = -\int_{X_b} i_{\tilde{v}_1}\omega \wedge i_{\tilde{v}_2} \operatorname{Im}\Omega.$$

Theorem 2.3. *(Hitchin) In local integral affine coordinates $y_1, \ldots, y_n$, the metric $g = (g_{ij})$ is the Hessian of a (locally defined) potential function K:*

$$g_{ij} = \frac{\partial^2 K}{\partial y_i \partial y_j}.$$

Suppose in addition that the Ricci-flat Kähler metric h on X is semi-flat*, that is, the induced metric on the fibres L of f is flat. Then*

$$\det(g_{ij}) = \textit{ constant},$$

that is, the potential function K satisifes the real Monge–Ampère equation. We say the integral affine manifold B^0 together with the metric g is a Monge–Ampère manifold.

Remark 2.4. In general, we do not expect the Kähler metric h to be semi-flat. However, for X close to a large complex structure limit, we expect the metric to be close to a semi-flat metric away from the singular fibres. For K3 surfaces, this is explained in [GW00].

The integral affine structure on B^0 is determined by the lattice $T^{\mathbb{Z}}B^0 \subset TB^0$ of integral tangent vectors. We can define a dual integral affine structure using the MacLean metric g by

$$\check{T}^{\mathbb{Z}}B^0 := \{v \in TB^0 \mid g(v,u) \in \mathbb{Z} \text{ for all } u \in T^{\mathbb{Z}}B^0\} \subset TB^0.$$

3 SYZ mirror symmetry

We are looking for another Calabi-Yau n-fold $\check{X}_t$ such that the SYZ fibration gives the same Monge–Ampère manifold but interchanges $T^{\mathbb{Z}}B^0$ and $\check{T}^{\mathbb{Z}}B^0$.

First consider the semi-flat case with no singular fibres. Then $X = TB/T^{\mathbb{Z}}B \to B$ and $\check{X} = TB/\check{T}^{\mathbb{Z}}B \to B$. Unfortunately Cheng-Yau showed that the only Calabi-Yaus that can arise in this way are complex tori.

Homework: The SYZ picture explains the interchange of Hodge numbers in classical mirror symmetry, see [G09, §1].

Reconstruction problem, I: Given a Monge–Ampère manifold B, produce Calabi–Yau manifolds X and $\check{X}$ SYZ fibred over it.

Unfortunately the analysis required to construct special Lagrangian fibrations in dimension $n \geq 3$ seems intractable. We will use a less direct approach inspired by the SYZ picture.

4 Gromov–Hausdorff collapse

Let X_t be a family of Calabi–Yau n-folds approaching a large complex structure limit. Let h_t be the Ricci flat Kähler metric on X_t. Rescale h_t so that the diameter of X_t is constant.

Definition 4.1. (Gromov–Hausdorff distance) Let (X, d_X), (Y, d_Y) be metric spaces. Let $f\colon X \to Y$ and $g\colon Y \to X$ be two maps (not necessarily continuous). Suppose that

$$|d_X(x_1, x_2) - d_Y(f(x_1), f(x_2))| < \epsilon \text{ for all } x_1, x_2 \in X,$$

$$d(x, g(f(x))) < \epsilon \text{ for all } x \in X,$$

and the two symmetric statements hold. Then the Gromov–Hausdorff distance $d_{GH}(X,Y) < \epsilon$. And $d_{GH}(X,Y)$ is the infimum of all such ϵ.

Theorem 4.2. *(Gromov) A family of Ricci flat manifolds of bounded diameter has a convergent subsequence with respect to d_{GH} (the limit being a metric space).*

Example 4.3. Elliptic curve. Let $X_t = \mathbb{C}/\langle 1, i/t\rangle$, $t \to 0$, with Ricci-flat metric induced from the standard metric on $\mathbb{C}$. Then X_t has diameter approximately $1/t$. Rescale to obtain $X_t = \mathbb{C}/\langle t, i\rangle$. Thus X_t is obtained from rectangle with height 1 and width $t \to 0$ by identifying opposite sides. So the GH limit is S^1.

A calculation shows that the fibres of the SYZ fibration have vanishing volume.

Conjecture 4.4. (Kontsevich) The GH limit of (X_t, h_t) is the base of the SYZ fibration with its Monge-Ampère metric.

Moreover, let us assume that X_t is simply connected and has full $SU(n)$ holonomy. Then we expect B is homeomorphic to S^n.

Conjecture 4.4 is proved for $n = 2$ (K3 surfaces) by Gross and Wilson [GW00].

Reconstruction problem II: Begin with $S^n \supset B^0$, $\Delta := S^n \setminus B^0$ codimension ≥ 2, B^0 Monge–Ampère, plus some conditions on Δ (for example, restriction on the monodromy of the integral affine structure around Δ). Construct a degenerating family of Calabi–Yaus with GH limit B.

References

[G09] M. Gross, The SYZ conjecture: From torus fibrations to degenerations, in Algebraic geometry - Seattle 2005, Proc. Sympos. Pure Math. 80, Part 1, p. 149–192, AMS 2009, and arXiv:0802.3407v1

[GW00] M. Gross and P. Wilson, Lagre complex structure limits of K3 surfaces, J. Differential Geom. 55 (2000), no. 3, 475–546, and arxiv:math/0008018v3 [math.DG] .