Special Lagrangian submanifolds of Calabi–Yau manifolds

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9/25/09

Notation: Ω^p= holomorphic p-forms, $\mathcal{A}^p$= smooth p-forms, $\mathcal{A}^{p,q}$= smooth (p,q)-forms.

Definition 0.1. A Calabi–Yau n-fold is a Kähler manifold (X,ω) with a nowhere vanishing global holomorphic n-form $\Omega \in \Gamma(X, \Omega^n_X)$ (a *holomorphic volume form*) such that $|\Omega|_\omega$ is constant (equivalently, Ω is covariantly constant).

Recall: A Kähler manifold (X,ω) of complex dimension n is

(1) A complex manifold $X = (M, I)$, where M is a smooth manifold of real dimension $2n$ and $I \in \Gamma(M, \operatorname{End} T_M)$, $I^2 = -1$, is a complex structure, and

(2) A symplectic manifold (M, ω), where $\omega \in \Gamma(M, \wedge^2 T^*_M)$ is nondegenerate (equivalently, ω^n is nowhere vanishing), and $d\omega = 0$, such that

(3) ω and I are compatible: $\omega(I\cdot, I\cdot) = \omega$ (ω is of type $(1,1)$), and $g := \omega(\cdot, I\cdot)$ is positive definite.

We have $T_M \subset T_M \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0} \oplus T^{0,1}$, the eigenbundles with eigenvalues $i, -i$ for I. Thus

$$\wedge^2 T_M \subset \wedge^2 (T_M \otimes \mathbb{C}) = \wedge^2 T^{1,0} \oplus T^{1,0} \otimes T^{0,1} \oplus \wedge^2 T^{0,1}.$$

Write $\Lambda^{1,1} = T^{1,0} \otimes T^{0,1}$. The Kähler form ω is a section of $\wedge^2 T_M \cap \Lambda^{1,1}$.

We have an isomorphism $T_M \xrightarrow{\sim} T^{1,0}$, $u \mapsto \frac{1}{2}[u - i \cdot Iu]$. (In coordinates $\frac{\partial}{\partial x} \mapsto \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) = \frac{\partial}{\partial z}$.)

We have a Riemannian metric $g := \omega(\cdot, I\cdot)$ on T_M and a Hermitian metric $h := g - i\omega$ on $T^{1,0}$

We require that $|\Omega|_\omega$ is constant, that is, $\Omega \wedge \overline{\Omega}/(\omega^n/n!)$ is constant.

Example 0.2. Let $M = \mathbb{R}^2$, $I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\omega = dx \wedge dy = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then

$$g = \omega(\cdot, I\cdot) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Example 0.3. Let $E = \mathbb{C}/\Lambda$ be an elliptic curve, $\omega = \frac{i}{2} dz \wedge d\bar{z}$, and $\Omega = dz$. Similarly for complex tori in arbitrary dimension. These are the only explicitly known compact examples.

Remark 0.4.

(1) If X is compact, then Ω is determined up to a $\mathbb{C}^\times$ factor (because $\Omega_X^n \simeq \mathcal{O}_X$ and $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$). In general this does not hold, but there are often preferred choices (e.g. for an affine hypersurface we can use residues).

(2) Yau's theorem (Calabi's conjecture): Let (X, ω) be a compact Kähler manifold and Ω a holomorphic volume form. Then there exists a Kähler metric with Kähler form ω' such that $[\omega'] = [\omega] \in H^2(X, \mathbb{R})$ and (X, ω', Ω) is Calabi–Yau.

Example 0.5. Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $n + 2$ in complex projective $(n+1)$-space. Then $\Omega_X^n \simeq \mathcal{O}_X$ by the adjunction formula.

Example 0.6. Let $X = (z_1^2 + \cdots + z_{n+1}^2 = 1) \subset \mathbb{C}^{n+1}$. Then X is diffeomorphic to T^*S^n, the cotangent bundle of the n-sphere. The real locus of X is $S^n \subset \mathbb{R}^{n+1}$. The complex manifold X is Kähler (because $\mathbb{C}^{n+1}$ is so). Let $\Omega = \sum (-1)^i z_i dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_{n+1}$, and $\omega_0 = \frac{i}{2} \partial\bar{\partial}|z|^2$ the standard Kähler form. Then $\omega = \frac{i}{2} \partial\bar{\partial} f |z|^2$ gives a Calabi–Yau metric for some smooth function f (which can be computed explicitly).

Example 0.7. The complex manifold $T^*\mathbb{P}^n$ is Calabi–Yau (Calabi showed this explicitly). In general, if Y is Kähler, then there exists an open neighbourhood $U \subset T^*Y$ of the zero section $Y \subset T^*Y$ that is Calabi–Yau.

Definition 0.8. If (X, ω, Ω) is a Calabi–Yau n-fold, a real submanifold $L \subset X$ is *special Lagrangian* if

(1) L is Lagrangian for ω, that is, $\omega|_L = 0$ and $\dim_{\mathbb{R}} L = \frac{1}{2} \dim_{\mathbb{R}} X = \dim_{\mathbb{C}} X$, and

(2) $\operatorname{Im} \Omega|_L = 0$.

Remark 0.9. Condition (2) is equivalent to requiring that $\operatorname{Re} \Omega|_L$ is the volume form on L. In other words, L is a calibrated submanifold of X for the calibration $\operatorname{Re} \Omega$ in the sense of Harvey and Lawson [?]. In particular, L minimizes volume in its homology class.

Remark 0.10. A Calabi–Yau n-fold comprises the data $(X, \omega, \Omega) = (M, I, \omega, \Omega)$. In fact, Ω determines I as follows. Suppose given $\Omega \in \Gamma(M, \mathcal{A}^n)$ such that $\Omega \wedge \overline{\Omega}$ is nowhere vanishing, Ω is locally decomposable, and $d\Omega = 0$. Then define $\mathcal{A}^{1,0}$ as the kernel of the map

$$\Omega \wedge (\cdot) \colon \mathcal{A}^1 \to \mathcal{A}^{n+1}.$$

There are few examples of special Lagrangian submanifolds of Calabi–Yau manifolds. The main sources of examples are

(1) Explicit metrics on non-compact Calabi–Yaus.

(2) Real locus of Calabi–Yaus with real structure.

(3) Complex Lagrangian submanifolds in hyperkähler manifolds.

We will discuss these in more detail next time.

Theorem 0.11. *(McLean) Let X be a Calabi–Yau n-fold and $L \subset X$ a compact special Lagrangian. Then the moduli space $\mathcal{M}_L$ of special Lagrangian deformations of L in M is smooth of dimension $b_1(L) = \dim H^1(L, \mathbb{R})$.*

More specifically, a global section $\xi \in \Gamma(L, N_{L/X})$ of the normal bundle of L in X defines a special Lagrangian deformation of L iff $\omega(\xi, \cdot) = \theta$ is harmonic.

Indeed, ξ determines a flow $\phi_t \colon L \to L_t$, $\phi_t = \exp(t\xi)$. We want $\phi_t^* \omega|_{L_t} = 0$ and $\phi_t^* \operatorname{Im} \Omega|_{L_t} = 0$. Infinitesimally

$$\mathcal{L}_\xi \omega = 0 = (i_\xi d + d i_\xi)\omega,$$

that is, $d(i_\xi \omega) = d\theta = 0$. Similarly $d(i_\xi \operatorname{Im} \Omega) = 0$. A local computation shows that $i_\xi(\operatorname{Im} \Omega) = - * (i_\xi \omega)$. Thus $d\theta = 0$ and $d(*\theta) = 0$, that is, θ is harmonic.

References

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