Special Lagrangian submanifolds of Calabi–Yau manifolds II

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Recall from last time: (X, ω, Ω) denotes a Calabi-Yau manifold of dimension n, with Kähler form ω and holomorphic volume form Ω. We say $L \subset X$ is *special Lagrangian* (SLag) if $\omega|_L = 0$, $\operatorname{Im}\Omega|_L = 0$, and $\dim_{\mathbb{R}} L = n = \frac{1}{2}\dim_{\mathbb{R}} X$. MacLean's theorem: If $L \subset X$ is compact then the moduli space $\mathcal{M}_L$ of special Lagrangian deformations of L is smooth at $[L] \in \mathcal{M}_L$ and the tangent space $T_{\mathcal{M}_L,[L]}$ is isomorphic to $H^1(L, \mathbb{R})$ via the map

$$\xi \in H^0(L, N_{L/X}) \to [\omega(\xi, \cdot)] \in H^1(L, \mathbb{R}).$$

Here $N_{L/X}$ denotes the normal bundle of L in X.

The moduli space $\mathcal{M}_L$ has a rich differential geometric structure. We now describe a local model for $\mathcal{M}_L$.

Let V be a vector space over $\mathbb{R}$. Then $V \oplus V^*$ has a canonical symplectic structure given by

$$\omega_{\mathrm{can}}((u, \alpha), (v, \beta)) = \beta(u) - \alpha(v).$$

Moreover $V \oplus V^*$ has a *pseudo-metric*, that is, a nondegenerate symmetric bilinear form $g \in \mathrm{Sym}^2(V \oplus V^*)$, given by

$$g((u, \alpha), (u, \alpha)) = \alpha(u)$$

or, equivalently,

$$g((u, \alpha), (v, \beta)) = \frac{1}{2}(\alpha(v) + \beta(u)).$$

(Warning: g is *not* definite — it has signature $(\dim V, \dim V)$.) Notice that the subspaces V, V^* are Lagrangian for ω_{can} and isotropic for g.

We will consider Lagrangian submanifolds $M \subset V \oplus V^*$ satisfying additional conditions.

Recall: If N is a smooth manifold, the cotangent bundle T^*N has a canonical symplectic structure. Indeed, at a point $p \in T^*N$ over $q \in N$, the tangent space T_pN is identified with $T_qM \oplus T_q^*M$, and we can define ω_{can} at p by the same formula as above. If $\alpha \colon N \to T^*N$ is a section (that is, a 1-form on N), then its image $M \subset T^*N$ is Lagrangian iff the 1-form α is closed, $d\alpha = 0$ (Exercise for the reader). So, if M is Lagrangian, then locally α is exact, $\alpha = d\phi$ (and the same is true globally if $H^1(N, \mathbb{R}) = 0$).

Now consider a Lagrangian submanifold $M \subset V \times V^*$ such the two projections define diffeomorphisms $M \to V$, $M \to V^*$. Note that $V \times V^*$ is a cotangent bundle in two ways: $V \times V^* = T^*V = T^*(V^*)$. So we obtain two smooth functions $\phi \colon V \to \mathbb{R}$, $\psi \colon V^* \to \mathbb{R}$, such that $M = \operatorname{im}(d\phi)$, $M = \operatorname{im}(d\psi)$. The derivative of $d\phi \colon V \to V \times V^*$ is the map

$$T_V \to d\phi^* T_{V \times V^*}, \quad v \mapsto v \oplus \operatorname{Hess}(\phi)(v, \cdot),$$

where $\operatorname{Hess}(\phi)$ denotes the Hessian of ϕ computed with respect to linear coordinates on V. That is, for $x_1, \dots, x_n$ linear coordinates on V,

$$\frac{\partial}{\partial x_i} \mapsto \frac{\partial}{\partial x_i} \oplus \sum_j \frac{\partial^2 \phi}{\partial x_i \partial x_j} dx_j.$$

Similarly for ψ.

We compute that $(d\phi)^* g = \operatorname{Hess}(\phi)$. Note that $\operatorname{Hess}(\phi)$ is nondegenerate: this follows from the formula for the derivative of $d\phi$ above and our assumption that $M = \operatorname{im}(d\phi)$ is transversal to the fibres of the projection $V \times V^* \to V^*$.

In summary

$$M = \{(v, d\phi_v) \mid v \in V\} = \{(d\psi_\alpha, \alpha) \mid \alpha \in V^*\} \subset V \times V^*,$$

and $\operatorname{Hess}(\phi)$ and $\operatorname{Hess}(\psi)$ are nondegenerate. So ϕ and ψ are related by the *Legendre transform*.

What is the analogue of the special Lagrangian condition for $M \subset V \times V^*$? We fix elements in $\wedge^{\dim V} V$ and $\wedge^{\dim V} V^*$ and require that a linear combination of these vanishes on M. (These forms correspond to the real and imaginary parts of a holomorphic volume form in the Calabi-Yau case. But we have not defined a complex structure on $V \times V^*$ at present.)

1 Affine structures

Let V be a real vector space. The group $\operatorname{Aff}(V)$ of affine linear transformations of V is a semidirect product $V \rtimes \operatorname{GL}(V)$, where the pair (b, A)

corresponds to the map $x \mapsto Ax + b$. In particular we have the exact sequence

$$0 \to V \to \mathrm{Aff}(V) \to \mathrm{GL}(V) \to 1.$$

We also have the homomorphism

$$\mathrm{Aff}(V) \to \mathrm{PGL}(V \oplus \mathbb{R}), \quad (b, A) \mapsto \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}.$$

Note that $\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Ax + b \\ 1 \end{pmatrix}$.

Definition 1.1. If M is a topological manifold or smooth manifold of dimension d, an *affine structure* on M is given by an atlas of distinguished charts $\phi_i \colon U_i \to V_i \subset \mathbb{R}^d$ for M such that the transition functions $\phi_j \circ \phi_i^{-1}$ are locally given by affine linear transformations of $\mathbb{R}^d$.

Remark 1.2. If M is a topological manifold, an affine structure on M defines a smooth structure on M. If M is a smooth manifold, an affine structure on M is required to be compatible with the smooth structure, that is, we require the charts of the affine structure to be smooth charts for M.

Remark 1.3. If M is a smooth manifold, the data of an affine structure on M is equivalent to a flat, torsion-free connection on the tangent bundle of M.

Example 1.4. The real torus $M = \mathbb{R}^d / \mathbb{Z}^d$ inherits an affine structure from $\mathbb{R}^d$.

Example 1.5. (Focus–focus singularity) We will describe an affine structure on the topological manifold $M = \mathbb{R}^2 \setminus \{(0,0)\}$ (which does *not* extend to an affine structure on $\mathbb{R}^2$). We take the open covering of M given by

$$U_1 = \mathbb{R}^2 \setminus [0, \infty) \times \{0\},$$

$$U_2 = \mathbb{R}^2 \setminus (-\infty, 0] \times \{0\},$$

and charts

$$\phi_1 \colon U_1 \to \mathbb{R}^2, \quad (x, y) \mapsto (x, y),$$

$$\phi_2 \colon U_2 \to \mathbb{R}^2, \quad (x, y) \mapsto (x + \max(y, 0), y).$$

(Note: the induced smooth structure is *not* the standard smooth structure on $\mathbb{R}^2 \setminus \{0, 0\}$.) The locally constant tangent vectors for the affine structure have monodromy $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}$ along an anticlockwise loop about the origin.

We can construct two affine structures on the moduli space $\mathcal{M}_L$ as follows. Let $U \subset \mathcal{M}_L$ be a connected, simply connected neighbourhood of $[L]$. We will construct charts

$$\phi \colon U \to H^1(L, \mathbb{R}), \quad \psi \colon U \to H^1(L, \mathbb{R})^* = H^{n-1}(L, \mathbb{R})$$

for the two affine structures. For $[N] \in U$ let $\gamma \colon [0,1] \to U$ be a path from $[L]$ to $[N]$ (unique up to isotopy by our assumption on U). The path γ lifts to a map $\Gamma \colon L \times [0,1] \to X$ (the restriction of the universal family of Lagrangian submanifolds over $\mathcal{M}_L$). We have $\Gamma^* \omega = \tilde{\theta} \wedge dt$ where $\tilde{\theta}$ is a 1-form on $L \times [0,1]$ which restricts to a closed form on each fibre $L \times \{t\}$. The chart ϕ is defined by

$$\phi \colon U \to H^1(L, \mathbb{R}), \quad [N] \mapsto \int_0^1 \tilde{\theta} \, dt.$$

(To be continued.)