How to glue a K3 surface from flat pieces, I

Paul Hacking

11/20/09

We describe the Gross–Siebert reconstruction theorem in dimension 2. Roughly speaking, given an integral affine structure with singularities on the 2-sphere, we produce a family of K3 surfaces $\mathcal{X}/\Delta$ over the disc with maximally degenerate special fibre X and smooth general fibre.

1 Input data

The input data is as follows:

(1) B a topological manifold homeomorphic to S^2, a finite set $\Delta \subset B$, and a distinguished atlas of charts for $B^0 := B \setminus \Delta$ with transition functions in $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$. We require that the monodromy of the integral affine structure around each singular point $\delta \in \Delta$ is of type

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$$

for some $m > 0$.

(2) $\mathcal{P}$ a polyhedral subdivision of B into lattice polytopes, such that each singular point δ lies in the interior of an edge of $\mathcal{P}$, with the edge being in the invariant direction. We also assume (for simplicity) that there is at most one singular point on each edge.

(3) ϕ a multivalued convex $\mathcal{P}$-piecewise integral affine function on B.

Remark 1.1. Note that in (1) we require that the translational parts of the transition functions of the integral affine structure are integral. So we have a well defined notion of integral points on B (after we fix a basepoint). This condition is not part of the usual definition of integral affine structure. If the integral affine structure is induced by a Lagrangian torus fibration of a

Kähler manifold (X, ω) over B, the condition corresponds to integrality of the class of the Kähler form ω (equivalently, $[\omega] = c_1(L)$ is the first Chern class of an ample line bundle).

Remark 1.2. Note that the data (1) and (2) can be specified by the set $\mathcal{P}^{\max}$ of maximal lattice polytopes in $\mathcal{P}$, the identifications of faces, and a fan structure at each vertex of $\mathcal{P}$. This was explained last time by Jenia.

Remark 1.3. We explain (3) in more detail. Let $i \colon B^0 \subset B$ denote the inclusion. Let $\operatorname{Aff}_B := i_* \operatorname{Aff}_{B^0}$ be the sheaf of integral affine functions on B. Let $\mathrm{ZPL}_{\mathcal{P}}$ denote the sheaf of continuous functions on B whose restriction to each cell of $\mathcal{P}$ is integral affine. We have an exact sequence of sheaves on B

$$0 \to \operatorname{Aff} \to \mathrm{ZPL}_{\mathcal{P}} \to \mathrm{ZPL}_{\mathcal{P}} / \operatorname{Aff} \to 0$$

Then $\phi \in \Gamma(\mathrm{ZPL}_{\mathcal{P}} / \operatorname{Aff})$ is a global section of the quotient sheaf. Note that ϕ determines a class in $H^1(\operatorname{Aff})$ which vanishes iff ϕ is induced by a single-valued piecewise integral affine function. In more down to earth terms, we may specify ϕ as follows. For $v \in \mathcal{P}$ a vertex, let Σ_v denote the fan given by the cells of $\mathcal{P}$ containing v. For each v, let ϕ_v be a convex piecewise integral linear function on Σ_v, satisfying the following compatibility condition for each edge $e = vw$ of $\mathcal{P}$. Let σ_1, σ_2 be the maximal cells containing e and let l_v be the "bend" of ϕ_v along e, that is, the linear function vanishing on e defined by

$$l_v = \phi_v|_{\sigma_2} - \phi_v|_{\sigma_1}.$$

Then we require that $l_v = l_w$.

Given B and $\mathcal{P}$, for each top dimensional cell $\sigma \in \mathcal{P}^{\max}$ let X_σ be the associated projective toric surface. Let

$$X = \bigcup_{\sigma \in \mathcal{P}^{\max}} X_\sigma$$

be the union of the X_σ glued along corresponding toric strata. We will construct a degenerating family of K3 surfaces with special fibre X.

For $v \in X$ a 0-stratum let U_v denote the open neighbourhood of $v \in X$ given by the complement of the toric strata not containing v. Let C be the cone in $\mathbb{Z}^2 \oplus \mathbb{Z}$ given by the graph of ϕ_v. Then we have local models for the deformation of X near v given by

$$U_v \subset \mathcal{U}_v := \operatorname{Spec} \mathbb{C}[C \cap \mathbb{Z}^3].$$

Here the deformation parameter t corresponds to $(0, 0, 1) \in \mathbb{Z}^3$. We also require:

(4) For each vertex $v \in \mathcal{P}$ and edge e emanating from v, a function $f_{e,v}$ on the patch $X_e \cap U_v \simeq \mathbb{C}^1$ of the component X_e of the double curve (singular locus) of X corresponding to e, satisfying a compatibility condition on each edge and at each vertex.

The compatibility condition along an edge $e = vw$ is as follows: let z be the coordinate on $X_e = \mathbb{P}^1$ such that $v = (z = 0)$ and $w = (z = \infty)$. If e contains a singular point δ let $m > 0$ be the integer occurring in the monodromy matrix, otherwise let $m = 0$. Then

$$f_{e,w} = z^{-m} f_{e,v}.$$

In other words, $f_{e,v}$, $f_{e,w}$ glue to give a section f_e of a line bundle on X_e of degree m. We assume that the m zeroes of f_e lie in the interior of X_e. Let $Z \subset X$ denote the zeroes of the f_e. Then the f_e together with the local deformations $U_v \subset \mathcal{U}_v$ define a *log smooth structure* on $X \setminus Z$. The local model for the deformation along the interior of X_e is

$$(xy = f_e t) \subset \mathbb{C}^2_{x,y} \times \mathbb{C}^\times_z \times \mathbb{C}^1_t.$$

So typically we have m singularities of the total space $\mathcal{X}$ along X_e of type $xy + zt = 0$, located at the zeroes $Z \cap X_e$ of f_e.

Remark 1.4. The use of log structures in the deformation theory of degenerations of Calabi–Yau manifolds was initiated in [KN94], following the earlier work [F83]. Roughly speaking, if X is a normal crossing degeneration of Calabi–Yau manifolds, then the deformation theory of the scheme X is obstructed (the deformation space is singular). However, in favorable circumstances, we can furnish X with a log smooth structure $\mathcal{M}$ such that the deformation theory of the pair $(X, \mathcal{M})$ is unobstructed. For example, on the local normal crossing surface $X = (xy = 0) \subset \mathbb{C}^3$, a smooth log structure corresponds to a choice of generators x', y' of the ideals of the irreducible components X_1, X_2 of X in $\mathcal{O}_X$, up to $x', y' \mapsto ux', u^{-1}y'$ for u a unit in $\mathcal{O}_X$. This is equivalent to the choice of a nowhere vanishing section of a natural line bundle on the double curve D of X, the so called infinitesimal normal bundle $\mathcal{N} = \mathcal{N}_{D/X_1} \otimes \mathcal{N}_{D/X_2}$. In the above setting this is the meaning of the function f_e.

Remark 1.5. WARNING. The integral affine manifold B described above is *not* the integral affine manifold described by Jenia last time (the Gromov Hausdorff limit of the degenerating family $(\mathcal{X}_t, g_t)$ of K3 surfaces). Rather it is related to it by a discrete version of Legendre duality defined using the piecewise linear convex function ϕ, see [GS06, §1.4]. In particular, this duality induces an inclusion reversing bijection between the cells of the polyhedral subdivisions of the two manifolds.

2 The construction

We inductively construct a deformation $\mathcal{X}^k$ of X over $A^k = \mathbb{C}[t]/(t^{k+1})$, a kth order infinitesimal thickening of X, for each $k \geq 0$. Moreover, the $\mathcal{X}^k$ carry compatible ample line bundles. It then follows from the Grothendieck Existence Theorem [G61, 5.4.5, p. 157] that the $\mathcal{X}^k/A^k$ are induced by a projective family $\mathcal{X}/\mathbb{C}[[t]]$ over the formal power series ring. The generic fibre of $\mathcal{X}/\mathbb{C}[[t]]$ is a smooth K3 surface over $\mathbb{C}((t))$, the fraction field of $\mathbb{C}[[t]]$. (We also expect that $\mathcal{X}$ is defined over the ring of convergent power series, in which case we obtain a projective family $\mathcal{X}/\Delta$ over the disc with general fibre a smooth K3, but this has not been proven.)

A key idea of Gross-Siebert is to construct $\mathcal{X}^k$ by gluing thickenings of closed toric strata of X.

Example 2.1. Let $\mathcal{X} = (xy = t) \subset \mathbb{C}^2_{x,y} \times \mathbb{C}^1_t$. Then the induced kth order thickening $\mathcal{X}^k = \operatorname{Spec} R^k$, where

$$R^k = R^k_1 \times_{R^k_{12}} R^k_2$$

(direct product of rings) and

$$R^k_1 = \mathbb{C}[x, y, t]/(xy = t, y^{k+1}),$$
$$R^k_2 = \mathbb{C}[x, y, t]/(xy = t, x^{k+1}),$$
$$R^k_{12} = \mathbb{C}[x, y, t]/(xy = t, x^{k+1}, y^{k+1})$$

are the coordinate rings of the kth order thickenings of the closed toric strata.

Similarly, we can construct any purely toric infinitesimal deformation in this way.

References

[F83] R. Friedman, Global smoothings of varieties with normal crossings, Ann. of Math. (2) 118 (1983), no. 1, 75–114.

[GS06] M. Gross and B. Siebert, Mirror symmetry via logarithmic degeneration data, I, J. Differential Geom. 72 (2006), no. 2, 169-338, and arXiv:math/0309070v3 [math.AG]

[G61] A. Grothendieck, Éléments de géométrie algébrique, III, Étude cohomologique des faisceaux cohérents, I, Inst. Hautes Études Sci. Publ. Math. 11 (1961), 5–167.

[KN94] Y. Kawamata and Y. Namikawa, Logarithmic deformations of normal crossing varieties and smoothing of degenerate Calabi–Yau varieties, Invent. Math. 118 (1994), no. 3, 395–409.