TROPICAL VERTEX

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A strange object appears in Kontsevich–Soibelman . . .

The group $\mathrm{Aut}(\mathbb{C}^* \times \mathbb{C}^*)$ is not very interesting, just $\mathrm{GL}(2, \mathbb{Z})$ semidirect product with translations. But

$$\mathrm{Aut}(\mathbb{C}[x^{\pm}, y^{\pm}] \otimes \mathbb{C}[[t]])$$

contains some interesting elements, for example

$$x^n y^m \mapsto (1 + yt)^n x^n y^m$$

Its inverse is

$$x^n y^m \mapsto (1 + yt)^{-n} x^n y^m.$$

More generally, consider

$$T : \, x^m y^n \mapsto (fx)^m (gy)^n.$$

Here

$$f = 1 + tf_1(x, y) + t^2 f_2(x, y) + \dots$$
$$g = 1 + tg_1(x, y) + t^2 g_2(x, y) + \dots$$

One can find an inverse of T order-by-order.

Want to consider a group G generated by elements of this sort. What will be the corresponding Lie algebra? $\mathrm{Lie}(G)$ is spanned by $\langle z^m \partial_n \rangle$, where $z^{(a,b)} = x^a y^b$ and $\partial_{a,b} x^c y^d = (ac + bd) x^c y^d$. For example, $\partial_{(0,1)} = y \frac{d}{dy}$, and $\exp(x \partial_{(0,1)}) y^n = e^x y^n$. What are the relations?

$$[z^m \partial_n, z^{m'} \partial_{n'}] = \langle n, m' \rangle z^{m+m'} \partial_{n'} - \langle n', m \rangle z^{m+m'} \partial_n$$

Consider a subalgebra in $\mathrm{Lie}(G)$ called $\mathfrak{h}$ spanned by elements $z^m \partial_n$ such that $\langle m, n \rangle = 0$. It is a subalgebra because it is a centralizer of some symplectic form.

0.1. DEFINITION. Let R be an Artinian ring with a maximal ideal m_R. Let $\mathfrak{h}_R = m_R \otimes \mathfrak{h}$. Group spanned by $\exp(x)$, $x \in \mathfrak{h}_R$ is called a tropical vertex.

Usually we just take R to be $\mathbb{C}[[t]]$ modulo a big power of t. Later on we can take an inverse limit.

What are the relations in the tropical vertex? Gross–Pandharipande–Siebert [GPS] realized that to understand relations, we need to study rational curves on toric surfaces. Introduce

$$\theta_{(a,b),\exp(f)} := \exp(f \partial_{(b,-a)}),$$

where

$$f = tf_1(x^a y^b) + t^2 f_2(x^a y^b) + \dots$$

Let

$$S_{l_1} = \theta_{(1,0),(1+tx)^{l_1}},$$
$$S_{l_2} = \theta_{(0,1),(1+ty)^{l_2}}.$$

For example, $S_{l_1}(x^n y^m) = x^n(1+tx)^{l_1 m} y^m$. Let's compute the commutator $[S_{l_1} S_{l_2} S_{l_1}^{-1} S_{l_2}^{-1}]$. It turns out that

$$[S_{l_1} S_{l_2} S_{l_1}^{-1} S_{l_2}^{-1}] = \overrightarrow{\prod} \theta_{(a,b),f_{a,b}}$$

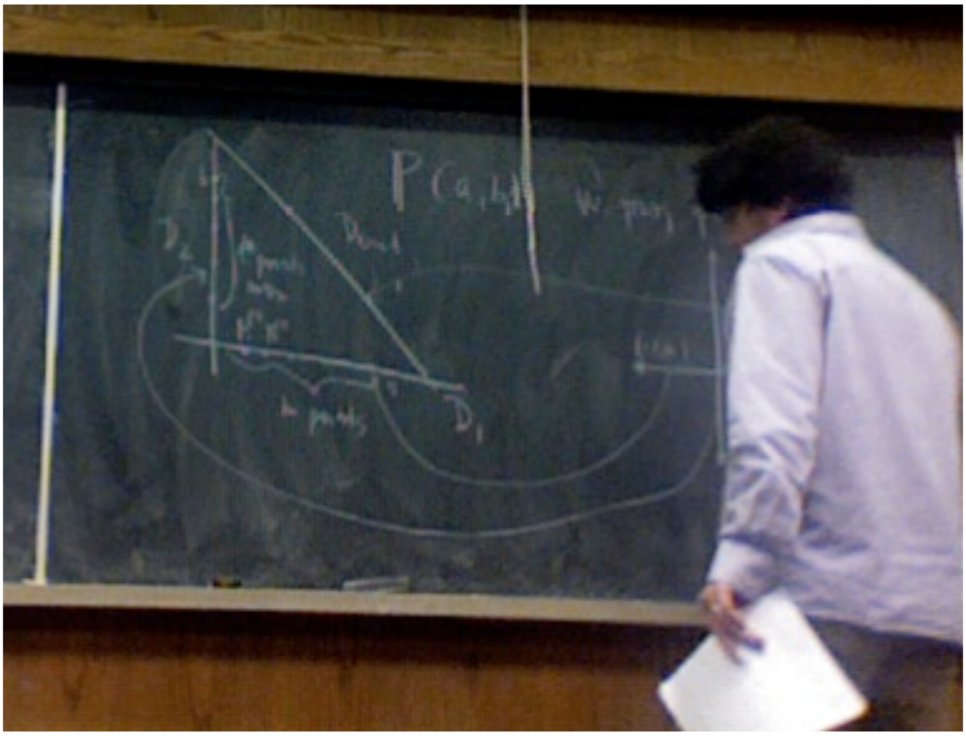
(the product is taken counterclockwise with respect to directions of positive primitive vectors (a,b)) and

$$\ln(f_{a,b}) = \sum_{k \geq 1} k c^k_{a,b}(l_1 l_2)(tx)^a (ty)^b,$$

where "structure constants" $c^k_{a,b}(l_1, l_2)$ count some curves.

Namely, let's define a GW-style invariant $N_{a,b}[P]$. Here $P = (P_1, P_2)$. P_1 is an ordered partition $p_1^{(1)} + \ldots + p_{l_1}^{(1)}$. P_2 is an ordered partition $p_1^{(2)} + \ldots + p_{l_2}^{(2)}$. We fix $|P_1| = ak$ and $|P_2| = bk$.

Let $\mathbb{P}(a,b,1)$ be a weighted projective plane. Fix l_1 points on D_1 and l_2 points in D_1. Partitions P^1 and P^2 describe multiplicities of these points.

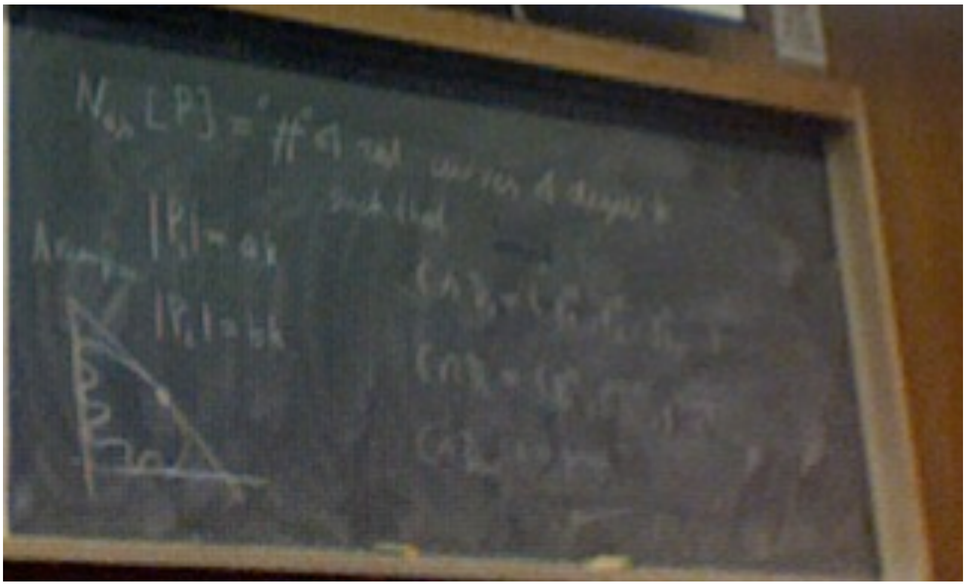


Then $N_{a,b}[P]$ is a "vurtual" number of rational curves C of degree k such that $C \cap D_1$ and $C \cap D_2$ are prescribed points with prescribed multiplicities and $C \cap D_{out}$, while not fixed, must be a single point of some large multiplicity.

Then

$$c^k_{ab}(l_1, l_2) = \sum_{P_1, P_2} N_{a,b}[(P_1, P_2)].$$

summation over $|P_1| = ak$, $|P_2| = bk$, $l(P_1) = l_1$, $l(P_2) = l_2$.



This computes $[S_{l_1} S_{l_2} S_{l_1}^{-1} S_{l_2}^{-1}]$. The proof is of combinatorial nature, it uses counts of tropical curves in the spirit of Mikhalkin.