Moduli of Special Lagrangian Submanifolds – III

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We continue using last time's notation: (X, ω, Ω) is a Calabi-Yau n-fold (with Kähler form ω and holomorphic volume form Ω) and $L \subset X$ is a special Lagrangian submanifold. That is,

$$\dim_{\mathbb{R}} L = n, \ \omega|_L = 0, Im\Omega|_L = 0$$

Next, $\mathcal{M}_L$ denotes McLean's moduli space of special Lagrangian deformations of L. It is a smooth manifold of (real) dimension $b_1(L) = \dim H^1(L, \mathbb{R})$, with a marked point, $[L] \in \mathcal{M}_L$.

1 Affine Structure on the Moduli Space

Assume $U \subset \mathcal{M}_L$ is a simply-connected, connected open subset (or, equivalently, assume we are working on the universal cover of $\mathcal{M}_L$). We construct a chart

$$\Phi : U \subset \mathcal{M}_L \to H^1(L, \mathbb{R})$$

as follows. Let $\mathfrak{U} \simeq U \times L$ be the local universal family. That is, by definition, a manifold, $\mathfrak{U}$, equipped with two maps, Γ and p (a submersion)

$$\begin{array}{ccc} & \mathfrak{U} & \\ {}^{p}\swarrow & & \searrow^{\Gamma} \\ U \subset \mathcal{M}_L & & X \end{array}$$

such that $\Gamma(p^{-1}([N])) = N \subset X, \ \forall [N] \in U$. Fix a basepoint, $[Y] \in U$. If U contains the marked point, $[L]$, we can take $Y = L$.

Idea: Construct first $d\Phi$.

That is, we construct a closed 1-form on U. It will be exact, as U is simply-connected, and we define $\Phi : U \to \mathbb{R}$ to be its potential, which will be well-defined up to an additive constant.

Here is the construction of the closed 1-form. Since U is contractible, the homology of $\mathfrak{U}$ is isomorphic to the homology of a single fibre, in particular, $H_1(\mathfrak{U},\mathbb{Z}) \simeq H_1(Y,\mathbb{Z})$. So choose one such isomorphism, which can be thought of as a trivialisation of the local system with fibre $H_1(Y,\mathbb{Z})$. Then for an element, $A \in H_1(Y,\mathbb{Z})$, we have a fibration of circles $\mathfrak{U}_A \subset \mathfrak{U}$, $p_A : \mathfrak{U}_A \to U$, such that $[p_A^{-1}([N])] = A \in H_1(Y,\mathbb{Z}), \forall [N] \in U$. I.e., the homology class of each fibre is the chosen element A.

Then the closed 1-form is $\zeta_A = p_{A*}(\Gamma^*\omega)$: we integrate $\Gamma^*(\omega)$ over the fibres of p_A. At a point $[N] \in U$,

$$(\zeta_A)_{[N]} = \int_{p_A^{-1}([N])} \Gamma^*\omega$$

Since pull-back of a closed form is closed, and integration along the fibre sends closed forms to closed forms, ζ_A is closed, and we have a function $\Phi_A : U \to \mathbb{R}$, well-defined up to an additive constant and such that $\zeta_A = d\Phi_A$.

Now if $\{A_i\}$ is a basis of $H_1(Y,\mathbb{Z})$ we can put together the different maps Φ_{A_i} to define a map $U \to \mathbb{R}^{b_1(L)}$. More canonically:

$$\Phi = \sum_i \Phi_{A_i}\alpha_i : U \to H^1(Y,\mathbb{R})$$

where $\{\alpha_i\}$ is the dual basis in $H^1(Y,\mathbb{R})$. By construction, this map has the following property. Given a vector ξ in the tangent space to U at $[Y]$, with a lift, $\tilde{\xi}$ to a tangent vector field to X, we have

$$d\Phi_{[Y]}(\xi) = \left[\Gamma^*\omega(\tilde{\xi},\ )|_Y\right] \in H^1(Y,\mathbb{R}),\ \xi \in T_{\mathcal{M}_L,[Y]}.$$

This is independent of the lift, and is, in fact, McLean's identification $T_{\mathcal{M}_L,[Y]} \simeq H^1(Y,\mathbb{R})$. In particular, by the inverse function theorem, this implies that Φ is a local diffeomorphism near $[Y]$.
Clarifications:

- The tangent space to the moduli space at the point $[Y]$ is contained in the space of global sections of the normal bundle $N_{Y/X}$ of Y in X, which fits in the exact sequence

$$0 \longrightarrow T_Y \longrightarrow T_X|_Y \longrightarrow N_{Y/X} \longrightarrow 0$$

- For any $[N] \in U$, $\Gamma^*\omega|_{p^{-1}([N])} = 0$ – this is the Lagrangian condition. This guarantees that the above expression for $d\Phi$ is well-defined, i.e., the result depends on ξ and not on its lift $\tilde{\xi}$.

- If $\{x_1, \dots x_n\}$ are local coordinates on L and $\{y_1, \dots y_m\}$ are coordinates on U, so that (y_i, x_j) are local coordinates on $\mathfrak{U}$ for which $p(y, x) = y$, then

$$\Gamma^* \omega = \sum_{ij} a_{ij} dx_i \wedge dy_j + \sum_{ij} b_{ij} dy_i \wedge dy_j.$$

We can repeat all of the above with $Im\Omega$, and obtain a map

$$\Psi : U \to H^{n-1}(L, \mathbb{R}) \simeq H^1(L, \mathbb{R})^\vee$$

where the last isomorphism is given by Poincaré duality (L is orientable).

2 Special Lagrangian Embedding of the Moduli Space

Last time we introduced a canonical symplectic form, ω_{can}, and a canonical (indefinite) metric, g_{can}, on $V \oplus V^\vee$, where V is a vector space. We also considered submanifolds, M, of $V \times V^\vee$ which are Lagrangian for ω_{can} and transverse to the two projections. We saw that in such a situation one can use any of the two projections as a coordinate chart, that such an M is given as the graph of a real-valued function, ϕ, on V (or a real-valued function, ψ, on $V^\vee$). The two functions are related by Legendre transform and the restriction of the canonical metric is the Hessian of ϕ (or ψ), after we pull it back to the chart V (or $V^\vee$). We also discussed the notion of such a manifold, M, being "special", which meant that a linear combination of volume forms on V and $V^\vee$ restricts to zero on M.

Now we shall state some of the main results of Hitchin, which are going to bring us to the discussion from last time, with $V = H^1(L, \mathbb{R})$. To make things easier to read, assume $[L] \in U$ is our base-point (otherwise, replace L with Y).

Theorem 2.1 ([Hit97]) *The map $F = (\Phi, \Psi) : U \subset \mathcal{M}_L \to H^1(L, \mathbb{R}) \times H^{n-1}(L, \mathbb{R})$ embeds U as a Lagrangian submanifold for the canonical form ω_{can}.*

Next, the moduli space $\mathcal{M}_L$ has a natural Riemannian metric. This is the Hodge metric: $[\alpha], [\beta] \mapsto \int_Y \alpha \wedge \star\beta$, where $[\alpha], [\beta] \in H^1(Y, \mathbb{R}) \simeq T_{\mathcal{M},[Y]}$.

Theorem 2.2 ([Hit97]) *The natural metric on $\mathcal{M}_L$ coincides with $F^* g_{can}$.*

In particular, there are two functions, $\phi : H^1(L, \mathbb{R}) \to \mathbb{R}$, and $\psi : H^{n-1}(L, \mathbb{R}) \to \mathbb{R}$, and

$$F^* g_{can} = \mathrm{Hess}(\phi) = \sum_{ij} \frac{\partial^2 \phi}{\partial u_i \partial u_j} du_i du_j, \ u_i = \Phi_{A_i}$$

Moreover, there are two torus fibrations over $\mathcal{M}_L$, whose fibre over $[N] \in \mathcal{M}_L$ is, respectively, $H^1(N, \mathbb{R})/H^1(N, \mathbb{Z})$ and $H^{n-1}(N, \mathbb{R})/H^{n-1}(N, \mathbb{Z})$.

Theorem 2.3 ([Hit97]) *The map F embeds $U \subset \mathcal{M}_L$ as a special Lagrangian submanifold of $H^1(L, \mathbb{R}) \times H^1(L, \mathbb{R})^\vee$ if and only if one of the following equivalent conditions holds:*

- $\det$ *$Hess(\phi) = const$*
- *The volume of the torus $H^1(N, \mathbb{R})/H^1(N, \mathbb{Z})$ is independent of the point $[N] \in \mathcal{M}_L$.*
- $\det$ *$Hess(\psi) = const$*
- *The volume of the torus $H^{n-1}(N, \mathbb{R})/H^{n-1}(N, \mathbb{Z})$ is independent of the point $[N] \in \mathcal{M}_L$.*

3 Complexified Moduli Space and its Kähler and Calabi-Yau Structure

For applications to mirror symmetry one needs to consider the complexified Kähler moduli, i.e., incorporate the B-field. We construct a complexification, $\mathcal{M}^{cx}$, of $\mathcal{M}_L$ as follows. Consider $U^{cx} = U \times H^1(Y, \mathbb{R})/H^1(Y, \mathbb{Z})$, with U and Y as before. The tangent space at the point $[Y]$ is $H^1(Y, \mathbb{R}) \times H^1(Y, \mathbb{R}) \simeq H^1(Y, \mathbb{R}) \otimes \mathbb{C}$. Thus each tangent space has a complex structure, I, and a hermitian metric obtained by extending the Riemannian metric.

Theorem 3.1 ([Hit97]) *The complex structure I is integrable. The hermitian structure on each tangent space gives rise to Kähler metric on $\mathcal{M}^{cx}$. The function ϕ is a Kähler potential.*

It is actually easy to describe a set of complex coordinates: the basis $\{\alpha_j\}$ determines coordinates, x_j, on the universal cover of the torus, and $z_j = u_j + ix_j$ are complex coordinates. One can also write an obvious holomorphic volume form, namely, $\widetilde{\Omega} = dz_1 \wedge dz_2 \wedge \ldots$.

Theorem 3.2 ([Hit97]) *The above Kähler metric and the holomorphic volume form $\widetilde{\Omega}$ define a Calabi-Yau structure on $\mathcal{M}^{cx}$ if and only if one of the above equivalent conditions holds. That is, if and only if the embedding, F, is special Lagrangian.*

It turns out that this condition is satisfied in the case when X is hyperkähler.

4 Brief Remarks on the Hyperkähler Case

This section has the aim of whetting the reader's appetite.

Suppose the Calabi-Yau n-fold X is hyperkähler. Then the special Lagrangian submanifold L is *complex* Lagrangian for some other complex structure, and $\mathcal{M}_L$ can be identified with the moduli space of complex Lagrangian submanifolds. In particular, the vector space $V = H^1(L, \mathbb{R})$ has now a symplectic structure, ω_0, and hence $V \times V^\vee$ has a natural symplectic structure $\Omega_1 = \omega_0 \oplus \omega_0^\vee$.

Theorem 4.1 ([Hit99]) *The embedding $F : U \subset \mathcal{M}_L \to H^1(L, \mathbb{R}) \times H^1(L, \mathbb{R})^\vee$ is Lagrangian for* both *symplectic structures, ω_{can} and $\Omega_1 = \omega_0 \oplus \omega_0^\vee$.*

Next, recall ([Fre99], [BCOV94]) that a *special Kähler* manifold, M, is a complex manifold, (M, I) endowed with:

- Kähler metric g with Kähler form ω
- Flat symplectic connection ∇ (i.e, $\nabla^2 = 0$, $\nabla\omega = 0$)
- $d^\nabla(id) = 0$ (i.e, ∇ is torsion-free)
- $d^\nabla(I) = 0$.

Caution: The last condition *does not* say $\nabla I = 0$. The bundle $End(T_M) = T_M \otimes T_M^\vee$ can be also thought of as $\mathcal{A}^1(T_M)$, the bundle of 1-forms with values in T_M. Given a connection on T_M, we obtain two different connections on $End(T_M)$: one is $\nabla \otimes 1 + 1 \otimes \nabla^\vee$, while the other is $d^\nabla : \mathcal{A}^1(T_M) \to \mathcal{A}^2(T_M)$.

Moreover, in the interesting examples ∇ is *not* the Levi-Civita connection – if it is, we just get a flat Kähler manifold.

Theorem 4.2 ([Hit99]) *A submanifold $M \subset V \times V^\vee$ which is Lagrangian for both Ω_1 and ω_{can} and is transversal to the two projections has a special (pseudo) Kähler metric induced from g_{can}. Conversely, any special (pseudo) Kähler structure arises locally in this way.*

If identify $V \times V^\vee \simeq T^\vee\mathbb{C}^m$, then a manifold M as above is given by the graph of $d\mathcal{F}$, $\mathcal{F} : \mathbb{C}^m \to \mathbb{C}$ a *holomorphic* function, called the *holomorphic prepotential*. Suppose the pseudo-metric $g_{can}|_M$ is an actual metric. Then $\mathrm{ImHess}(\mathcal{F}) > 0$, so we have a map from M to the Siegel upper-half space. This is the classifying map for the family of tori (polarised abelian varieties) carried by M.

The cotangent bundle to a special Kähler manifold is hyperkähler ([Fre99], [CFG89]). In particular, the metric on the complexified moduli space of special Lagrangians is hyperkähler (and hence, Calabi-Yau) when X is hyperkähler. This is the starting point of the "semi-flat" mirror symmetry.

By multiplying the complex symplectic form by a complex number of modulus 1, one obtains a family of flat connections on M, parametrised by S^1. This corresponds to a Higgs bundle on $T_M \oplus T_M^\vee$, i.e., an (integrable) section of $End(T_M \oplus T_M^\vee) \otimes T_M^\vee$. The only non-zero component of the Higgs field is a section of $Sym^3(T_M)$ – the Donagi-Markman cubic. The cubic is given by the third derivatives of $\mathcal{F}$, i.e., by the derivative of the period map.

The Hermite-Yang-Mills equations for the Higgs bundle are the tt^*-equations introduced by [CV91], and studied by C.Hertling, C.Sabbah and others. This data can also be repackaged as certain variation of Hodge structures of weight one.

References

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