Overview of classical mirror symmetry

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(1) Physics

(2) Quintic 3-fold

(3) Math

1

String theory is a $N = 2$ superconformal field theory (SCFT) which models elementary particles by loops propagating in spacetime.

super — supersymmetry.

$N = 2$ — number of symmetries.

field theory — instance of a quantum field theory.

conformal — conformal structure on the worldsheet (surface in space time traced out by motion of string), coming from minimization of energy.

Some live on

(1) Calabi–Yau 3-fold

(2) Point. Here have *superpotential* $F(\Phi_1, \ldots, \Phi_n)$, a weighted homogeneous polynomial in variables $\Phi_1, \ldots, \Phi_n$, and *vacuum states*

$$\mathbb{C}[\Phi_1, \ldots, \Phi_n]/(\{\partial F/\partial \Phi_i\}),$$

an Artinian ring. This is a *Landau–Ginzburg* theory.

Also have orbifold Landau–Ginzburg theories — fields are invariant fields of Landau–Ginzburg theory under finite group action.

In the Calabi–Yau case, the data of a Calabi–Yau 3-fold V together with a complexified Kähler class ω gives rise to the SCFT. We have a $U(1) \times U(1)$ action on a Hilbert space. The (p, q)-eigenspace is

$$H^q(V, \wedge^p T) \simeq H^q(V, \Omega^{3-p}).$$

(Note that $\wedge^3\Omega$ is trivial because V is Calabi–Yau.) The $(-p,q)$-eigenspace is $H^q(V,\Omega^p)$.

The SCFT moduli space is (locally) a product of complex and Kähler moduli. There is a formal symmetry of the SCFT — a sign change in $U(1)\times U(1)$ (meaning, compose with inverse map on one factor) gives new $N=2$ SCFT.

Expectation: New theory comes from another Calabi–Yau 3-fold V^0, the so called “mirror”.

$$H^{p,q}(V) = H^q(V,\Omega^p) \simeq H^q(V^0,\Omega^{3-p}) = H^{3-p,q}(V^0)$$

The Hodge diamond of a Calabi–Yau 3-fold has the form

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{2,2} & & 0 & \\ 1 & & h^{1,2} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

(Recall that a Calabi–Yau n-fold V is a complex manifold such that Ω^n is trivial and $h^i(\mathcal{O}_V)=0$ for $0<i<n$.) So we see that the Hodge diamonds of V and V^0 are related by reflection in the line $y=-x$. This is the origin of the term “mirror symmetry”.

Also $H^{2,1}(V) = H^1(V,\Omega^2) \simeq H^1(V,T)$, the space of deformations of complex structure, and $H^{1,1}(V)$ is the space of deformations of (complexified) Kähler structure. Thus the Kähler moduli of V is identified with the complex moduli of V^0.

There is a “physics proof” of Mirror symmetry in the Landau–Ginzburg case. There is a deformation argument that applies to some Calabi–Yau 3-folds (including the quintic) to deduce mirror symmetry from the Landau–Ginzburg case. (Roughly, we consider deforming the Kähler class to a class defining a contraction to a space of lower dimension. The Landau–Ginzburg case corresponds to the totally degenerate case where the Calabi-Yau is contracted to a point.)

2

Let V be the quintic 3-fold, that is, the locus of zeroes of a homogeneous polynomial of degree 5 in complex projective 4-space.

The mirror of V is V'/G, where V' is a quintic 3-fold and G is a group of order 125 acting linearly on $\mathbb{P}^4$. Here we choose the quintic V' so that it is invariant under the action of G. This gives a 1-parameter family of choices. Remark: Strictly speaking, we should pass to a crepant resolution of V'/G here, that is, a resolution of singularities $W \to V'/G$ such that Ω^3_W is trivial.

The isomorphism of SCFT's induces an isomorphism of moduli spaces of SCFT's, the *mirror map*, which identifies the 3-*point functions* on these moduli spaces. We now describe the 3-point functions on the Kähler and complex moduli spaces.

(1) (Kähler). Given (V, ω), for $\omega_1, \omega_2, \omega_3 \in H^{1,1}(V)$ we have

$$\langle \omega_1, \omega_2, \omega_3 \rangle = \int_V \omega_1 \wedge \omega_2 \wedge \omega_3 + \sum_{\beta \in H_2(V)} n_\beta \int_\beta \omega_1 \int_\beta \omega_2 \int_\beta \omega_3 \frac{q}{1-q}.$$

where n_β is the "number of rational curves" in class β (*instantons*), and $q = \exp(2\pi i \int_\beta \omega)$. The terms involving q are *non-perturbative worldsheet corrections*.

(2) (Complex). Given V^0, for $\theta_1, \theta_2, \theta_3 \in H^1(V^0, T) \simeq H^{2,1}(V^0)$, we have

$$\langle \theta_1, \theta_2, \theta_3 \rangle \simeq \int \Omega \wedge (\nabla_{\theta_3} \nabla_{\theta_2} \nabla_{\theta_1} \Omega)$$

where ∇ is the Gauss–Manin connection. This 3-point function can be computed by classical methods (Hodge theory). There are no non-perturbative corrections. (Roughly, this is because the integral $\int_\beta \theta_i$ mirror to $\int_\beta \omega_i$ is zero by dimensions.)

For the quintic 3-fold V and its mirror V^0: The Kähler moduli of V is 1-dimensional, $q = \exp(2\pi i \int_\beta \omega)$, where β is the positive generator of $H_2(V) \simeq \mathbb{Z}$. The point $q = 0$ is a "boundary point" of Kähler moduli (the limit as the imaginary part of the complexified Kähler form ω tends to ∞, the *large volume limit*). The complex moduli of V^0 is also 1-dimensional, with parameter $x = \psi^{-5}$ and universal family the quotient of the family of invariant quintics

$$x_1^5 + \cdots + x_5^5 + \psi x_1 \cdots x_5 = 0$$

by the finite group G. The boundary point $x = 0$ has *maximally unipotent monodromy*. That is, for a small loop around $x = 0$ in the moduli space, parallel transport induces an isometry T of the middle cohomology $H^3(W, \mathbb{Z})$

of the fibre W of the universal family over the basepoint (the *monodromy*), and, possibly after a basechange, $(T-I)^4 = 0$ but $(T-I)^3 \neq 0$. This gives rise to natural coordinates on the complex moduli space near $x = 0$.

The mirror map $q = q(x)$ is given by

$$q = -x \cdot \exp\left(\frac{5}{y_0(x)} \cdot \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5} \left(\sum_{j=n+1}^{5n} \frac{1}{j}\right)(-1)^n x^n\right)$$

where

$$y_0(x) = \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5}(-1)^n x^n.$$

Now the identification of 3-point functions under the mirror map gives

$$5 + \sum_{d=1}^{\infty} n_d \cdot d^3 \cdot \frac{q^d}{1-q^d} = \frac{5}{(1+5^5 x) y_0(x)^2} \cdot \left(\frac{q}{x}\frac{dx}{dq}\right)^3,$$

and we find $n_d = 2875, 609250, 317206375, \ldots$ for $d = 1, 2, 3, \ldots$

2.1 Complications

(1) It is not known that n_d is finite for all d (but this has been conjectured by Clemens). We do know that n_d is finite for $d \leq 10$.

(2) We can take an almost complex structure J on (V, ω) and count pseudoholomorphic curves. Then for generic choice of J, all the curves should be smooth. Vainsenscher observed that, in the algebraic case, $n_5 = 229305888887625$, and 17601000 of these curves are 6-nodal. For each of these singular curves and a choice of node we can define an étale double cover by a reducible curve with two rational components meeting in a single node. These covers contribute to n_{10}, and it is now known that these account exactly for the discrepancy between n_{10} and the actual number of rational curves of degree 10 on V. Kontsevich introduced the notion of stable maps to give a precise definition of the n_β.

3

Mirror symmetry has had a huge impact on mathematics. We list a few examples.

In 1994 Batyrev used reflexive polytopes to “explain” mirror symmetry. In particular mirror symmetry for the quintic fits naturally in this framework. This led to an emphasis on toric varieties.

Chen-Ruan, McDuff, and Kontsevich-Manin introduced Gromov-Witten invariants. This led to the theories of stable maps, quantum cohomology, virtual fundamental classes, orbifold cohomology, and a renewed interest in algebraic stacks.

In 1994 Kontsevich proposed a homological version of mirror symmetry. If V,V^0 is a mirror pair, we expect that the derived category of coherent sheaves on V^0 (“complex”) is equivalent to the “derived category of Lagrangian submanifolds of V” (“Kähler”). A naive consequence is the following: If $p \in V^0$ is a point we have the structure sheaf $\mathcal{O}_{\{p\}}$ of the closed subvariety $\{p\} \subset V^0$, a skyscraper sheaf at p with stalk $\mathbb{C}$. According to the homological mirror symmetry proposal, this sheaf should correspond to a Lagrangian submanifold L_p of V. Now, tautologically, V^0 is the moduli space of points $p \in V^0$, hence V^0 is the moduli space of the $L_p \subset V$. Thus we obtain a construction of the mirror V^0 in terms of V. This is related to T-duality (Strominger–Yau–Zaslow), Fourier–Mukai transforms on K3 surfaces, etc.