The Strominger–Yau–Zaslow conjecture

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1 Background

1.1 Kähler metrics

Let X be a complex manifold of dimension n, and M the underlying smooth manifold with (integrable) almost complex structure I. If h is a hermitian metric on X, then

$$h = g - i\omega$$

where $g = \mathrm{Re}(h)$ is a Riemannian metric on M and $\omega = -\mathrm{Im}(h)$ is a real 2-form on M. Note that

$$g(I\cdot, I\cdot) = g, \quad \omega(I\cdot, I\cdot) = \omega$$

and

$$g = \omega(\cdot, I\cdot), \quad \omega = g(I\cdot, \cdot)$$

since h is hermitian. In particular it follows that any two of g, ω, I determine the third.

The metric g is *Kähler* if the 2-form ω is closed, $d\omega = 0$. We say (X, g) is a *Kähler manifold*. So, a Kähler manifold is simultaneously a Riemannian manifold (M, g), a complex manifold $X = (M, I)$, and a symplectic manifold (M, ω), and the structures are compatible.

The metric g is Kähler iff the almost complex structure I is covariantly constant for the Levi-Civita connection ∇^{LC}, that is, $\nabla^{\mathrm{LC}} I = 0$. So if (X, g) is Kähler then the holonomy $\mathrm{Hol}(g)$ of g is contained in the unitary group $U(n)$. Conversely, if (M, g) is a Riemannian manifold of dimension $2n$ and $\mathrm{Hol}(g) \subset U(n)$ then we can define an almost complex structure I on M by parallel transport. Then I is integrable and so (M, I, g) is a Kähler manifold. See [KN69, IX, Thm. 2.5, Thm. 4.3].

1.2 Calabi–Yau metrics

Let X be a Calabi–Yau manifold, that is, a compact complex manifold of dimension n such that there exists a nowhere zero holomorphic n-form Ω. Note that Ω is determined up to a $\mathbb{C}^\times$ factor.

Let $\kappa \in H^2(X, \mathbb{R})$ be a Kähler class, that is, the de Rham cohomology class of the real 2-form ω_0 associated to a Kähler metric g_0 on X.

Theorem 1.1. *(Calabi's conjecture = Yau's theorem) There exists a unique Kähler metric g with associated form ω in the class κ such that g is Ricci flat, that is, the Ricci tensor* $\mathrm{Ric}(g) = 0$.

The condition $\mathrm{Ric}(g) = 0$ is equivalent to Ω being covariantly constant, $\nabla^{\mathrm{LC}}\Omega = 0$ [KN69, IX, Thm. 4.6], [J00, 6.2.4]. So the holonomy $\mathrm{Hol}(g)$ of g is contained in the special unitary group $SU(n)$. Conversely, if (M, g) is a Riemannian manifold of dimension $2n$ such that $\mathrm{Hol}(g) \subset SU(n)$, then we can define an (integrable) almost complex structure I and nowhere zero holomorphic n-form Ω by parallel transport, so (M, I, g) is a Calabi–Yau manifold.

2 Mirror symmetry and the SYZ conjecture

Notation: X is a Calabi–Yau manifold of complex dimension n, Ω is a nowhere zero holomorphic n-form, ω is the Kähler form of a Ricci-flat Kähler metric g on X.

Mirror symmetry is a correspondence between pairs (X, Ω, ω), $(\check{X}, \check{\Omega}, \check{\omega})$ of Calabi–Yau manifolds. Roughly speaking, the complex geometry of X is related to the symplectic geometry of $\check{X}$, and vice versa. The most precise formulation is Kontsevich's homological mirror symmetry conjecture [K95].

Recall that a submanifold $L \subset X$ is *special Lagrangian* if $\omega|_L = 0$ and $\mathrm{Im}\,\Omega|_L = 0$.

Remark 2.1. Note that the special Lagrangian condition depends on the choice of Ω (it is different for $\exp(i\theta)\Omega$). In what follows we will choose Ω to suit our purposes.

Conjecture 2.2. (Strominger–Yau–Zaslow) [SYZ96] There exist dual fibrations $f\colon X \to B$, $\check{f}\colon \check{X} \to B$ of X and $\check{X}$ by special Lagrangian tori.

In more detail: a smooth fibre L of $f\colon X \to B$ is a real n-torus such that $\omega|_L = 0$ and $\mathrm{Im}\,\Omega|_L = 0$. Similarly for $\check{f}$. The fibrations are dual in the following sense. A real torus L is a quotient V/Λ where V is a real vector space and $\Lambda \subset V$ is a lattice, that is, Λ is a free abelian group and

$$\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} V.$$

Note that canonically $\Lambda = H_1(L, \mathbb{Z})$. The dual torus is V^*/Λ^*, where $\Lambda^* = \operatorname{Hom}(\Lambda, \mathbb{Z})$.

Let $B^0 \subset B$ denote the locus of smooth fibres of f and $f^0 \colon X^0 \to B^0$ the restriction. Let Λ be the local system of lattices on B^0 with fibres $H_1(X_b, \mathbb{Z})$, $b \in B^0$. (That is, Λ is a locally trivial family of free abelian groups parametrised by B^0.) Then, as a map of smooth manifolds, f^0 can be identified with

$$\Lambda \otimes_{\mathbb{Z}} \mathbb{R}/\Lambda \to B^0.$$

The *dual fibration* is

$$\Lambda^* \otimes_{\mathbb{Z}} \mathbb{R}/\Lambda^* \to B^0,$$

and the conjecture asserts that this is isomorphic to $\check{f}^0$. Equivalently, in terms of the local systems, $\check{\Lambda}$ is isomorphic to Λ^*.

Remark 2.3. Given $f \colon X \to B$, we construct the dual fibration as a map of smooth manifolds. So we only describe the topology of the mirror manifold. The construction of the mirror as a Calabi–Yau manifold is in general very difficult, but there is a straightforward construction in the semi-flat case (meaning, the fibres of f are flat), see [G09, §3].

Remark 2.4. It is now expected that the SYZ conjecture is only valid in the limit as the Calabi–Yau X approaches a boundary point of the moduli space, see [G09] for more details.

3 Examples of SYZ fibrations

3.1 Elliptic curve

Let $X = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$, $\Omega = dz = dx + idy$, $\omega = h dx \wedge dy = h\frac{i}{2} dz \wedge d\bar{z}$, some $h \in \mathbb{R}, h > 0$. (Note: In this case, the Ricci flat metric is the flat metric, with constant Kähler form.)

A special Lagrangian submanifold $L \subset X$ is a submanifold of real dimension 1 such that $\omega|_L = 0$ (this is automatic because ω is a 2-form) and $\operatorname{Im}\Omega|_L = dy|_L = 0$. So $L = (y = c) \subset X$. The map

$$f \colon X = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \to B = \mathbb{R}/\operatorname{Im}(\tau)\mathbb{Z} \simeq S^1, \quad z \mapsto \operatorname{Im} z = y$$

is a fibration by special Lagrangian tori $L = \mathbb{R}/\mathbb{Z} \simeq S^1$.

3.2 Complex torus

This is similar. We have $X = V/\Lambda$ where V is a complex vector space of dimension n and $\Lambda \subset V$ is a lattice, that is,

$$\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} V$$

is an isomorphism of real vector spaces. (So X is diffeomorphic to the real $2n$-torus $(S^1)^{2n}$.) In coordinates $V = \mathbb{C}^g$, $\Omega = dz_1 \wedge \cdots \wedge dz_n$, and $\omega = \frac{i}{2}\sum h_{ij} dz_i \wedge d\bar{z}_j$ where $h = (h_{ij})$ is a hermitian matrix (corresponding to a flat hermitian metric on X).

A Lagrangian submanifold $L \subset X$ is given by a Lagrangian subspace $W \subset V$ of the real vector space V for the symplectic form ω such that $W \cap \Lambda$ has full rank n. (Such a subspace W exists if for example the class of ω in $H^2(X, \mathbb{R})$ is integral, equivalently $\omega = c_1(L)$ for some holomorphic line bundle L on X.) Then $L = W/W \cap \Lambda$ is a Lagrangian torus. Pick a $\mathbb{Z}$-basis of $W \cap \Lambda$. This is a $\mathbb{C}$-basis of V (because $\omega(v, Iv) = g(v, v) > 0$ for $v \neq 0$, so $W \cap IW = 0$ for Lagrangian W). Let $z_1, \ldots, z_n$ be the corresponding coordinates, so $X = \mathbb{C}^n/\mathbb{Z}^n + \tau\mathbb{Z}^n$ for some $n \times n$ complex matrix τ. Let $\Omega = dz_1 \wedge \cdots \wedge dz_n$. Then the map

$$f \colon X = \mathbb{C}^n/\mathbb{Z}^n + \tau\mathbb{Z}^n \to B = \mathbb{R}^n/\operatorname{Im}(\tau)\mathbb{Z}^n \simeq (S^1)^n, \quad z \mapsto \operatorname{Im}(z)$$

is a fibration by special Lagrangian tori $L = \mathbb{R}^n/\mathbb{Z}^n \simeq (S^1)^n$.

3.3 K3 surface

A *K3 surface* is a simply connected Calabi–Yau manifold of dimension 2. These are the only Calabi–Yaus in dimension 2 besides complex tori. Kodaira proved that all K3 surfaces are diffeomorphic. An example is a quartic hypersurface in $\mathbb{P}^3_{\mathbb{C}}$.

Let (X, g) be a K3 surface with Ricci flat Kähler metric g. Then the holonomy $\operatorname{Hol}(g)$ of g is equal to $SU(2)$. We have $SU(2) = Sp(1)$, the group of unit quaternions acting on $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ by left multiplication. So parallel transport defines complex structures I, J, K satisfying the quaternion relations

$$I^2 = J^2 = K^2 = IJK = -1$$

such that the metric g is Kähler with respect to each complex structure. In fact we have a 2-sphere of complex structures $aI + bJ + cK$, $a^2 + b^2 + c^2 = 1$. We say that (X, g) is a *hyperkähler* manifold.

As we explain below, a special Lagrangian fibration for X is the same as a holomorphic fibration with respect to one of the other complex structures. It is relatively easy to construct a holomorphic fibration (using algebraic geometry). In this way we obtain a special Lagrangian fibration.

Let ω be the Kähler form of g and Ω a nowhere zero holomorphic 2-form. At a point $p \in X$ we can choose complex coordinates dz_1, dz_2 on the tangent space T_pX such that

$$\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$$

and $\Omega = \lambda dz_1 \wedge dz_2$ for some $\lambda \in \mathbb{R}, \lambda > 0$. We normalise Ω (replacing Ω by $\lambda^{-1}\Omega$) so that $\Omega = dz_1 \wedge dz_2$ at p (equivalently, globally we have $\Omega \wedge \bar{\Omega} = 2\omega^2$). We identify $T_pX = \mathbb{C}^2$ with $\mathbb{H}$ via

$$\mathbb{C}^2 \to \mathbb{H}, \quad e_1, ie_1, e_2, ie_2 \mapsto 1, i, j, k.$$

Let I, J, K be the induced complex structures. Then, the forms $(\omega, \operatorname{Re}\Omega, \operatorname{Im}\Omega)$ are the Kähler forms $(\omega_I, \omega_J, \omega_K)$ for the complex structures I, J, K. (Exercise for the reader. Note it is enough to check at p and recall $\omega_I = g(I\cdot, \cdot)$ etc.) Similarly, the form $\Omega' := \operatorname{Im}\Omega + i\omega$ is a nowhere zero holomorphic 2-form for the complex structure J. So, writing $\omega' = \omega_J$, we have

$$(\omega', \operatorname{Re}\Omega', \operatorname{Im}\Omega') = (\operatorname{Re}\Omega, \operatorname{Im}\Omega, \omega).$$

Let $X = (M, I)$, $X' = (M, J)$ denote the complex manifolds with complex structures I, J. Then a submanifold $L \subset X$ is special Lagrangian iff $L \subset X'$ is a complex submanifold. Indeed, by definition, $L \subset X$ is special Lagrangian if $\omega|_L = 0$ and $\operatorname{Im}\Omega|_L = 0$, equivalently, $\Omega'|_L = (\operatorname{Im}\Omega + i\omega)|_L = 0$. For $q \in L$ a point, one can check that $\Omega'|_{T_qL} = 0$ iff the tangent space $T_qL \subset T_qX'$ is a complex subspace. So $\Omega'|_L = 0$ iff $L \subset X'$ is a complex submanifold, as required. This fact was observed in the original paper of Harvey and Lawson on special Lagrangian manifolds [HL82, V.3].

Now suppose given $\gamma \in H_2(X, \mathbb{Z})$ a 2-cycle such that $\gamma^2 = 0$, $\gamma \cdot [\omega] = 0$, and γ is primitive (indivisible). Our aim is to construct a special Lagrangian fibration $f\colon X \to B$ such that $\gamma = [L]$ is the class of a fibre L, a real 2-torus. (Note: If $\gamma = [L]$ is the class of a fibre of a fibration $X \to B$ then, since L is homologous to a distinct fibre L',

$$\gamma^2 = [L]^2 = [L] \cdot [L'] = [L \cap L'] = 0.$$

The second condition $\gamma \cdot [\omega] = 0$ is implied by the Lagrangian condition $\omega|_L = 0$.)

Replacing Ω by $\exp(i\theta)\cdot\Omega$ we may assume that $[\mathrm{Re}\,\Omega]\cdot\gamma > 0$ and $[\mathrm{Im}\,\Omega]\cdot\gamma = 0$. Now, for the complex structure $X' = (M, J)$ (with notation as above) we have

$$[\Omega']\cdot\gamma = [\mathrm{Im}\,\Omega + i\omega]\cdot\gamma = 0, \quad [\omega']\cdot\gamma = [\mathrm{Re}\,\Omega]\cdot\gamma > 0.$$

The first equality shows that the 2-cycle γ is of type $(1,1)$. Indeed, we have

$$H^2(X', \mathbb{C}) = H^{2,0}\oplus H^{1,1}\oplus H^{0,2} = \mathbb{C}\cdot[\Omega']\oplus H^{1,1}\oplus\mathbb{C}\cdot[\bar{\Omega}']$$

and $H^{1,1}$ is the orthogonal complement of $H^{2,0}\oplus H^{0,2}$. So, by the Lefschetz theorem on $(1,1)$-classes, $\gamma = c_1(L)$ is the first Chern class of a holomorphic line bundle M on X'.

We now show (by a standard argument) that M has a nonzero global holomorphic section $s\in\Gamma(X', M)$. (Then, roughly speaking, the zero locus $(s=0)\subset X'$ is a holomorphic submanifold and so a special Lagrangian submanifold $L\subset X$.) If Y is a compact complex surface and L is a holomorphic line bundle on Y then we have the *Riemann–Roch formula*

$$\chi(L) = \frac{1}{12}(c_1^2 + c_2) + \frac{1}{2}c_1(L)(c_1(L) + c_1).$$

Here c_1, c_2 are the Chern classes of the holomorphic tangent bundle of Y, and $\chi(L)$ is the holomorphic Euler-Poincaré characteristic of the line bundle L:

$$\chi(L) := \sum(-1)^i \dim H^i(Y, L)$$

where $H^i(Y, L)$ is the Cech cohomology of the sheaf of holomorphic sections of L (in particular $H^0(Y, L)$ is the space of global sections). For a K3 surface, $c_1 = 0$ and $c_2 = 24$ so the Riemann Roch formula becomes

$$\chi(L) = 2 + \frac{1}{2}c_1(L)^2.$$

Moreover, by Serre duality (and using the Calabi–Yau condition)

$$\chi(L) \le \dim H^0(L) + \dim H^2(L) = \dim H^0(L) + \dim H^0(L^*).$$

In our case $Y = X'$, $L = M$ we have $c_1(M)^2 = \gamma^2 = 0$, so

$$\dim H^0(M) + \dim H^0(M^*) \ge 2. \tag{1}$$

Also, by construction, $c_1(M)\cdot[\omega'] = \gamma\cdot[\omega'] > 0$. Now if $s\in H^0(L)$ is a global holomorphic section of a line bundle L on a complex surface Y

then the zero locus $D = (s = 0) \subset Y$ is a union of analytic curves with homology class $[D]$ Poincaré dual to $c_1(L)$. Moreover, if ω is a Kähler form then $c_1(L) \cdot [\omega] = \int_D \omega$ is the volume of D with respect to the Kähler metric. In our case $c_1(M^*) \cdot [\omega'] = -c_1(M) \cdot [\omega'] < 0$ so M^* has no nonzero global holomorphic sections, $H^0(M^*) = 0$. So $\dim H^0(M) \geq 2$ by (1).

Typically, we have $\dim H^0(M) = 2$, and the global sections of M define a holomorphic map $f \colon X' \to \mathbb{P}^1_{\mathbb{C}}$, with general fibre a smooth curve of genus 1. In general, one can show that there is a holomorphic map f with the same properties, but the zero locus of a global section of M will be a union of fibres of f together with some (-2)-curves (a *(-2)-curve* C is a copy of $\mathbb{P}^1_{\mathbb{C}}$ with $C^2 = -2$). See for example [H08, Thm. 12.6(3)]. So, in the typical case, the class of the fibre is equal to γ, but this can fail in general.

Finally the holomorphic elliptic fibration $f \colon X' \to \mathbb{P}^1_{\mathbb{C}}$ constructed above is a special Lagrangian fibration $f \colon X \to B = S^2$ for the original complex structure.

References

[G09] M. Gross, The SYZ conjecture: From torus fibrations to degenerations, in Algebraic geometry - Seattle 2005, Proc. Sympos. Pure Math. 80, Part 1, p. 149–192, AMS 2009, and arXiv:0802.3407v1

[H08] P. Hacking, Compact complex surfaces, Lecture notes from a graduate course, 2008, available at http://www.math.umass.edu/ hacking/surfaces.pdf

[HL82] R. Harvey, H.B. Lawson, Calibrated geometries, Acta Math. 148 (1982), 47–157.

[J00] D. Joyce, Compact manifolds with special holonomy, O.U.P. 2000.

[K95] M. Kontsevich, Homological algebra of mirror symmetry, in Proceedings of the International Congress of Mathematicians, Zurich 1994, Vol. 1, 120–139, Birkhauser 1995, and arXiv:alg-geom/9411018v1

[KN69] S. Kobayashi and K. Nomizu, Foundations of differential geometry, II, Wiley 1969.

[SYZ96] A. Strominger, S-T. Yau and E. Zaslow, Mirror symmetry is T-duality, Nucl. Phys. B 479 (1996), no. 1–2, 243–259, and arXiv:hep-th/9606040v2