PAUL'S SECOND TALK

Last time we constructed an SYZ fibration for a K3 surface

$$f:\, X \to B \simeq S^2,$$

where fibers of f are special Lagrangians of X. We used the fact that there exists another complex structure on X such that f is an elliptic fibration with respect to this new complex structure.

Input data: $\gamma \in H_2(X, \mathbb{Z})$, $\gamma^2 = 0$, $\gamma \cdot \omega = 0$, γ is primitive. Typically, γ is the class of a fiber of f.

Recall that a general fiber X_b of an elliptic fibration is an elliptic curve (=complex torus) and typically there are 24 singular fibers (copies of a "pinched torus"="nodal cubic"). In less typical cases it could happen that singular fibers are more involved.

Now we discuss monodromy. Take a pinched torus and a nearby smooth fiber. Going along a loop in the base gives an automorphism of a general fiber (as a C^∞ manifold) defined up to homotopy. It can be described as a "Dehn twist": remove a little cylinder from a torus (around a vanishing cycle that gets crushed to a point in a pinched torus). Then the Dehn twist is identity outside the removed torus and twists a little cylinder by $360°$.

On the level of homology, this gives Picard–Lefschetz formula: an automorphism T of $H_1(X_b, \mathbb{Z})$ acts as $T(\alpha) = \alpha + \langle \alpha, \delta \rangle \delta$, where δ is the vanishing cycle. If we choose a standard basis,

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Now we discuss the integral affine structure.

Let $f^0 :\, X^0 \to B^0$ be the restriction to the smooth locus. B^0 is S^2 without 24 points. B^0 has a distinguished atlas of charts with transition functions in $\mathrm{SL}(2, \mathbb{Z})$ semidirect product with $\mathbb{R}^2$. Namely, by a basic theorem of symplectic geometry (action-angles coordinates), near a smooth fiber f has a local model given by the first projection $\mathbb{R}^2 \times (\mathbb{R}^2/\mathbb{Z}^2) \to \mathbb{R}^2$ and $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$, where x_1, x_2 are coordinates on $\mathbb{R}^2$ and y_1, y_2 are coordinates on the torus. This gives integral affine coordinates x_1, x_2 on the base. Intrinsically, $H_1(X_b, \mathbb{Z})$ maps to the cotangent space $T_b^* B$, namely α maps to the functional that takes $v \in T_b B$ to $\int_\alpha \omega(\tilde{v}, -\cdot)$, where $\tilde{v}$ is a lift of v to X. In coordinates this gives 1-forms dx_1, dx_2 on the base B.

B is a singular $\mathbb{Z}$-affine manifold with 24 "focus-focus" singularities. The picture of a hocus-pocus from Kontsevich–Soibelman appeared on the board (with a cut along the monodromy-invariant direction and the picture of how straight lines look like). There was a long and violent discussion of affine structures and what the hell do they mean.

Finally, we are going to address the question of how to see SYZ fibration algebraically.

Consider a degeneration $\mathcal{X} \to \Delta$ of K3 surfaces over the disc. Let $X = \mathcal{X}_0$ be the special fiber. We assume that this degeneration has a "maximally unipotent monodromy", whatever this means.

In fact, we are just going to consider a simple example: Consider a family of quartic surfaces

$$\mathcal{X} = (x_0x_1x_2x_3 + tF_4 = 0) \subset \mathbb{P}^3 \times \Delta.$$

Its special fibre is a tetrahedron (union of coordinate planes). Its general fibre is a smooth K3 (for a good choice of F_4).

$[\omega]$ is just the class of the restriction of the Fubini–Study form of $\mathbb{P}^3$. Put it simply, $[\omega] = c_1(\mathcal{O}(1)|_{\mathcal{X}_t})$.

We want to see SYZ fibration from here. Look at the vertex of X. The local model is $(xyz = t) \subset \mathbb{C}^3 \times \Delta$ It gives a cycle $\gamma = (|x| = |y| = |z| = |t|^{1/3}) \subset X_t$ and a class $[\gamma] \in H_2(\mathcal{X}_t)$, which we are going to call a "vanishing cycle", because it shrinks to a point of the special fiber. We claim that it is an SYZ fiber. Indeed, $\omega \cdot \gamma = 0$ because ω extends to the family but γ shrinks to the point of the special fiber.

Why is $\gamma^2 = 0$? We can map $\mathcal{X}_t$ to $\mathbb{R}^2$ by $|x|, |y|$. Then γ is a fiber (locally near a vertex), so $\gamma^2 = 0$.

What is the fibration here? On each component $\mathbb{P}^2_{\mathbb{C}}$ of X, we have a moment map μ from $\mathbb{P}^2$ to a 2-simplex. In coordinates,

$$\mu(x, y, z) = (|X|^2, |Y|^2, |Z|^2)/(|X|^2 + |Y|^2 + |Z|^2).$$

Topologically, μ is a quotient by the compact torus $S^1 \times S^1 \subset \mathbb{C}^* \times \mathbb{C}^*$.

We can glue these fibrations to get a map from X to the boundary of the tetrahedron. Fibers over the edges and vertices are not tori but circles and points, but the hope is to deform this fibration to an SYZ fibration on $\mathcal{X}_t$.

Compare with a 1-dim case: we have a moment map from $\mathbb{P}^1$ (a sphere) to the interval which can be degenerated to the moment map of the pinched sphere to the interval that sends a pinched point to a point.

Now we want to find 24 singular points. The total space $\mathcal{X}$ has four A_1 singularities along any of the six coordinate lines $l_{ij} = \{x_i = x_j = 0\} \subset X$ at points where $\{F_4 = 0\}$ intersects l_{ij}. Locally, these singularities have equations $x_ix_j + zt = 0$ in $\mathbb{C}^4$, where z is a differential of F_4 at the intersection point.