How to glue a K3 surface from flat pieces, II

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1 Recap

Recall from part I: The aim is to construct a 1-parameter family $\mathcal{X}/\Delta$ of degenerating K3 surfaces associated to some input data of a combinatorial nature. We first describe the special fibre X of the family. It is a union of polarized toric varieties (X_σ, L_σ) associated to lattice polygons σ. The polygons σ are glued along faces to form a polyhedral subdivision $\mathcal{P}$ of S^2. The components X_σ are glued along their toric boundaries to form X, with the combinatorics of the gluing being dictated by $\mathcal{P}$. (Also the ample line bundles L_σ glue to give an ample line bundle L on X.) We inductively construct infinitesimal deformations $\mathcal{X}^k/\operatorname{Spec}(\mathbb{C}[t]/(t^{k+1}))$ of X of order k, for each $k \geq 0$. Then a general result shows that these are induced by a family $\mathcal{X}/\operatorname{Spec}\mathbb{C}[[t]]$ over the formal power series ring, and one can check that the generic fibre is a smooth K3 surface.

2 Comparison with semistable degenerations

It is instructive to compare the families constructed by Gross–Siebert to the more familiar semistable degenerations of K3 surfaces studied by Kulikov, Friedman, Kawamata–Namikawa, etc. Let $\mathcal{X}^\times/\Delta^\times$ be a family of polarized K3 surfaces over the punctured disc. By a general theorem of Mumford, after a base change

$$\Delta \to \Delta, \quad t \mapsto t^k,$$

there exists a completion $\tilde{\mathcal{X}}/\Delta$ which is a *semistable family*, that is, the total space $\tilde{\mathcal{X}}$ is smooth and the special fibre $\tilde{X} = \tilde{\mathcal{X}}_0$ is a normal crossing divisor (locally looks like a union of coordinate hyperplanes in $\mathbb{C}^3$). Moreover, Kulikov showed that we may assume that the total space $\tilde{\mathcal{X}}$ satisfies the Calabi–Yau condition $K_{\tilde{\mathcal{X}}} = 0$ (a modern proof of this fact can be obtained via the 3-fold minimal model program). By a general Hodge-theoretic result, the monodromy

$$T\colon H^2(\mathcal{X}_t, \mathbb{Z}) \to H^2(\mathcal{X}_t, \mathbb{Z})$$

of $\mathcal{X}^\times/\Delta^\times$ satisfies $(T-I)^3 = 0$. If T is maximally unipotent, that is $(T-I)^2 \neq 0$, then the special fibre $\tilde{X}$ of $\tilde{\mathcal{X}}/\Delta$ is a union of rational surfaces $\tilde{X}_\sigma$ glued along boundary divisors $\Delta_\sigma \subset \tilde{X}_\sigma$. Each Δ_σ is a cycle of smooth rational curves (copies of $\mathbb{P}^1_{\mathbb{C}}$) such that $K_{\tilde{X}_\sigma} + \Delta_\sigma = 0$, and the gluing is such that the dual complex Σ of $\tilde{X}$ is a triangulation of S^2. Necessarily, some of the components $(\tilde{X}_\sigma, \Delta_\sigma)$ are *not* toric. However, typically, each $(\tilde{X}_\sigma, \Delta_\sigma)$ is obtained from a toric surface together with its toric boundary (X_σ, B_σ) by blowing up a number of smooth points of the toric boundary B_σ (with Δ_σ being the strict transform of B_σ). Thus X_σ is obtained from $\tilde{X}_\sigma$ by contracting a number of disjoint (-1)-curves each meeting the boundary Δ_σ in a unique point. There is an induced contraction $\tilde{\mathcal{X}} \to \mathcal{X}/\Delta$ of the 3-fold $\tilde{\mathcal{X}}/\Delta$. Indeed, each of the (-1)-curves $C \subset \tilde{X}_\sigma$ has normal bundle in $\tilde{\mathcal{X}}$ of type

$$\mathcal{N}_{C/\tilde{\mathcal{X}}} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$$

and it follows that it can be contracted to a singularity of type

$$(xy + zt = 0) \subset \mathbb{C}^3 \times \Delta$$

(the 3-fold ordinary double point). The resulting family $\mathcal{X}/\Delta$ is of the type constructed by Gross–Siebert. The ODP singularities of the total space $\mathcal{X}$ correspond to the singularities of the associated integral affine structure on S^2. The dual complex Σ of $\tilde{X}$ coincides with the dual complex of X, so is dual to the polyhedral subdivision $\mathcal{P}$. Gross-Siebert study $\mathcal{X}/\Delta$ instead of $\tilde{\mathcal{X}}/\Delta$ because it is possible to understand the non-toric "corrections" associated to the singularities of $\mathcal{X}$ but at present it is not known how to deal with the non-toric components of $\tilde{\mathcal{X}}_0$ directly.

Remark 2.1. The deformation theory of normal crossing degenerations of K3 surfaces was described by Friedman and is well understood. In particular, there is a simple criterion for smoothability. What is new in the approach of Gross–Siebert is the *explicit* construction of a deformation associated to the combinatorial input data. The hope is that this description will "explain" mirror symmetry for K3 surfaces, in particular the relation between complex moduli (given by periods of the holomorphic 2-form) and counts of holomorphic curves and discs on the mirror surface, and more generally the homological mirror symmetry conjecture of Kontsevich.

3 Construction of $xy + zt = 0$ singularity

Recall from part I: Gross-Siebert construct toric local models of the infinitesimal deformations $\mathcal{X}^k/\operatorname{Spec}(\mathbb{C}[t]/(t^{k+1}))$ by gluing infinitesimal thickenings of the components of X. However, as described above, the family $\mathcal{X}/\Delta$ will have singularities of type

$$(xy + zt = 0) \subset \mathbb{C}^3 \times \Delta$$

in the interior of 1-dimensional toric strata of X. To produce the local models at these points we need to add gluing corrections to the toric construction. This section is based on [GS08, §2.3].

We consider a focus-focus singularity (monodromy $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$) of the integral affine structure on $|\mathcal{P}| = S^2$. By assumption, the singularity is in the interior of an edge of $\mathcal{P}$, with the direction of the edge being the monodromy invariant direction. We obtain two local charts I, II for a punctured neighbourhood of the singular point by cutting along one of the rays emanating from the singularity along the edge. In both charts, the subdivision $\mathcal{P}$ near the singularity corresponds to the subdivision of $\mathbb{R}^2$ into the left and right half planes $\sigma_1 = \mathbb{R}_{\leq 0} \times \mathbb{R}$, $\sigma_2 = \mathbb{R}_{\geq 0} \times \mathbb{R}$ with common boundary $\rho = \{0\} \times \mathbb{R}$ and the singularity at $(0, 0)$. In chart I we cut along $\{0\} \times \mathbb{R}_{\geq 0}$, in chart II along $\{0\} \times \mathbb{R}_{\leq 0}$. The transition function from chart I to chart II is given by the identity on σ_1 and $(u, v) \mapsto (u, u + v)$ on σ_2. We fix generators x_1, y_1, z of the ring

$$R^k_{\sigma_1} = \mathbb{C}[x_1, y_1, z^{\pm 1}, t]/(x_1 y_1 = t, y_1^{k+1})$$

giving the thickening of the component X_{σ_1} of X. The generators x_1, y_1, z correspond to the integral tangent vectors $(-1, 0), (0, 1), (1, 0)$ to σ_1 in either chart. (Note: to obtain the t-dependence, we lift these vectors to the graph of the piecewise linear function ϕ in $\mathbb{R}^2 \times \mathbb{R}$, the vertical vector $(0, 0, 1)$ corresponding to t. In our example we take $\phi(u, v) = \max(0, u)$ on $\mathbb{R}^2_{u,v}$.) Let x_2, y_2, z be the generators of $R^k_{\sigma_2}$ obtained from x_1, y_1, z by parallel transport in chart I. We have surjections

$$R^k_{\sigma_i} \twoheadrightarrow R^k_{\rho,\sigma_i} := \mathbb{C}[x_i, y_i, z^{\pm 1}, t]/(x_i y_i = t, x_i^{k+1}, y_i^{k+1})$$

corresponding to restriction to the thickening of the toric stratum X_ρ. We need to choose an identification

$$R^k_{\rho,\sigma_1} \xrightarrow{\sim} R^k_{\rho,\sigma_2}$$

and form the fibre product of rings

$$R^k_{\sigma_1} \times_{R^k_{\rho,\sigma_2}} R^k_{\sigma_2}$$

to obtain the coordinate ring of the deformation. However, the identifications given by parallel transport in the two charts are different: they are

$$h_I \colon x_1, y_1 \mapsto x_2, y_2$$

$$h_{II} \colon x_1, y_1 \mapsto zx_2, z^{-1}y_2$$

(and $z \mapsto z$ for both charts). We can correct this inconsistency by introducing the automorphisms g_I, g_{II} of the localized ring $(R^k_{\rho,\sigma_2})_{1+z}$ given by

$$g_I \colon x_2, y_2 \mapsto (1+z)x_2, (1+z)^{-1}y_2$$

$$g_{II} \colon x_2, y_2 \mapsto (1+z^{-1})x_2, (1+z^{-1})^{-1}y_2$$

then we have the equality

$$g_I h_I = g_{II} h_{II} \colon (R^k_{\rho,\sigma_1})_{1+z} \xrightarrow{\sim} (R^k_{\rho,\sigma_2})_{1+z}$$

and the corresponding fibre product

$$R^k_{\sigma_1} \times_{(R^k_{\rho,\sigma_2})_{1+z}} R^k_{\sigma_2}$$

can be identified with

$$\mathbb{C}[x, y, z^{\pm 1}, t]/(xy = (1+z)t, t^{k+1})$$

as we want.

4 Scattering

Once we introduce the local gluing corrections of Section 3 we are forced to introduce further corrections to ensure global consistency of the gluing. These corrections are encoded by a *scattering diagram* drawn on the integral affine manifold with singularities $B = |\mathcal{P}|$. This is closely related to an earlier construction of Kontsevich–Soibelman [KS06].

Let us first analyze the situation at a vertex $v \in \mathcal{P}$ such that one of the edges incident to v contains a singularity of the integral affine structure (cf. [GS08, Ex. 3.3]). Assume that v corresponds to a semistable point $P \in \mathcal{X}/\Delta$ of the deformation, that is, $(P \in \mathcal{X})/\Delta$ is locally of the form

$$(xyz = t) \subset \mathbb{C}^3_{x,y,z} \times \Delta.$$

Then locally near v the polyhedral subdivision $\mathcal{P}$ can be identified with the complete fan Σ in $\mathbb{R}^2$ generated by $(-1,0),(0,-1),(1,1)$, with the piecewise linear function ϕ given by $\phi(-1,0)=\phi(0,-1)=0$, $\phi(1,1)=1$. The coordinates x,y,z correspond to the integral tangent vectors $(-1,0),(0,-1),(1,1)$ (or rather their lifts to the graph of ϕ). Let $\sigma_1=\langle(0,-1),(1,1)\rangle,\sigma_2=\langle(1,1),(-1,0)\rangle,\sigma_3=\langle(-1,0),(0,-1)\rangle$ be the maximal cones of Σ. We have thickenings of the components X_{σ_i} of X which are glued along the thickenings of the 1-dimensional strata $X_{\rho_{ij}}$, $\rho_{ij}:=\sigma_i\cap\sigma_j$. In the purely toric case, this is encoded by a system of rings $R^k_{\sigma_i}$, $R^k_{\rho_{ij}}$ with maps

$$R^k_{\sigma_i}\to R^k_{\rho_{ij}}$$

given by the natural projections, such that the inverse limit R^k is the coordinate ring of the kth order deformation.

Suppose now that there is a singularity of the affine structure on the ray ρ_{23}. Then the gluing along $R^k_{\rho_{23}}$ is modified by an automorphism, for example

$$\theta_{23}\colon (R^k_{\rho_{23}})_{1+x}\to(R^k_{\rho_{23}})_{1+x},\quad x,y,z\mapsto x,(1+x)^{-1}y,(1+x)z,$$

as in Section 3. This destroys consistency of the inverse limit R^k with the toric model at the vertex. To correct this, we extend the ray ρ_{23} through the origin, subdividing σ_1 into two cones $\sigma_1'=\langle(0,-1),(1,0)\rangle$, $\sigma_1''=\langle(1,0),(1,1)\rangle$ and define $R^k_{\sigma_1'}$, $R^k_{\sigma_1''}$ to be two copies of $R^k_{\sigma_1}$, connected by the isomorphism

$$\theta_{1''1'}\colon R^k_{\sigma_1''}\to R^k_{\sigma_1'}$$

defined by the same formulas as θ_{23}. Note that $1+x$ is invertible in $R^k_{\sigma_1}$ (because x is nilpotent). Now the inverse limit of the modified system of rings agrees with the toric model at the vertex.

In general, for each singularity of the integral affine manifold, we draw the rays emanating from the singularity in the monodromy invariant direction, together with an attached function f ($1+x$ in the above example) which describes the correction to the toric gluing. These rays pass into the interior of maximal cells as described above. When the ray crosses a wall between two maximal cells, the order (with respect to t) of the function $f-1$ in the corresponding coordinate ring increases. So (at finite order k), after passing through finitely many cells we have $f=1$, that is, the attached automorphism is the identity, and we may terminate the ray.

When two such rays intersect there is a canonical scattering procedure which produces rays with attached functions emanating from the point of intersection such that the associated automorphisms preserve consistency of the gluing. Suppose for simplicity the rays intersect in the interior of a maximal cell of $\mathcal{P}$. The corresponding toric chart of the deformation is the trivial deformation of the torus $(\mathbb{C}^\times)^2$. The two incident rays define infinitesimal automorphisms of this torus, and the induced gluing is consistent iff the two automorphisms commute. This typically does not hold, but the commutator can be factored uniquely into automorphisms of the same form, see [GPS09, Thm. 1.4]. This produces the scattering diagram.

References

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