MIRROR SYMMETRY AND CLUSTER ALGEBRAS

PAUL HACKING AND SEAN KEEL

Abstract. We explain our proof, joint with Mark Gross and Maxim Kontsevich, of conjectures of Fomin–Zelevinsky and Fock–Goncharov on canonical bases of cluster algebras. We interpret a cluster algebra as the ring of global functions on a non-compact Calabi-Yau variety obtained from a toric variety by a blow up construction. We describe a canonical basis of a cluster algebra determined by tropical counts of holomorphic discs on the mirror variety, using the algebraic approach to the Strominger–Yau–Zaslow conjecture due to Gross and Siebert.

1. Introduction

We say a complex variety U is *log Calabi–Yau* if it admits a smooth projective compactification X with normal crossing boundary[1] D such that $K_X + D = 0$, that is, there is a nowhere zero holomorphic top form Ω on U with simple poles along D. The mirror symmetry phenomenon for compact Calabi–Yau manifolds extends to the case of log Calabi–Yau varieties, see [A09] and §4. We say U has *maximal boundary* if D has a 0-stratum (a point cut out by $n = \dim_{\mathbb{C}} X$ branches of D) and is *positive* if D is the support of an ample divisor[2] (so in particular U is affine) . The *tropical set* $U^{\mathrm{trop}}(\mathbb{R})$ of U is the cone over the dual complex of D; we write $U^{\mathrm{trop}}(\mathbb{Z})$ for its integral points.

Conjecture 1.1. *Mirror symmetry defines an involution on the set of positive log Calabi–Yau varieties with maximal boundary. For a mirror pair U and V, there is a basis ϑ_q, $q \in U^{\mathrm{trop}}(\mathbb{Z})$ of $H^0(V, \mathcal{O}_V)$ parametrized by the integral points of the tropical set of U, which is canonically determined up to multiplication by scalars $\lambda_q \in \mathbb{C}^\times$, $q \in U^{\mathrm{trop}}(\mathbb{Z})$.*

For example, if $U \simeq (\mathbb{C}^\times)^n$ is an algebraic torus, then the mirror V is the dual algebraic torus, and the canonical basis is given by the characters of V (up to scalars), which may be characterized as the units of $H^0(V, \mathcal{O}_V)$. The set of characters of V corresponds under the duality to the set of 1-parameter subgroups of U, which is

The authors were partially supported by NSF grants DMS-1601065 (P.H.) and DMS-1561632 (S.K.).

[1] More generally, (X, D) has $\mathbb{Q}$-factorial divisorial log terminal singularities ([KM98], Definition 2.37).

[2] More generally, D is the support of a big and nef divisor.

identified with $U^{\mathrm{trop}}(\mathbb{Z})$. The heuristic justification for Conjecture 1.1 coming from mirror symmetry is explained in §4.

Cluster algebras were introduced by Fomin and Zelevinsky as a tool to understand the constructions of canonical bases in representation theory by Lusztig [FZ02]. In §2 we review a description of cluster varieties in terms of toric and birational geometry [GHK15b]. Roughly speaking, a cluster variety is a log Calabi–Yau variety U which carries a non-degenerate holomorphic 2-form and is obtained from a toric variety $\bar{X}$ by blowing up codimension 2 centers in the toric boundary and removing its strict transform. The existence of the 2-form greatly constrains the possible centers and accounts for the combinatorial description of cluster varieties. The mutations of cluster theory are given by elementary transformations of $\mathbb{P}^1$-bundles linking different toric models.

For a cluster variety U, Fock and Goncharov defined a dual cluster variety V by an explicit combinatorial recipe, and stated the analogue of Conjecture 1.1 in this setting [FG06]. In §5 we use an algebraic version of the Strominger–Yau–Zaslow mirror construction [SYZ96] to explain that if U is positive then V should be its mirror. (If U is not positive, then we expect that the mirror of U is an open analytic subset of V and the Fock–Goncharov conjecture is false, cf. [GHK15b].) Under a hypothesis on U related to positivity, our construction proves Conjecture 1.1 in this case. In particular, the hypothesis is satisfied in the case of the mirror of the base affine space G/N for $G = \mathrm{SL}_m$ studied by Fomin and Zelevinsky, so we obtain canonical bases of representations of G by the Borel–Weil–Bott theorem.

Acknowledgements. This paper is based on joint work with Mark Gross and Maxim Kontsevich [GHK15a], [GHK15b], [GHKK14]. The algebraic approach to the Strominger–Yau–Zaslow conjecture in §5 is due to Gross and Siebert [GS11].

2. Log Calabi–Yau varieties

Definition 2.1. A *log Calabi–Yau pair* (X, D) is a smooth complex projective variety X together with a reduced normal crossing divisor $D \subset X$ such that $K_X + D = 0$. Thus there is a nowhere zero holomorphic top form Ω on $U = X \setminus D$ (a *holomorphic volume form*) such that Ω has a simple pole along each component of D, uniquely determined up to multiplication by a nonzero scalar.

We say a variety U is *log Calabi–Yau* if there exists a log Calabi–Yau pair (X, D) such that $U = X \setminus D$.

Remark 2.2. Note that if U is a smooth variety and (X, D) is any normal crossing compactification of U, the subspace $H^0(\Omega^p_X(\log D)) \subset H^0(\Omega^p_U)$ for each $p \geq 0$ is independent of (X, D) [D71]. In particular, if U is a log Calabi–Yau variety then there is a holomorphic volume form Ω on U such that Ω has at worst a simple pole along each boundary divisor of any normal crossing compactification (X, D), uniquely determined up to a scalar.

Definition 2.3. We say a log Calabi–Yau pair (X, D) has *maximal boundary* if the boundary D has a 0-stratum, that is, a point $p \in D \subset Y$ cut out by $n = \dim_{\mathbb{C}} X$ analytic branches of the divisor D, so that we have a local analytic isomorphism

$$(p \in D \subset X) \simeq (0 \in (z_1 \cdots z_n = 0) \subset \mathbb{C}^n).$$

We say a log Calabi–Yau variety U has maximal boundary if some (equivalently, any [dFKX12], Proposition 11) log Calabi–Yau compactification (X, D) of U has maximal boundary.

Definition 2.4. We say a log Calabi–Yau variety U is *positive* if there exists a log Calabi–Yau compactification $(X, D = \sum D_i)$ and positive integers a_i such that $A = \sum a_i D_i$ is ample. In particular, $U = X \setminus D$ is affine.

Example 2.5. The algebraic torus $(\mathbb{C}^\times)^n$ is a log Calabi-Yau variety, with holomorphic volume form $\Omega = \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$. Any toric compactification (X, D) satisfies $K_X + D = 0$.

Example 2.6 (Non-toric blow up)**.** Let (X, D) be a log Calabi–Yau pair and $Z \subset X$ a smooth subvariety of codimension 2 which is contained in a unique component of D and meets the other components transversely. Let $\pi \colon \tilde{X} \to X$ be the blow up of Z and $\tilde{D} \subset \tilde{X}$ the strict transform of D. Then the pair $(\tilde{X}, \tilde{D})$ is log Calabi–Yau.

Definition 2.7. A *toric model* of a log Calabi–Yau variety U is a log Calabi–Yau compactification (X, D) of U together with a birational morphism $f \colon (X, D) \to (\bar{X}, \bar{D})$ such that $(\bar{X}, \bar{D})$ is a toric variety together with its toric boundary and f is a composition of non-toric blow ups as in Example 2.6.

Remark 2.8. In the description of a log Calabi–Yau variety U in terms of a toric model $(X, D) \to (\bar{X}, \bar{D})$, one can replace the projective toric variety $\bar{X}$ with the toric open subset $\bar{X}' \subset \bar{X}$ given by the union of the big torus $T \subset \bar{X}$ and the open T-orbit in each boundary divisor containing the center of one of the blow ups. Thus the fan Σ' of $\bar{X}'$ is the subset of the fan of $\bar{X}$ consisting of $\{0\}$ and the rays corresponding to these boundary divisors. Cf. [GHK15b], §3.2.

Example 2.9. Let $X = \mathbb{P}^2$ and $D = Q + L$ the union of a smooth conic Q and a line L meeting transversely. We describe a toric model of the log Calabi–Yau surface $U = X \setminus D$. First, choose a point $p \in Q \cap L$ and blow up at p. Second, blow up at the intersection point of the exceptional divisor and the strict transform of Q. Let $\tilde{X} \to X$

be the composition of the two blow ups and $\tilde{D} = \pi^{-1}D$. The strict transform in $\tilde{X}$ of the tangent line to Q at p is a (-1)-curve E meeting $\tilde{D}$ transversely at a point of the exceptional divisor of the second blow up. Contracting E yields a toric pair $(\bar{X}, \bar{D})$ with $\bar{X} \simeq \mathbb{F}_2 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$.

Proposition 2.10. *(*[GHK15a]*, Proposition 1.3) Let U be a log Calabi-Yau surface with maximal boundary. Then U has a toric model.*

The proof is an exercise in the birational geometry of surfaces.

Example 2.11. We describe an example of a log Calabi–Yau 3-fold with maximal boundary which is irrational. In particular, it does not have a toric model.

Smooth quartic 3-folds are irrational [IM71]. Let $X \subset \mathbb{P}^4$ be a smooth quartic 3-fold with hyperplane section

$$D = (X_1^4 + X_2^4 + X_3^4 + X_1X_2X_3X_4 = 0) \subset \mathbb{P}^3.$$

The surface D has a unique singular point $p = (0:0:0:1)$. The minimal resolution of D is obtained by blowing up p and has exceptional locus a triangle of (-3)-curves (in particular, $p \in D$ is a *cusp singularity*).

We describe a sequence of blow ups $\pi\colon \tilde{X} \to X$ such that the inverse image of D is a normal crossing divisor. First blow up the point p. The inverse image D^1 of D consists of two components, the exceptional divisor $E \simeq \mathbb{P}^2$ and the strict transform D' of D (which is its minimal resolution). The intersection $E \cap D'$ is the exceptional locus of $D' \to D$, a triangle of smooth rational curves. Blow up the nodes of the triangle. For each of the exceptional divisors $E_i \simeq \mathbb{P}^2$, the strict transforms of E and D meet E_i in a common line $l_i \subset E_i$. Finally blow up each line l_i to obtain $\tilde{X}$.

Define the divisor $\tilde{D} \subset \tilde{X}$ to be the union of the strict transforms of D, E, and the exceptional divisors over the lines l_i. Then $K_{\tilde{X}} + \tilde{D} = 0$. The variety $\tilde{U} = \tilde{X} \setminus \tilde{D}$ is an irrational log Calabi–Yau 3-fold with maximal boundary.

Remark 2.12. On the other hand, if (X, D) is a log Calabi-Yau pair with maximal boundary then X is rationally connected [KX16], (18).

3. Cluster varieties

3.1. **Birational geometric description of cluster varieties.**

Definition 3.1. We say a log Calabi–Yau variety U is a *cluster variety* if

(1) There is a non-degenerate holomorphic 2-form σ on U such that for some (equivalently, any [D71]) normal crossing compactification (X, D) we have $\sigma \in H^0(\Omega^2_X(\log D))$.

(2) U has a toric model.

Remark 3.2. It is customary in the theory of cluster algebras to allow the 2-form to be degenerate. However, the non-degenerate case is the essential one (cf. §5.4).

Example 3.3. Every log Calabi–Yau surface with maximal boundary is a cluster variety by Proposition 2.10.

Suppose U is a cluster variety with 2-form σ and toric model $f\colon (X,D) \to (\bar{X},\bar{D})$. Then $\sigma = f^*\bar{\sigma}$ for some $\bar{\sigma} \in H^0(\Omega^2_{\bar{X}}(\log \bar{D}))$ by Hartogs' theorem. The sheaf $\Omega_{\bar{X}}(\log \bar{D})$ is freely generated by $\frac{dz_1}{z_1}, \ldots, \frac{dz_n}{z_n}$, where $z_1, \ldots, z_n$ is a basis of characters for the algebraic torus $T = \bar{X} \setminus \bar{D} \simeq (\mathbb{C}^\times)^n_{z_1,\ldots,z_n}$. See [F93], Proposition, p. 87. Thus $\bar{\sigma} = \frac{1}{2}\sum a_{ij}\frac{dz_i}{z_i} \wedge \frac{dz_j}{z_j}$ for some non-degenerate skew matrix (a_{ij}). The following lemma is left as an exercise.

Lemma 3.4. *Let (X,D) be a normal crossing pair, $Z \subset X$ a smooth codimension 2 subvariety contained in a unique component F of D and meeting the remaining components transversely, $\pi\colon \tilde{X} \to X$ the blow up with center Z, and $\tilde{D}$ the strict transform of D. Let $\sigma \in H^0(\Omega^2_X(\log D))$ be a log 2-form on X. Let $D_F = (D-F)|_F$ and let $\operatorname{Res}_F\colon \Omega^2_X(\log D) \to \Omega_F(\log D_F)$ be the Poincaré residue map. Then σ lifts to a log 2-form on $(\tilde{X},\tilde{D})$ if and only if $(\operatorname{Res}_F \sigma)|_Z = 0$.*

Now let $Z \subset F \subset \bar{D}$ be the center of one of the blow ups for the toric model f. We may choose coordinates on T so that $F \setminus \bar{D}_F = (z_1 = 0) \subset \mathbb{A}^1_{z_1} \times (\mathbb{C}^\times)^{n-1}_{z_2,\ldots,z_n}$, then $\operatorname{Res}_F(\bar{\sigma}) = \sum_{j>1} a_{1j}\frac{dz_j}{z_j}$. Using the lemma, we deduce that

(1) $\operatorname{Res}_F(\bar{\sigma})$ is proportional to an integral log 1-form, that is, $\operatorname{Res}_F(\bar{\sigma}) = \nu \cdot \sum_{j>1} b_j \frac{dz_j}{z_j}$ for some $\nu \in \mathbb{C}^\times$ and pairwise coprime $b_j \in \mathbb{Z}$. Equivalently, writing $\chi\colon T \to \mathbb{C}^\times$ for the character $\prod z_j^{b_j}$, $\operatorname{Res}_F(\bar{\sigma}) = \nu\frac{d\chi}{\chi}$.
(2) $Z = F \cap \overline{(\chi = \lambda)}$ for some $\lambda \in \mathbb{C}^\times$.

Remark 3.5. Note that, after a change of coordinates, we may assume $\chi = z_2$. Thus, if f is a single blow up, then $U = X \setminus D$ decomposes as a product $U' \times (\mathbb{C}^\times)^{n-2}_{z_3,\ldots,z_n}$. In general U does not globally decompose as a product.

Conversely, any sequence of non-toric blow ups of $(\bar{X},\bar{D},\bar{\sigma})$ with the above properties yields a cluster variety.

3.2. **Atlas of tori and elementary transformations.** The usual description of a cluster variety U is as follows: The variety U is the union of a countable collection of open subsets T_α (indexed by *seeds* α) which are copies of a fixed algebraic torus

$T \simeq (\mathbb{C}^\times)^n$. The glueing maps between the open subsets are compositions of *mutations*, given (for some choice of coordinates $z_1, \ldots, z_n$ on T) by the formula

$$\mu \colon T \dashrightarrow T', \quad (z_1, z_2, \ldots, z_n) \mapsto (z_1(1+cz_2)^{-1}, z_2, z_3, \ldots, z_n)$$

for some $c \in \mathbb{C}^\times$.

There is the following geometric interpretation. First, note that a toric model $f \colon (X, D) \to (\bar{X}, \bar{D})$ determines an open inclusion of the torus $T = \bar{X} \setminus \bar{D}$ in $U = X \setminus D$ via f^{-1}. This is the origin of the torus charts of a cluster variety: seeds correspond to toric models. Second, mutations correspond to birational transformations between toric models given by elementary transformations of $\mathbb{P}^1$-bundles. In the above notation, let $(\bar{X}, \bar{D})$ be the toric partial compactification of T given by $\mathbb{P}^1_{z_1} \times (\mathbb{C}^\times)^{n-1}_{z_2,\ldots,z_n}$. Let $Z = (z_1 = 0) \cap (1 + cz_2 = 0)$, let $\pi \colon X \to \bar{X}$ be the blow up of Z, and D the strict transform of $\bar{D}$. Write $H = (1 + cz_2 = 0) \subset \bar{X}$ and let $H' \subset X$ be its strict transform. Then H' can be blown down, yielding a morphism $\pi' \colon (X, D) \to (\bar{X}', \bar{D}')$ to a second toric pair such that $\bar{X}'$ is also isomorphic to $\mathbb{P}^1_{z_1} \times (\mathbb{C}^\times)^{n-1}_{z_2,\ldots,z_n}$ and π' is the blow up of $Z' = (z_1 = \infty) \cap (1 + cz_2 = 0)$. The birational map $(\bar{X}, \bar{D}) \dashrightarrow (\bar{X}', \bar{D}')$ is an *elementary transformation* of $\mathbb{P}^1$-bundles over $(\mathbb{C}^\times)^{n-1}$. Writing $U = X \setminus D$, $T = \bar{X} \setminus \bar{D}$, and $T' = \bar{X}' \setminus \bar{D}'$, we have $T \cup T' = U \setminus W$ where $W \simeq Z \simeq Z'$ is the intersection of the exceptional divisors of π and π'. The mutation $\mu \colon T \dashrightarrow T'$ is the restriction of the birational map $\bar{X} \dashrightarrow \bar{X}'$.

Remark 3.6. In general the union of tori in the original definition of a cluster variety is an open subset of a cluster variety in the sense of Definition 3.1 with complement of codimension at least 2 provided that the parameters $\lambda \in \mathbb{C}^\times$ are very general [GHK15b], Theorem 3.9. For simplicity we will always assume that this is the case.

3.3. **Combinatorial data for toric model of a cluster variety.** We can give an intrinsic description of the data for a toric model of a cluster variety as follows. (We use the notation of [F93] for toric varieties.) Let $T = \bar{X} \setminus \bar{D}$ be the big torus acting on $\bar{X}$. Let $N = H_1(T, \mathbb{Z}) = \mathrm{Hom}(\mathbb{C}^\times, T)$ be the lattice of 1-parameter subgroups of T and $M = N^* = \mathrm{Hom}(T, \mathbb{C}^\times)$ the dual lattice of characters of T. Then $T = N \otimes_{\mathbb{Z}} \mathbb{C}^\times$. We sometimes use the multiplicative notation z^m for characters.

Let U be a cluster variety and $f \colon (X, D) \to (\bar{X}, \bar{D})$ be a toric model for U. With notation as in §3.1, let $Z = F \cap \overline{(\chi = \lambda)}$ be the center of one of the blow ups. The toric boundary divisor $F \subset \bar{D}$ corresponds to a primitive vector $v \in N$ (the generator of the corresponding ray of the fan of $\bar{X}$). The character χ corresponds to a primitive element $m \in v^\perp \subset M$ (primitive because Z is assumed irreducible). The 2-form $\bar{\sigma}$ lies in $H^0(\Omega^2_{\bar{X}}(\log \bar{D})) = \wedge^2 M_{\mathbb{C}}$. The condition $\mathrm{Res}_F(\bar{\sigma}) = \nu \cdot \frac{d\chi}{\chi}$ is equivalent to

$\bar{\sigma}(v,\cdot) = \nu \cdot m$. The associated mutation is given by

$$\mu = \mu_{(m,v)} \colon T \dashrightarrow T, \quad \mu^*(z^{m'}) = z^{m'}(1 + cz^m)^{-\langle m',v\rangle} \tag{3.7}$$

where $c = -1/\lambda$.

3.4. **The tropicalization of a log Calabi–Yau variety.**

Definition 3.8. Let U be a log Calabi–Yau variety. We define the *tropicalization* $U^{\mathrm{trop}}(\mathbb{R})$ of U as follows. Let (X, D) be a log Calabi–Yau compactification of U. We may assume (blowing up boundary strata if necessary) that D is a simple normal crossing divisor, that is, each component D_i of D is smooth and each intersection $D_{i_1} \cap \cdots \cap D_{i_k}$ is either irreducible or empty. The *dual complex* of D is the simplicial complex with vertex set indexed by components of D, such that a set of vertices spans a simplex if and only if the intersection of the corresponding divisors is non-empty. Let $U^{\mathrm{trop}}(\mathbb{R})$ be the cone over the dual complex of D, and $U^{\mathrm{trop}}(\mathbb{Z})$ its integral points. One can show using [AKMW02] that $U^{\mathrm{trop}}(\mathbb{R})$ is independent of the choice of (X, D) up to $\mathbb{Z}$PL-homeomorphism [KS06], §6.6, [dFKX12], Proposition 11.

The set $U^{\mathrm{trop}}(\mathbb{Z})$ has the following intrinsic description: Let Ω be a holomorphic volume form on U as in Remark 2.2. Then $U^{\mathrm{trop}}(\mathbb{Z}) \setminus \{0\}$ is identified with the set of pairs (ν, k) consisting of a divisorial valuation $\nu \colon \mathbb{C}(U)^\times \to \mathbb{Z}$ such that $\nu(\Omega) < 0$ and a positive integer k. Thus roughly speaking $U^{\mathrm{trop}}(\mathbb{Z}) \setminus \{0\}$ is the set of all pairs (F, k) where F is a boundary divisor in some log Calabi-Yau compactification (X, D) of U and $k \in \mathbb{N}$.

Example 3.9. If $U = T = N \otimes \mathbb{C}^\times$ is an algebraic torus then we have an identification $U^{\mathrm{trop}}(\mathbb{Z}) = N$: Given $0 \neq v \in N$ write $v = kv'$ where $k \in \mathbb{N}$ and $v' \in N$ is primitive. Then v' corresponds to a toric boundary divisor associated to the ray $\rho = \mathbb{R}_{\geq 0} \cdot v'$ in $N_{\mathbb{R}}$, with associated valuation $\nu \colon \mathbb{C}(T)^\times \to \mathbb{Z}$ determined by $\nu(z^m) = \langle m, v' \rangle$. These are the only divisors along which Ω has a pole, by [KM98], Lemmas 2.29 and 2.45.

If (X, D) is a toric compactification of T then the cone over the dual complex of D is identified with the fan Σ of X in $U^{\mathrm{trop}}(\mathbb{R}) = N_{\mathbb{R}}$.

If f is a nonzero rational function on a log Calabi–Yau variety U, then we have a $\mathbb{Z}$PL map $f^{\mathrm{trop}} \colon U^{\mathrm{trop}}(\mathbb{R}) \to \mathbb{R}$ defined on primitive integral points $\nu = (\nu, 1)$ by $f^{\mathrm{trop}}(\nu) = \nu(f)$. If $f \colon U \dashrightarrow V$ is a birational map between log Calabi–Yau varieties that is compatible with the holomorphic volume forms, then there is a canonical $\mathbb{Z}$PL identification $f^{\mathrm{trop}} \colon U^{\mathrm{trop}}(\mathbb{R}) \xrightarrow{\sim} V^{\mathrm{trop}}(\mathbb{R})$ defined by $f^{\mathrm{trop}}(\nu) = \nu \circ f^*$.

Example 3.10. For the mutation (3.7), we have

$$\mu^{\text{trop}} \colon N_{\mathbb{R}} \to N_{\mathbb{R}}, \quad \mu^{\text{trop}}(w) = \begin{cases} w & \text{if } \langle m, w\rangle \geq 0 \\ w - \langle m, w\rangle v & \text{if } \langle m, w\rangle < 0. \end{cases}$$

4. Mirror symmetry

Mirror symmetry is a phenomenon arising in theoretical physics which predicts that Calabi–Yau varieties (together with a choice of Kähler form) come in mirror pairs U and V such that the symplectic geometry of U is equivalent to the complex geometry of V, and vice versa.

4.1. **The Strominger–Yau–Zaslow conjecture.** Recall that a submanifold L of a symplectic manifold (U, ω) is *Lagrangian* if $\dim_{\mathbb{R}} L = \frac{1}{2}\dim_{\mathbb{R}} U$ and $\omega|_L = 0$. Let U be a log Calabi–Yau manifold with holomorphic volume form Ω and Kähler form ω. We say a Lagrangian submanifold L of (U, ω) is *special Lagrangian* if $\operatorname{Im}\Omega|_L = 0$. The Strominger–Yau–Zaslow conjecture asserts that mirror Calabi-Yau varieties admit dual special Lagrangian torus fibrations [SYZ96]. More precisely, there exist continuous maps $f \colon U \to B$ and $g \colon V \to B$ with common base B and a dense open set $B^o \subset B$ such that

(1) The restrictions $f^o \colon U^o \to B^o$ and $g^o \colon V^o \to B^o$ are C^∞ real n-torus fibrations such that the fibers are special Lagrangian, and
(2) The associated local systems $R^1 f^o_* \mathbb{Z}$ and $R^1 g^o_* \mathbb{Z}$ on B^o are dual.

Example 4.1. Let $U = (\mathbb{C}^\times)^n_{z_1,\ldots,z_n}$, $\Omega = (\frac{1}{2\pi i})^n \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$, and $\omega = \frac{1}{2\pi}\frac{i}{2}\sum_{j=1}^n \frac{dz_j}{z_j} \wedge \frac{d\bar{z}_j}{\bar{z}_j}$. Then the map $f \colon U \to \mathbb{R}^n$, $f(z_1, \ldots, z_n) = (\log|z_1|, \ldots, \log|z_n|)$ is a special Lagrangian torus fibration. (Topologically f is the quotient by the compact torus $(S^1)^n \subset (\mathbb{C}^\times)^n$.)

Example 4.2. Let $U = T = N \otimes \mathbb{C}^\times \simeq (\mathbb{C}^\times)^n$ and let (X, D) be a smooth projective toric compactification. Let A be an ample line bundle on X. Let $K \subset T$ be the compact torus. Then, using the description of X as a GIT quotient of affine space, the Kempf–Ness theorem ([MFK94], Theorem 8.3), and symplectic reduction, one can construct a K-invariant Kähler form ω on X in class $c_1(A) \in H^2(X, \mathbb{R})$. If μ is the associated moment map, then μ maps X onto the lattice polytope $P \subset M_{\mathbb{R}}$ associated to (X, A), and is topologically the quotient by K. The restriction of μ to T is a special Lagrangian torus fibration for the Kähler form $\omega|_U$ and holomorphic volume form Ω as in Example 4.1.

Construction 4.3. If $f \colon (U, \omega) \to B$ is a Lagrangian torus fibration, then the locus $B^o \subset B$ of smooth fibers inherits an *integral affine structure* (an atlas of charts with transition functions of the form $\mathbf{x} \mapsto A\mathbf{x}+\mathbf{b}$ for some $A \in \mathrm{GL}(n,\mathbb{Z})$ and $\mathbf{b} \in \mathbb{R}^n$). This may be constructed as follows. Fix $b_0 \in B^o$ and let $W \subset B^o$ be a small contractible neighborhood of b_0. For $\gamma \in H_1(f^{-1}(b_0),\mathbb{Z})$ define $y_\gamma \colon W \to \mathbb{R}$ by $y_\gamma(b) = \int_\Gamma \omega$ where $\Gamma \subset X$ is a cylinder fibering over a path from b_0 to b in W swept out by a loop in the class γ. Applying this construction to a basis of $H_1(f^{-1}(b_0),\mathbb{Z}) \simeq \mathbb{Z}^n$ gives a system of integral affine coordinates $y_1,\ldots,y_n$ on $W \subset B^o$. In Examples 4.1 and 4.2 this integral affine structure is the restriction of the standard integral affine structure on $\mathbb{R}^n$.

Example 4.4. Let U, (X,D), etc. be as in Example 4.2 and consider a non-toric blow up $(\tilde{X},\tilde{D})$ of (X,D) as in Example 2.6. Then we can modify the moment map $\mu\colon X \to P$ to obtain a map $\tilde{\mu}\colon \tilde{X} \to \tilde{P}$ such that the restriction $f\colon \tilde{U} \to B$ to the interior B of $\tilde{P}$ is a Lagrangian torus fibration with singular fibers [AAK16].

Assume first that $n = \dim_{\mathbb{C}} X = 2$. Thus we have a smooth point p of D, $\pi\colon \tilde{X} \to X$ is the blow up of p, and $\tilde{D} \subset \tilde{X}$ is the strict transform of D. Let $S^1 \subset K$ be the stabilizer of the point $p \in X$, so that S^1 acts on $\tilde{X}$. Let $e \subset P$ be the edge of P containing $\mu(p)$, and choose integral affine coordinates y_1, y_2 on $M_{\mathbb{R}} \simeq \mathbb{R}^2$ such that $p = (0,0)$, $e \subset (y_1 = 0)$, and $P \subset (y_1 \geq 0)$. Let $\epsilon > 0$ be sufficiently small so that the triangle T with vertices $(0,-\epsilon/2),(0,+\epsilon/2),(\epsilon,0)$ is contained in P and its intersection with the boundary of P is contained in the interior of e. Let $\tilde{P}$ be the topological space obtained by collapsing $T \subset P$ via the map $y_1\colon T \to [0,\epsilon]$. Then $\tilde{P}$ is the base of an S^1-invariant map $\tilde{\mu}\colon (\tilde{X},\tilde{\omega}) \to \tilde{P}$ such that the restriction to the interior B of $\tilde{P}$ is a Lagrangian torus fibration with a unique singular fiber (a pinched torus) over the image $q \in B$ of the point $(\epsilon,0) \in P$. The fibration has monodromy around q given by the Dehn twist in the vanishing cycle (the class of the S^1-orbits). The exceptional (-1)-curve $E \subset \tilde{X}$ fibers over the interval $I \subset \tilde{P}$ given by the image of $T \subset P$. The class of the symplectic form in $H^2(\tilde{X},\mathbb{R})$ is $[\tilde{\omega}] = \pi^*[\omega] - \epsilon c_1(E) = c_1(\pi^*A - \epsilon E)$. The symplectic form $\tilde{\omega}$ and the fibration $\tilde{\mu}$ agree with ω and μ over the complement of a tubular neighborhood of $I \subset \tilde{P}$.

A similar construction applies in dimension $n > 2$ [AAK16], §4. (Here we work over the open set $X' \subset X$ given by the complement of the codimension two strata, cf. Remark 2.8.) Applying this construction repeatedly, we can construct a Lagrangian torus fibration on any log Calabi-Yau variety with a toric model. (Note that the fibration is *not* special Lagrangian (but cf. [AAK16], Remark 4.6).)

4.2. **Homological mirror symmetry.** The homological mirror symmetry conjecture of Kontsevich [K95] asserts the following mathematical formulation of mirror symmetry: For mirror compact Calabi–Yau varieties U and V, the derived Fukaya category $\mathcal{F}(U)$ of U is equivalent to the derived category of coherent sheaves $D(V)$ on

V. Roughly speaking, the objects of the Fukaya category of U are Lagrangian submanifolds L together with a unitary local system, and the morphisms are given by Lagrangian Floer cohomology. See [A14] for an introduction.

If U is a log Calabi–Yau variety then, at least if U is positive (Definition 2.4), the HMS conjecture is expected to hold with the following adjustments. First, we must allow non-compact Lagrangian submanifolds with controlled behaviour at infinity. Second, the definition of the morphisms in the Fukaya category is modified at infinity using a Hamiltonian vector field associated to a function $H\colon U \to \mathbb{R}$ such that $H \to \infty$ sufficiently fast at infinity. The resulting category $\mathcal{F}(U)$ is called the wrapped Fukaya category [A14], §4.

The HMS and SYZ conjectures are related as follows. Suppose U and V are mirror log Calabi–Yau varieties with dual Lagrangian torus fibrations $f\colon U \to B$ and $g\colon V \to B$. Let $L = f^{-1}(b)$ be a smooth fiber of f. The rank 1 unitary local systems ∇ on L are classified by their holonomy $\mathrm{hol}(\nabla) \in \mathrm{Hom}(\pi_1(L), U(1)) = L^*$ (the dual torus). It is expected that the pairs $[(L,\nabla)] \in \mathcal{F}(U)$ correspond under the equivalence $\mathcal{F}(U) \simeq D(V)$ to the skyscraper sheaves $\mathcal{O}_p \in D(V)$ for $p \in g^{-1}(b) \simeq L^*$. (More generally, the equivalence should be given by a real version of the relative Fourier–Mukai transform for the dual torus fibrations, cf. [KS01], §9, [P03], §6.)

It follows that, to a first approximation, one can regard the mirror V of U as the moduli space of pairs $[(L,\nabla)]$ where L is a fiber of f and ∇ is a $U(1)$ local system. We define local holomorphic coordinates on V as follows (a complexified version of the integral affine coordinates y_γ of Construction 4.3). For $L_0 = f^{-1}(b_0)$ a smooth fiber, $\gamma \in H_1(L_0,\mathbb{Z})$, and $(L = f^{-1}(b), \nabla)$ a nearby fiber together with a $U(1)$ local system, let Γ be a cylinder over a short path from b_0 to b with initial fiber Γ_{b_0} in class γ and final fiber Γ_b. We define

$$z^\gamma([(L,\nabla)]) = \exp\left(-2\pi y_\gamma(b)\right) \cdot \mathrm{hol}_\nabla(\Gamma_b) = \exp\left(-2\pi \int_\Gamma \omega\right) \cdot \mathrm{hol}_\nabla(\Gamma_b).$$

Suppose now that U is a log Calabi–Yau variety, and (X,D) is a log Calabi–Yau compactification such that ω extends to a 2-form on X. The homology groups $H_2(X, L = f^{-1}(b))$ form a local system over B^o; let $T\colon H_2(X,L_0) \to H_2(X,L)$ be the local trivialization given by parallel transport. Then, for $\beta \in H_2(X,L_0)$, we define

$$z^\beta([(L,\nabla)]) = \exp\left(-2\pi \int_{T(\beta)} \omega\right) \cdot \mathrm{hol}_\nabla(\partial T(\beta)).$$

Then $z^\beta = c z^{\partial\beta}$ where $c = \exp(-2\pi \int_\beta \omega) \in \mathbb{R}_{>0}$.

We can attempt to define global holomorphic functions ϑ_q on V for each $q = (F,k) \in U^{\mathrm{trop}}(\mathbb{Z}) \backslash \{0\}$ as follows [CO06], [A09]. Let (X,D) be a log Calabi–Yau compactification of U such that F is a component of the boundary D and ω extends to X. Let L be a smooth fiber of f. For $\beta \in H_2(X, L, \mathbb{Z})$, let N_β be the (virtual) count of holomorphic discs $h\colon (\mathbb{D}, \partial\mathbb{D}) \to (X, L)$ such that h meets F with contact order k and is disjoint from the remaining boundary divisors, and $h(\partial\mathbb{D})$ passes through a general point $p \in L$. We assume that N_β is well defined (independent of the choice of $p \in L$). We define

$$\vartheta_{(F,k)}([(L, \nabla)]) = \sum_{\beta \in H_2(X,L,\mathbb{Z})} N_\beta z^\beta([(L, \nabla)]).$$

(Note that the sum may not converge in general.)

Example 4.5. Let $U = T = N \otimes \mathbb{C}^\times \simeq (\mathbb{C}^\times)^n$. Let $(\bar{X}, \bar{D})$ be a toric compactification and $F \subset \bar{D}$ a boundary divisor corresponding to a primitive vector $v \in N$. Let $g\colon \mathbb{C}^\times \to T$ be the associated 1-parameter subgroup of T. Then g extends to a morphism $\bar{g}\colon \mathbb{C} \to \bar{X}$ such that $\bar{g}$ meets F transversely at a single point and is disjoint from the other boundary divisors. Let h be the restriction of $\bar{g}$ to the closed unit disc $\mathbb{D} \subset \mathbb{C}$. Let $K \subset T \subset \bar{X}$ be the compact torus. Then $h\colon (\mathbb{D}, \partial\mathbb{D}) \to (\bar{X}, K)$ is a holomorphic disc ending on the fiber K of the moment map and passing through the point $e \in K$. It is the unique such disc and counts with multiplicity 1. The same applies to any choice of fiber and marked point (because they are permuted simply transitively by T). Similarly, there is a unique disc meeting F with contact order k given by the multiple cover $\tilde{h}(z) = h(z^k)$. See [CO06], Theorems 5.3 and 6.1.

The functions ϑ_q as defined above are discontinuous in general, because the counts of holomorphic discs N_β ending on an SYZ fiber $L = f^{-1}(b)$ vary discontinuously with b. This is due to the existence of SYZ fibers which bound holomorphic discs in U. Such fibers lie over (thickened) real codimension 1 walls in the base B. In more detail, suppose $\{L_t\}_{t \in [0,1]}$ are the SYZ fibers over a path crossing a wall in the base. If L_{t_0} bounds a holomorphic disc in U, then there may exist a family of holomorphic discs h_t in X ending on L_t for $t < t_0$, such that the limit of h_t as $t \to t_0$ is a *stable disc* given by the union of two discs ending on L_{t_0}, one of which is contained in U, and such that this stable disc does *not* deform to a holomorphic disc in X ending on L_t for $t > t_0$.

Example 4.6. ([A09], Example 3.1.2.) Let $(\bar{X}, \bar{D}) = (\mathbb{C}^2_{z_1,z_2}, (z_1 z_2 = 0))$ with Kähler form $\frac{1}{2\pi}\frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$. Let $\pi\colon X \to \bar{X}$ be the blow up of $p = (1, 0) \in \bar{X}$ with exceptional curve E, and $D \subset X$ the strict transform of $\bar{D}$. As in Example 4.4, we have a Kähler form ω on X and a map $\bar{f}\colon X \to \bar{B}$ which restricts to a Lagrangian torus fibration $f\colon U \to B$ over the interior B of $\bar{B}$. Moreover, over the complement of a small neighborhood N of $\bar{f}(E)$ the map $\bar{f}$ agrees with the moment map $\mu\colon \bar{X} \to \mathbb{R}^2_{\geq 0}$, $(z_1, z_2) \mapsto \frac{1}{2}(|z_1|^2, |z_2|^2)$. The map $\bar{f}\colon X \to \bar{B} \simeq \mathbb{R}^2_{\geq 0}$ is defined by $\frac{1}{2}|\pi^* z_1|^2$ and μ_{S^1}, the moment map for the S^1 action on (X, ω), normalized so that $\mu_{S^1} = 0$ on the strict transform of the z_1-axis. (Then on the singular fiber $\mu_{S^1} = \int_E \omega = \epsilon > 0$.)

There is a real codimension 1 wall H in the base B defined by $|\pi^* z_1| = 1$. Note that $(\pi^* z_1 = 1) \subset U$ is the union of two copies of $\mathbb{A}^1$ meeting in a node: $E \setminus E \cap D$ and the strict transform of $(z_1 = 1)$. (The node is the singular point of the pinched torus fiber of f.) These curves are S^1-equivariant and map to the wall in the base with fibers the S^1-orbits (which collapse to the singular point at the pinched torus fiber). Thus each smooth fiber over the wall bounds a holomorphic disc in U contained in one of the two curves.

Now let $D_1 = (\pi^* z_1 = 0) \subset X$ and consider the associated function ϑ_{D_1} defined by counting holomorphic discs in X meeting D transversely at a point of D_1, ending on an SYZ fiber L, and passing through a marked point $p \in L$. We assume L lies over the region $R = B \setminus N$ where the fibration f agrees with the moment map μ. Note that $f^{-1}R \subset U \setminus E \simeq \bar{U}$. Let $\pi(p) = (\nu_1, \nu_2) \in \bar{U} = (\mathbb{C}^\times)^2_{z_1,z_2}$, so $L = (|\pi^* z_1| = |\nu_1|, |\pi^* z_2| = |\nu_2|)$. If $|\nu_1| < 1$ then there is a unique disc given by the strict transform of the disc $\mathbb{D} \to \bar{X} = \mathbb{C}^2$, $z \mapsto (\nu_1 z, \nu_2)$. If $|\nu_1| > 1$ there are two discs, one described as before and the second given by the strict transform of the disc $z \mapsto (\nu_1 z, \nu_2(\frac{\nu_1 z - 1}{\bar{\nu}_1 - z})/(\frac{\nu_1 - 1}{\bar{\nu}_1 - 1}))$. See [CO06], Theorem 5.2. Writing $\beta_1, \beta_1' \in H_2(X, L)$ for the classes of the two discs, observe that $\beta_1' = \beta_1 + \alpha \in H_2(X, L)$ where α is the parallel transport of the class of the disc associated to the portion of the wall meeting R. (More precisely, if $\{L_t\}_{t \in [0,1]}$ are the fibers over a path γ in $R \subset B$ crossing the wall at time t_0 from $|\pi^* z_1| > 1$ to $|\pi^* z_1| < 1$, then the limit of the holomorphic disc in class β_1' ending on L_t as $t \to t_0$ from below is the union of the holomorphic discs ending on L_{t_0} in classes β_1 and α.) We thus have

$$\vartheta_{D_1} = \begin{cases} z^{\beta_1} & \text{if } |\pi^* z_1| < 1 \\ z^{\beta_1} + z^{\beta_1'} = z^{\beta_1} \cdot (1 + z^{\alpha}) & \text{if } |\pi^* z_1| > 1. \end{cases}$$

On the other hand, let D_2 be the strict transform of $(z_2 = 0)$. Then, with notation as above, there is a unique disc meeting D transversely at a point of D_2, ending on L, and passing through $p \in L$, given by the inverse image of the disc $z \mapsto (\nu_1, \nu_2 z)$ in $\bar{X} = \mathbb{C}^2$. (For $\nu_1 = 1$, this is the stable disc given by the union of the strict transform of disc in $\bar{X}$ (which is the disc associated to the wall) and the exceptional curve E.) Thus, writing $\beta_2 \in H_2(X, L)$ for the class of this disc, we have $\vartheta_{D_2} = z^{\beta_2}$.

We have defined (using the local holomorphic coordinates z^γ, $\gamma \in H_1(L, \mathbb{Z})$) a complex structure on the total space of the dual fibration $V^o \to B^o$ of the smooth locus of the SYZ fibration $f \colon U \to B$. However it is expected that this does *not* extend to a complex structure on a fibration $V \to B$. Roughly speaking, if $V \to B$ is a topological extension of the fibration $V^o \to B^o$, and $W \subset V$ is a neighborhood of a point $p \in V \setminus V^o$, there are too few holomorphic functions defined on $W \cap V^o$ for the complex structure to extend. For instance, some of the z^γ are not well defined due to the monodromy action on $H_1(L, \mathbb{Z})$. The naive definition of the mirror V^o must be corrected to account for discs ending on SYZ fibers. These glueing corrections are such that the ϑ_q define global holomorphic functions on V^o, and can be used to define an extension $V^o \subset V$.

For instance, in Example 4.6, the corrected mirror V^o is an analytic open subset of $\hat{V}^o = (\mathbb{C}^\times)^2_{w_1,w_2} \cup (\mathbb{C}^\times)^2_{w_1',w_2'}$, where the two torus charts correspond to the two connected components $|\pi^* z_1| < 1$ and $|\pi^* z_1| > 1$ of the complement of the wall H in the base B, the glueing is given by

$$(w_1, w_2) \mapsto (w_1(1 + cw_2)^{-1}, w_2),$$

and $w_1 = w_1' = z^{\beta_1}$ and $w_2 = w_2' = z^{\beta_2}$ on the naive mirror (and we have trivialized the local system $H_2(X, L)$ over the the region $R \subset B$ as above). The parameter c is given by $c = z^{-E}$, $z^E := \exp(-2\pi \int_E \omega)$, so that $cw_2 = z^{-E} z^{\beta_2} = z^\alpha$ (since $\beta_2 = \alpha + [E]$ in $H_2(X, L)$). Then ϑ_{D_1} and ϑ_{D_2} are the global functions on V^o which restrict to w_1 and w_2 in the first chart. In fact, defining $\hat{V} = \operatorname{Spec} H^0(\mathcal{O}_{\hat{V}^o})$, we have an isomorphism

$$\hat{V} \xrightarrow{\sim} (uv = 1 + cw) \subset \mathbb{A}^2_{u,v} \times \mathbb{C}^\times_w$$

given by $u \mapsto w_1$, $v \mapsto {w_1'}^{-1}$, $w \mapsto w_2$, and $\hat{V}^o = \hat{V} \setminus \{q\}$ where $q \mapsto (0, 0, -1/c)$ (cf. §3.2). The mirror $V \subset \hat{V}$ equals $V^o \cup \{q\}$, an analytic open subset of the affine variety $\hat{V}$.

Remark 4.7. The point $q \in V$ should correspond under HMS to the pinched torus fiber of the SYZ fibration $f : U \to B$ regarded as an immersed Lagrangian S^2 (with the trivial $U(1)$ local system). See [S13], Lecture 11.

In general, the wall crossing transformations should take the following form, cf. [AAK16], p. 207. Let $\{L_t\}_{t \in [0,1]}$ be the fibers of $f : U \to B$ over a path crossing a wall in the base B. Assume for simplicity that all the holomorphic discs in U bounded by the L_t have relative homology class some fixed $\alpha \in H_2(U, L)$. The boundaries of these discs sweep out a cycle $c \in H_{n-1}(L)$. Then the wall crossing transformation in the local coordinates z^γ, $\gamma \in H_1(L)$, is given by

$$z^\gamma \mapsto z^\gamma \cdot f(z^\alpha)^{c \cdot \gamma}$$

where $f(z^\alpha) = 1 + z^\alpha + \cdots \in \mathbb{Q}[[z^\alpha]]$ is a power series encoding virtual counts of multiple covers of the discs.

4.3. **Symplectic cohomology.** Suppose that U is a positive log Calabi–Yau variety with maximal boundary. Suppose V is HMS mirror to U, so that we have an equivalence $\mathcal{F}(U) \simeq D(V)$ between the wrapped Fukaya category of U and the derived category of coherent sheaves on V. Symplectic cohomology SH^* is a version of Hamiltonian Floer cohomology for noncompact symplectic manifolds [S08]. There is a *closed-open string map* $SH^*(U) \to HH^*(\mathcal{F}(U))$ which is conjectured to be an isomorphism, cf. [S02], §4. (Recently, Ganatra–Pardon–Shende and Chantraine–Dimitroglou-Rizell–Ghiggini–Golovko have announced results which, combined with [G13], would establish this result.) Recall that $HH^n(D(V)) \simeq \oplus_{p+q=n} H^p(\wedge^q T_V)$ (the Hochschild–Kostant–Rosenberg isomorphism), in particular, $HH^0(D(V)) \simeq H^0(\mathcal{O}_V)$. Thus the above conjecture and HMS would yield an isomorphism of $\mathbb{C}$-algebras $SH^0(U) \simeq H^0(\mathcal{O}_V)$. In particular, assuming the mirror V is affine, it can be constructed as $V = \operatorname{Spec} SH^0(U)$.

Conjecturally, $SH^0(U)$ has a natural basis parametrized by $U^{\mathrm{trop}}(\mathbb{Z})$. (This was proved by Pascaleff in dimension 2 [P13]; there is ongoing work of Ganatra–Pomerleano on the general case.) We expect that this basis corresponds to the global functions ϑ_q, $q \in U^{\mathrm{trop}}(\mathbb{Z})$ under the above isomorphism $SH^0(U) \simeq H^0(\mathcal{O}_V)$ (where we define $\vartheta_0 = 1$). In particular, we expect that the ϑ_q, $q \in U^{\mathrm{trop}}(\mathbb{Z})$ form a basis of $H^0(\mathcal{O}_V)$.

4.4. **The Fock–Goncharov mirror of a cluster variety.** Fock and Goncharov [FG06] defined a candidate for the mirror V of a cluster variety U by a simple combinatorial recipe which we reproduce in our notation here. We will give a partial justification for the Fock–Goncharov construction in §5.

Recall that $U = X \setminus D$ is described (up to codimension two) as a union of copies T_α, $\alpha \in A$ of the algebraic torus $T = N \otimes \mathbb{C}^\times \simeq (\mathbb{C}^\times)^n$ with transition maps given by compositions of mutations

$$\mu = \mu_{(m,v)} \colon T_\alpha \dashrightarrow T_\beta, \quad \mu^*(z^{m'}) = z^{m'} \cdot (1 + cz^m)^{-\langle m', v\rangle}$$

for some $c \in \mathbb{C}^\times$. In addition we have a non-degenerate log 2-form σ on U such that $\sigma|_{T_\alpha} = \bar{\sigma} \in \wedge^2 M_{\mathbb{C}}$ for each α. We assume that the sign of m above has been chosen according to the convention of §5.3.

The Fock–Goncharov mirror $(V, \sigma^\vee)$ is described as follows. Let $T^\vee = N^* \otimes \mathbb{C}^\times$ be the dual algebraic torus to $T = N \otimes \mathbb{C}^\times$. We write $N^\vee = H_1(T^\vee, \mathbb{Z}) = N^* = M$ and $M^\vee = (N^\vee)^* = N$. Then (up to codimension two) V is a union of copies $T^\vee_\alpha$, $\alpha \in A$ of $T^\vee$, with transition maps given by

$$\mu^\vee = \mu_{(v,-m)} \colon T^\vee_\alpha \dashrightarrow T^\vee_\beta \quad \mu^{\vee *}(z^{v'}) = z^{v'} \cdot (1 + c^\vee z^v)^{\langle v', m\rangle}$$

for some $c^\vee \in \mathbb{C}^\times$. Let

$$\phi \colon N_{\mathbb{C}} \to M_{\mathbb{C}}, \quad \phi(v) = \bar{\sigma}(v, \cdot)$$

be the isomorphism determined by the non-degenerate form $\bar{\sigma}$ on $N_{\mathbb{C}}$ and $\bar{\sigma}^\vee$ the form on $N^\vee_{\mathbb{C}} = M_{\mathbb{C}}$ given by

$$\bar{\sigma}^\vee(m_1, m_2) = \bar{\sigma}(\phi^{-1}(m_1), \phi^{-1}(m_2)).$$

Then the log 2-form $\sigma^\vee$ on V is given by $\sigma^\vee|_{T^\vee_\alpha} = \bar{\sigma}^\vee \in \wedge^2 M^\vee_{\mathbb{C}}$.

Equivalently, given the data N, $\bar{\sigma} \in \wedge^2 M_{\mathbb{C}}$, $m_i \in M$, $v_i \in N$, $\lambda_i \in \mathbb{C}^\times$, $i = 1, \ldots, r$ of §3.3 determining the cluster variety $U = X \setminus D$ in terms of a toric model $\pi\colon (X, D) \to (\bar{X}, \bar{D})$, the Fock–Goncharov mirror is associated to the data $N^\vee = M$, $\bar{\sigma}^\vee$, $v_i \in M^\vee$, $-m_i \in N^\vee$, and some $\lambda_i^\vee \in \mathbb{C}^\times$, $i = 1, \ldots, r$.

Remark 4.8. Recall that, for mirror Calabi–Yau varieties U and V, symplectic deformations of U correspond to complex deformations of V, and vice versa. In particular, if we regard U as a symplectic manifold (forgetting the complex structure) and V as a complex manifold (forgetting the Kähler form), then the parameters $\lambda_i^\vee$ for V are determined by the class of the symplectic form on U (and the parameters λ_i for U are irrelevant).

Remark 4.9. The Fock–Goncharov mirror construction is an involution. The isomorphism between U and the mirror of the mirror of U is given in the torus charts by the map $T \to T$, $t \mapsto t^{-1}$.

Remark 4.10. We expect that the mirror of a log Calabi–Yau variety U with maximal boundary is of the same type if and only if U is positive. If U is a positive cluster variety, we expect that the Fock–Goncharov mirror is the mirror in the sense of SYZ and HMS. For a general cluster variety U, we expect that the true mirror is an analytic open subset of the Fock–Goncharov mirror. Cf. the discussion of completion of the mirror via symplectic inflation in the positive case in [A09], §2.2.

Example 4.11. Let $\bar{X}$ be the smooth projective toric surface given by the complete fan in $\mathbb{R}^2$ with rays generated by $(1,0)$, $(0,1)$, $(-1,2)$, $(-1,1)$, $(-1,0)$, $(-1,-1)$, $(0,-1)$, $(1,-1)$, $(2,-1)$. The toric boundary $\bar{D} \subset \bar{X}$ is a cycle of smooth rational curves with self-intersection numbers $-2, -2, -1, -2, -2, -1, -2, -2, -1$. Let $\pi\colon X \to \bar{X}$ be the blow up of three points in the smooth locus of $\bar{D}$, one point on each of the (-1)-curves, and D the strict transform of $\bar{D}$. Then $U = X \setminus D$ is a cluster variety. The divisor $D = \sum D_i$ is a cycle of nine (-2)-curves; in particular the intersection matrix $(D_i \cdot D_j)$ is negative semi-definite, and U is *not* positive. It is expected (cf. [AKO06], [A09], §5) that the mirror of U is the log Calabi–Yau surface $V = Y \setminus E$ where $Y = \mathbb{P}^2$ and $E \subset Y$ is a smooth elliptic curve. In particular, there does not exist an open inclusion of an algebraic torus $(\mathbb{C}^\times)^2$ in V, so V is not a cluster variety.

Remark 4.12. In dimension 2, we may assume (multiplying σ by a non-zero scalar) that $\mathbb{Z} \cdot \bar{\sigma} = \wedge^2 M$. Let $\psi\colon N \to M$ be the isomorphism given by $\psi(v) = -\bar{\sigma}(v, \cdot)$. Then $\psi(v_i) = -m_i$ and $\psi^*(v_i) = -\psi(v_i) = m_i$. So if we take $\lambda_i^\vee = \lambda_i$ then the isomorphism $\psi \otimes \mathbb{C}^\times \colon T \xrightarrow{\sim} T^\vee$ extends to an isomorphism $U \to V$. That is, in dimension two the Fock-Goncharov mirror V of U is deformation equivalent to U.

Note that 2-torus fibrations are self-dual by Poincaré duality, so SYZ mirrors are diffeomorphic in dimension 2. For U a log Calabi–Yau surface with maximal boundary, the Fock–Goncharov mirror construction is valid if and only if U is positive (cf. [GHK15a], [K15]), and in that case the mirror V is deformation equivalent to U.

5. Scattering diagrams

Given a cluster variety U together with a choice of toric model, we explain how to build a *scattering diagram* in $U^{\text{trop}}(\mathbb{R})$. Heuristically, this is the tropicalization of the collection of walls in the base of the SYZ fibration together with the attached generating functions encoding counts of holomorphic discs in U ending on SYZ fibers described in §4.1. We use the scattering diagram to construct a canonical topological basis ϑ_q, $q \in U^{\text{trop}}(\mathbb{Z})$ of the algebra of global functions on a formal completion of the Fock–Goncharov mirror family. We expect that when U is positive this basis is algebraic and defines a canonical basis of global functions on the Fock–Goncharov mirror. We prove this under certain hypotheses on U related to positivity.

5.1. **Definitions and algorithmic construction of scattering diagrams.** Let $A = \mathbb{C}[t_1, \ldots, t_r]$ and $\mathfrak{m} = (t_1, \ldots, t_r) \subset A$. We write $\hat{M}$ for the $\mathfrak{m}$-adic completion $\varprojlim M/\mathfrak{m}^l M$ of an A-module M.

Let $N \simeq \mathbb{Z}^n$ be a free abelian group of rank n, and write $M = N^*$. Let $\bar{\sigma} \in \wedge^2 M_{\mathbb{C}}$ be a non-degenerate skew form.

Definition 5.1. A *wall* is a pair $(\mathfrak{d}, f)$ consisting of a codimension 1 rational polyhedral cone $\mathfrak{d} \subset N_{\mathbb{R}}$ together with an attached function $f \in \widehat{A[N]}$ satisfying the following properties. Let $m \in M$ be a primitive vector (determined up to sign) such that $\mathfrak{d} \subset m^{\perp}$. Then there exists a primitive vector $v \in N$ such that

(1) $\bar{\sigma}(v, \cdot) = \nu \cdot m$ for some $\nu \in \mathbb{C}^{\times}$,
(2) $f \in \widehat{A[z^v]} \subset \widehat{A[N]}$, and
(3) $f \equiv 1 \bmod \mathfrak{m} z^v \widehat{A[z^v]}$.

(Note in particular $v \in m^{\perp}$ because $\bar{\sigma}$ is skew-symmetric.)

The cone $\mathfrak{d}$ is called the *support* of the wall. The vector $-v$ is called the *direction* of the wall. We say a wall $(\mathfrak{d}, f)$ is *incoming* if $v \in \mathfrak{d}$, otherwise, we say it is *outgoing*. (The terminology comes from the dimension 2 case, where the support of an outgoing wall is necessarily the ray $\mathbb{R}_{\geq 0} \cdot (-v)$ in the direction of the wall.)

Crossing a wall $(\mathfrak{d}, f)$ defines an associated automorphism θ of $\widehat{A[N]}$ over $\hat{A}$ such that $\theta \equiv \text{id} \bmod \mathfrak{m}$. Let $m \in M$ be as in Definition 5.1. Then the automorphism associated to crossing the wall from $(m > 0)$ to $(m < 0)$ is given by

$$\theta\colon \widehat{A[N]} \to \widehat{A[N]}, \quad z^u \mapsto z^u \cdot f^{\langle u,m\rangle}.$$

A *scattering diagram* $\mathfrak{D}$ is a collection of walls such that for all $l \in \mathbb{N}$, there are finitely many walls $(\mathfrak{d}, f)$ such that $f \not\equiv 1 \bmod \mathfrak{m}^l$ (so that the associated automorphism is non-trivial modulo $\mathfrak{m}^l$).

The *support* $\operatorname{Supp}\mathfrak{D}$ of $\mathfrak{D}$ is the union of the supports of the walls. A *joint* of $\mathfrak{D}$ is an intersection of walls of codimension 2 in $N_{\mathbb{R}}$. The *singular locus* $\operatorname{Sing}\mathfrak{D}$ is the union of the joints of $\mathfrak{D}$ and the relative boundaries of the walls of $\mathfrak{D}$. A *chamber* of $\mathfrak{D}$ is the closure of an open connected component of $N_{\mathbb{R}} \setminus \operatorname{Supp}\mathfrak{D}$.

If $\gamma\colon [0,1] \to N_{\mathbb{R}} \setminus \operatorname{Sing}\mathfrak{D}$ is a smooth path such that $\gamma(0), \gamma(1) \notin \operatorname{Supp}(\mathfrak{D})$ and γ is transverse to each wall it crosses, it defines an automorphism $\theta_{\mathfrak{D},\gamma}$ given by composing wall crossing automorphisms. In more detail, let $\mathfrak{D}_l \subset \mathfrak{D}$ be the finite subset of the scattering diagram consisting of walls $(\mathfrak{d}, f)$ such that $f \not\equiv 1 \bmod \mathfrak{m}^l$. Let $0 < t_1 < \ldots < t_k < 1$ be the times at which γ crosses a wall of $\mathfrak{D}_l$, and θ_i the composition of the automorphisms associated to the walls crossed at time t_i (note that if two walls lie in the same hyperplane then the associated automorphisms commute, so θ_i is well defined). Let $\theta^l_{\mathfrak{D},\gamma}$ be the automorphism $\theta_k \circ \cdots \circ \theta_1$ of $(A/\mathfrak{m}^l)[N]$. Then $\theta_{\mathfrak{D},\gamma} = \varprojlim \theta^l_{\mathfrak{D},\gamma}$.

We say two scattering diagrams $\mathfrak{D}$, $\mathfrak{D}'$ are *equivalent* if $\theta_{\mathfrak{D},\gamma} = \theta_{\mathfrak{D}',\gamma}$ for all paths γ such that $\theta_{\mathfrak{D},\gamma}$ and $\theta_{\mathfrak{D}',\gamma}$ are defined.

A version of the following result was proved in dimension two in [KS06], §10. The general case follows from [GS11].

Theorem 5.2. *(*[GHKK14]*, Theorem 1.12) Let $\mathfrak{D}_{\text{in}}$ be a scattering diagram such that the support of each wall is a hyperplane. Then there is a scattering diagram $\mathfrak{D} = \operatorname{Scatter}(\mathfrak{D}_{\text{in}})$ containing $\mathfrak{D}_{\text{in}}$ such that*

(1) $\mathfrak{D} \setminus \mathfrak{D}_{\text{in}}$ *consists of outgoing walls, and*
(2) $\theta_{\mathfrak{D},\gamma} = \mathrm{id}$ *for all loops γ such that $\theta_{\mathfrak{D},\gamma}$ is defined.*

Moreover, $\mathfrak{D}$ is uniquely determined up to equivalence by these properties.

The theorem is proved modulo $\mathfrak{m}^l$ for each $l \in \mathbb{N}$ by induction on l. The inductive step is an explicit algorithmic construction. A self-contained proof in dimension two is given in [GPS10], Theorem 1.4. The basic construction in the general case is the same, cf. [GHKK14], Appendix C.

5.2. **Initial scattering diagram for cluster variety.** Let (U, σ) be a cluster variety. Recall the combinatorial data from §3.3 describing U in terms of a toric model $\pi\colon (X, D) \to (\bar{X}, \bar{D})$: Let $T = \bar{X} \setminus \bar{D} \simeq (\mathbb{C}^{\times})^n$ be the big torus, $N = H_1(T, \mathbb{Z}) \simeq \mathbb{Z}^n$, and $M = N^*$. We have $\bar{\sigma} = \sigma|_T \in H^0(\Omega^2_{\bar{X}}(\log \bar{D})) = \wedge^2 M_{\mathbb{C}}$ a non-degenerate skew matrix. We have primitive vectors $m_i \in M$, $v_i \in N$, $i = 1, \ldots, r$ such that $\bar{\sigma}(v_i, \cdot) = \nu_i m_i$, some $\nu_i \in \mathbb{C}^\times$. The rays $\mathbb{R}_{\geq 0} \cdot v_i$ are contained in the fan of $\bar{X}$ so correspond to components $\bar{D}_i \subset \bar{D}$. Then π is given by the blow up of the smooth centers

$$Z_i = \bar{D}_i \cap \overline{(z^{m_i} = \lambda_i)} \subset \bar{X}$$

for some $\lambda_i \in \mathbb{C}^\times$.

For the cluster variety U, we define

$$\mathfrak{D}_{\text{in}} = \{(m_i^\perp, 1 + t_i z^{v_i}) \mid i = 1, \ldots, r\}.$$

The enumerative interpretation is as follows. The strict transform of the divisor $\overline{(z^{m_i} = \lambda_i)} \subset \bar{X}$ in U is swept out by holomorphic discs ending on SYZ fibers L with boundary class $v_i \in H_1(L, \mathbb{Z}) = N$. These are the holomorphic discs corresponding to the ith initial wall. The two dimensional case is explained in Example 4.6. In dimension $n > 2$, U is locally isomorphic to a product $U' \times (\mathbb{C}^\times)^{n-2}$ (see Remark 3.5).

Example 5.3. Let $r = 2$ and $v_1, v_2 = (1,0), (0,1) \in N = \mathbb{Z}^2$. Then $\mathfrak{D} = \text{Scatter}(\mathfrak{D}_{\text{in}})$ consists of the two incoming walls $(\mathbb{R} \cdot (1,0), 1 + t_1 z^{(1,0)})$ and $(\mathbb{R} \cdot (0,1), 1 + t_2 z^{(0,1)})$ and one outgoing wall $(\mathbb{R}_{\geq 0}(-1,-1), 1 + t_1 t_2 z^{(1,1)})$.

Here is the enumerative interpretation of the outgoing wall. The cluster variety U has toric model $\pi \colon (X, D) \to (\bar{X}, \bar{D})$ where $\bar{X} = \mathbb{P}^2$ with toric boundary $\bar{D} = \bar{D}_1 + \bar{D}_2 + \bar{D}_3$, and π is given by blowing up two points p_1, p_2 in the smooth locus of $\bar{D}$, with $p_1 \in \bar{D}_1$ and $p_2 \in \bar{D}_2$. Let C be the strict transform of the line through p_1 and p_2. Then C meets D in a single point p. Holomorphic discs associated to the outgoing wall are approximated by holomorphic discs contained in $C \setminus \{p\}$. (One can also give an explicit description using [CO06] as in Example 4.6.)

Note that in general the walls of the scattering diagram may be dense in some regions of $N_{\mathbb{R}}$, and the attached functions are not polynomial. See e.g. [GPS10], Example 1.6 and Remark 5.5 below.

For $U = X \setminus D$ a log Calabi–Yau surface with maximal boundary, [GPS10] proves an enumerative interpretation of the scattering diagram in terms of virtual counts of maps $f \colon \mathbb{P}^1 \to X$ meeting the boundary D in a single point. A similar interpretation in terms of log Gromov-Witten invariants is expected in general [GS16], §2.4.

The following lemma (which will be needed in §5.4 below) is left as an exercise.

Lemma 5.4. *Let U be a cluster variety with associated combinatorial data $v_i \in N$, $m_i \in M$, $i = 1, \ldots, r$. Then*

$$\text{Pic}(U) = \text{im}(H^2(X, \mathbb{Z}) \to H^2(U, \mathbb{Z})) = \text{coker}((v_1, \ldots, v_r)^T \colon M \to \mathbb{Z}^r)$$

and

$$\pi_1(U) = H_1(U,\mathbb{Z}) = \operatorname{coker}((v_1,\dots,v_r)\colon \mathbb{Z}^r \to N).$$

5.3. **Reduction to irreducible case and sign convention.** For (U,σ) a cluster variety such that $H_1(U,\mathbb{Q}) = 0$, there is an étale cover $\tilde{U} \to U$ which decomposes as a product of cluster varieties (U_i,σ_i), $U_i = X_i \setminus D_i$, such that $H^0(\Omega^2_{X_i}(\log D_i)) = \mathbb{C}\cdot\sigma_i$ for each i (cf. the Bogomolov decomposition theorem in the compact setting).

We now assume that $H^0(\Omega^2_X(\log D)) = \mathbb{C}\cdot\sigma$. It follows from §3.1 that the subspace $H^0(\Omega^2_X(\log D)) \subset H^0(\Omega^2_{\bar{X}}(\log \bar{D})) = \wedge^2 M_{\mathbb{C}}$ is defined over $\mathbb{Q}$. So we may assume, multiplying by a nonzero scalar, that $\sigma \in \wedge^2 M$. Then $\bar{\sigma}(v_i,\cdot) = \nu_i m_i$ for some $\nu_i \in \mathbb{Q}$. Note that the blow up description of $U = X \setminus D$ depends on m_i through the center $Z_i = D_i \cap \overline{(z^{m_i} = \lambda_i)}$. So we may assume (replacing m_i by $-m_i$ and λ_i by λ_i^{-1} if necessary) that $\nu_i > 0$.

In this case, it follows from the proof of Theorem 5.2 that for all walls $(\mathfrak{d}, f)$ in $\mathfrak{D} = \operatorname{Scatter}(\mathfrak{D}_{\text{in}})$ we have $\mathfrak{d} \subset m^\perp$ for some nonzero $m \in M$ such that $m = \sum a_i m_i$ with $a_i \geq 0$ for each i. In particular, if the m_i are linearly independent then $\mathfrak{D}$ has two chambers given by

$$\mathcal{C}^+ = \{v \in N_{\mathbb{R}} \mid \langle m_i, v\rangle \leq 0 \text{ for all } i = 1,\dots,r\}$$

and $\mathcal{C}^- := -\mathcal{C}^+$.

5.4. **Reduction to the case of linearly independent m_i: Universal deformation and universal torsor.** We now explain how to reduce to the case that the $m_i \in M$ are linearly independent. Assume for simplicity that M is generated by the m_i. Equivalently, by Lemma 5.4, the Fock–Goncharov mirror V of U is simply connected.

The surjection $(m_1,\dots,m_r)\colon \mathbb{Z}^r \to M$ determines a surjective homomorphism $\varphi\colon (\mathbb{C}^\times)^r \to T^\vee$ and, dually, an injective homomorphism $T \hookrightarrow (\mathbb{C}^\times)^r$

We have the universal deformation $p\colon \mathcal{U} \to S := (\mathbb{C}^\times)^r/T$ of $U = p^{-1}([(\lambda_i)])$ given by varying the parameters λ_i.

Our assumption implies that $\operatorname{Pic} V = \operatorname{coker}(m_1,\dots,m_r)^T$ is torsion-free by Lemma 5.4. Let $L_1,\dots,L_s$ be a basis of $\operatorname{Pic} V$. The *universal torsor* $q\colon \tilde{V} \to V$ is the fiber product of the $\mathbb{C}^\times$-bundles $L_i^\times$ over V. It is a principal bundle with group $\operatorname{Hom}(\operatorname{Pic} V, \mathbb{C}^\times) = \ker(\varphi)$.

The 2-form σ on U lifts canonically to a *relative* 2-form $\sigma_{\mathcal{U}}$ on $\mathcal{U}/S$ (non-degenerate on each fiber). Equivalently, $\sigma_{\mathcal{U}}$ defines a Poisson bracket on $\mathcal{U}$ with symplectic leaves the fibers of p. The 2-form $\sigma^\vee$ on V pulls back to a degenerate 2-form on $\tilde{V}$.

Write $N_{\mathcal{U}} = \mathbb{Z}^r$ with standard basis $e_1,\dots,e_r$ and $M_{\mathcal{U}} = N_{\mathcal{U}}^*$ with dual basis $f_1,\dots,f_r$. We have the inclusion $(m_1,\dots,m_r)^T\colon N \subset N_{\mathcal{U}}$. The variety $(\mathcal{U},\sigma_{\mathcal{U}})$ is a (generalized) cluster variety with toric model given by the combinatorial data $N_{\mathcal{U}}$,

$\bar{\sigma}_{\mathcal{U}} = \bar{\sigma} \in \wedge^2 M_{\mathbb{C}}$, $v_i \in N_{\mathcal{U}}$, $f_i \in M_{\mathcal{U}}$, $i = 1, \ldots, r$. The variety $(\tilde{V}, q^*\sigma^\vee)$ is the (generalized) Fock–Goncharov mirror cluster variety. Cf. [GHK15b], §4.

Roughly speaking, in the terminology of Fock and Goncharov, $\mathcal{U}$ is the $\mathcal{X}$-variety for the given combinatorial data and $\tilde{V}$ is the $\mathcal{A}$-variety for the Langlands dual data. The Fomin–Zelevinsky (upper) cluster algebra is the ring of global functions $H^0(\tilde{V}, \mathcal{O}_{\tilde{V}})$.

One can construct the scattering diagram associated to $\mathcal{U}$ in $(N_{\mathcal{U}})_{\mathbb{R}} \supset N_{\mathbb{R}}$ as before using the relative 2-form $\bar{\sigma}_{\mathcal{U}} = \bar{\sigma} \in \wedge^2 M_{\mathbb{C}}$. (Condition (1) in Definition 5.1 can be rewritten $\bar{\sigma}^\vee(-m, \cdot) = \nu^\vee v$, where $\nu^\vee = \nu^{-1}$, that is, the corresponding condition for the Fock–Goncharov mirror V. In this form it generalizes to the above setting.)

Note that $H^0(\tilde{V}, \mathcal{O}_{\tilde{V}}) = \oplus_{L \in \operatorname{Pic} V} H^0(V, L)$ is the Cox ring of V. The torus $\operatorname{Hom}(\operatorname{Pic} V, \mathbb{C}^\times)$ acts with weight $L \in \operatorname{Pic} V$ on the summand $H^0(V, L)$. Our construction of a canonical basis of $H^0(\tilde{V}, \mathcal{O}_{\tilde{V}})$ is equivariant for the torus action. In particular we obtain a canonical basis of $H^0(V, \mathcal{O}_V)$ (and also $H^0(V, L)$ for each $L \in \operatorname{Pic} V$). Thus we may replace (U, σ) and $(V, \sigma^\vee)$ by $(\mathcal{U}, \sigma_{\mathcal{U}})$ and $(\tilde{V}, q^*\sigma^\vee)$ and assume that the m_i are linearly independent.

5.5. **Mutation invariance of the support of the scattering diagram and cluster complex.** The support of the scattering diagram $\mathfrak{D}$ is invariant under mutation [GHKK14], §1.3. That is, if $\pi \colon (X, D) \to (\bar{X}, \bar{D})$ and $\pi' \colon (X, D) \to (\bar{X}', \bar{D}')$ are two toric models for (X, D) related by a mutation $\mu \colon T \dashrightarrow T'$ as in §3.2, and $\mathfrak{D}$, $\mathfrak{D}'$ are the scattering diagrams associated to π, π' then $\mu^{\text{trop}}(\operatorname{Supp} \mathfrak{D}) = \operatorname{Supp} \mathfrak{D}'$. Heuristically, this is so because $\operatorname{Supp} \mathfrak{D}$ is the union of the tropicalizations of all holomorphic discs in U ending on SYZ fibers, viewed in $N_{\mathbb{R}}$ using the $\mathbb{Z}$PL identification $U^{\text{trop}}(\mathbb{R}) \simeq N_{\mathbb{R}}$ corresponding to the open inclusion $T = \bar{X} \setminus \bar{D} \subset U = X \setminus D$ of log Calabi–Yau varieties.

Let $\mu = \mu_{(m,v)}$ as in (3.7). Let (m_i, v_i), $i = 1, \ldots, r$, be the combinatorial data for the toric model π, with $(m, v) = (m_1, v_1)$. Recall the explicit formula in Example 3.10 for μ^{trop}. In particular, μ^{trop} is linear on the halfspaces $\mathcal{H}_+ = (m \geq 0)$ and $\mathcal{H}_- = (m \leq 0)$. Let T_+, T_- be the linear automorphisms of $N_{\mathbb{R}}$ which agree with μ^{trop} on $\mathcal{H}_+$ and $\mathcal{H}_-$. Then $T_+ = \text{id}$ and T_- is the symplectic transvection

$$T_-(w) = w - \langle m, w \rangle v = w - \nu^{-1} \cdot \bar{\sigma}(v, w) v.$$

Then the combinatorial data (m_i', v_i'), $i = 1, \ldots, r$ for π' is given by

$$(m_i', v_i') = \begin{cases} (-m_1, -v_1) & \text{if } i = 1 \\ ((T_+^*)^{-1}(m_i), T_+(v_i)) & \text{if } v_i \in \mathcal{H}_+ \text{ and } i > 1 \\ ((T_-^*)^{-1}(m_i), T_-(v_i)) & \text{if } v_i \in \mathcal{H}_- \text{ and } i > 1. \end{cases}$$

(The sign reversal $v_1' = -v_1$ follows from the description of the elementary transformation in §3.2. The signs of the m_i' are determined by the sign convention of §5.3.)

Using the explicit formula in Example 3.10 for μ^{trop}, we see that the chambers $\mu^{\mathrm{trop}}(\mathcal{C}^+)$ and $\mathcal{C}^{+'}$ in $\mathfrak{D}'$ meet along the codimension 1 face defined by $m = 0$.

Applying elementary transformations repeatedly, we obtain a simplicial fan $\Delta^+ \subset U^{\mathrm{trop}}(\mathbb{R}) \simeq N_{\mathbb{R}}$ with maximal cones the positive chambers $\mathcal{C}^+_\alpha$ associated to each torus chart T_α, such that two maximal cones meet along a codimension 1 face if and only if the torus charts are related by a mutation. This is the Fock–Goncharov *cluster complex* (the dual graph is the Fomin–Zelevinsky *exchange graph*). The maximal cones of Δ^+ are chambers of the scattering diagram $\mathfrak{D}$. Thus the scattering diagram is discrete in the interior of the support of Δ^+ in $N_{\mathbb{R}}$.

Remark 5.5. Note that here we are using our assumption that the m_i are linearly independent (see §5.4). Without this assumption, the scattering diagram can be everywhere dense in $N_{\mathbb{R}}$. For example, this is the case for $U = X \setminus D$ where $X \subset \mathbb{P}^3$ is a smooth cubic surface and D is a triangle of lines on X (equivalently, (X, D) is obtained from $\bar{X} = \mathbb{P}^2$ together with its toric boundary $\bar{D}$ by blowing up six general points in the smooth locus of $\bar{D}$, two on each line). To see this, first observe that we can construct another toric model of U of the same combinatorial type as follows. Let $\bar{D}_1, \bar{D}_2, \bar{D}_3$ be the components of $\bar{D}$. Let $\bar{X}^1$ be the blowup of the point $p = \bar{D}_1 \cap \bar{D}_2 \in \bar{X}$. Then $\bar{X}^1$ is a ruled surface with sections the exceptional divisor E and the strict transform of $\bar{D}_3$. Let p_1, p_2 be the two centers of $\pi \colon X \to \bar{X}$ on $\bar{D}_3$ and let $\bar{X}^1 \dashrightarrow \bar{X}^2$ be the composite of the elementary transformations with centers p_1 and p_2. Finally, blow down the strict transform of $\bar{D}_3$ to obtain $\bar{X}' \simeq \mathbb{P}^2$ with toric boundary $\bar{D}'$ given by the strict transforms of $\bar{D}_1$, $\bar{D}_2$, and E. Then by construction we have another toric model $\pi' \colon (X', D') \to (\bar{X}', \bar{D}')$ of U given by blowing up two points on each boundary divisor. The rational map $T = \bar{X} \setminus \bar{D} \dashrightarrow T' = \bar{X}' \setminus \bar{D}'$ is a composite of two mutations. Let $v_1, v_2 \in N$ correspond to the boundary divisors $\bar{D}_1, \bar{D}_2$ of $\bar{X}$ under the identification $U^{\mathrm{trop}}(\mathbb{Z}) = N$ given by $T \subset U$, then the boundary divisor E of $\bar{X}'$ correponds to $v_1 + v_2 \in N$ under this identification. Recall that the scattering diagram associated to a toric model has an incoming wall associated to each blow up. The support of the wall contains the ray corresponding to the boundary divisor containing the center of the blow up. Moreover, the support of the scattering diagram is invariant under mutation. In particular, it follows that the rays generated by v_1, $v_1 + v_2$, and v_2 lie in $\mathrm{Supp}(\mathfrak{D})$. Repeating the above construction one can prove by induction that every rational ray lies in $\mathrm{Supp}(\mathfrak{D})$.

In terms of the construction of the versal deformation $\mathcal{U}$ of U in §5.4, the scattering diagram $\mathfrak{D}$ for U is the slice of the scattering diagram $\mathfrak{D}_{\mathcal{U}}$ for $\mathcal{U}$ by the subspace

$(m_1, \ldots, m_r)\colon N_{\mathbb{R}} \hookrightarrow \mathbb{R}^r$. This slice can miss the discrete part of $\mathfrak{D}_{\mathcal{U}}$ so that there are no chambers in $\mathfrak{D}$.

The functions attached to walls of the scattering diagram change in a simple way under mutation. Each wall of the cluster complex corresponds to a portion of an incoming wall for some seed (and there are no outgoing walls with support contained in an incoming wall [GHKK14], Remark 1.29). So one can describe the functions attached to walls of the cluster complex explicitly. One finds that they are polynomials (in fact of the form $1 + cz^v$ for $c \in A = \mathbb{C}[t_1, \ldots, t_r]$ a monomial and $v \in N$). So one can define an algebraic family $\mathcal{V}/\mathbb{A}^r_{t_1,\ldots,t_r}$ as follows: $\mathcal{V}$ is a union of copies of $T^\vee \times \mathbb{A}^r$ indexed by chambers $\mathcal{C}^+ \in \Delta^+$, with transition maps $T^\vee \times \mathbb{A}^r \dashrightarrow T^\vee \times \mathbb{A}^r$ given by $\theta_{\mathfrak{D},\gamma}$ for γ a path in $\operatorname{Supp} \Delta^+$ from the first chamber to the second. One finds that the restriction of $\mathcal{V}/\mathbb{A}^r$ to $(\mathbb{C}^\times)^r$ is the family of Fock–Goncharov mirrors to U. Moreover, because the automorphisms are trivial modulo $\mathfrak{m} = (t_1, \ldots, t_r)$, the special fiber $\mathcal{V}_0$ equals the torus $T^\vee$.

5.6. **Broken lines.** We now describe the construction of global functions ϑ_q, $q \in U^{\mathrm{trop}}(\mathbb{Z})$ on the mirror family $\mathcal{V}/\mathbb{A}^r$ using a tropical analogue of the heuristic construction described in §4.2.

Definition 5.6. Let $v \in U^{\mathrm{trop}}(\mathbb{Z}) = N$ be a nonzero vector and $p \in N_{\mathbb{R}}$ a general point. A *broken line* for v with endpoint p is a continuous piecewise-linear path $\gamma\colon (-\infty, 0] \to N_{\mathbb{R}}$ together with, for each domain of linearity $L \subset (-\infty, 0]$, a monomial $c_L \cdot z^{v_L}$, $c_L \in A = \mathbb{C}[t_1, \ldots, t_r]$, $v_L \in N$, such that

(1) There is an initial unbounded domain of linearity with attached monomial $1 \cdot z^v$.
(2) For all L and $t \in L$, $\gamma'(t) = -v_L$.
(3) If γ is not linear at $t \in (-\infty, 0]$ then γ crosses a wall at time t. Let L and L' be the domains of linearity before and after crossing the wall and θ the wall crossing automorphism. Then $c_{L'} z^{v_{L'}}$ is a monomial term in $\theta(c_L z^{v_L}) \in \widehat{A[N]}$.
(4) $\gamma(0) = p$.

We write $M(\gamma)$ for the final monomial attached to a broken line γ.

Recall that the Fock–Goncharov mirror V of U is a union $V = \bigcup_{\mathcal{C}^+ \in \Delta^+} T^\vee_{\mathcal{C}^+}$ of copies of the dual torus $T^\vee = M \otimes \mathbb{C}^\times$ indexed by the maximal cones $\mathcal{C}^+$ of the cluster complex Δ^+. We have the family $\mathcal{V} \to \operatorname{Spec} A = \mathbb{A}^r_{t_1,\ldots,t_r}$, $\mathcal{V} = \bigcup_{\mathcal{C}^+ \in \Delta^+} T^\vee_{\mathcal{C}^+} \times \mathbb{A}^r$, with fiber V over the point $t_i = c_i^\vee = -1/\lambda_i^\vee$, and its formal completion $\hat{\mathcal{V}} \to \operatorname{Spf} \hat{A}$ over $0 \in \mathbb{A}^r$.

We now define theta functions ϑ_v for $v \in U^{\mathrm{trop}}(\mathbb{Z}) \simeq N$ on $\hat{\mathcal{V}}$. We define $\vartheta_0 = 1$. Let $v \in N$ be a nonzero vector. For $p \in \operatorname{Supp} \Delta^+$ a general point, we define

$$\vartheta_{v,p} = \sum_{\gamma} M(\gamma) \in \widehat{A[N]},$$

where the sum is over broken lines γ for v with endpoint p. For general points $p, p' \in \operatorname{Supp} \Delta^+$, and γ a path from p to p', we have $\theta_{\mathfrak{D},\gamma}(\vartheta_{v,p}) = \vartheta_{v,p'}$ [GHKK14], Theorem 3.5, [CPS], §4. So, by the definition of $\mathcal{V}/\mathbb{A}^r$, the $\vartheta_{v,p}$ for $p \in \operatorname{Supp} \Delta^+$ general define a global function ϑ_v on $\hat{\mathcal{V}}$.

Example 5.7. Consider the scattering diagram $\mathfrak{D}$ for the data $r = 1$, $v_1 = (1, 0) \in N = \mathbb{Z}^2$. Write $z_1 = z^{(1,0)}$ and $z_2 = z^{(0,1)}$. The scattering diagram $\mathfrak{D} = \mathfrak{D}_{\mathrm{in}}$ consists of the single wall $(\mathbb{R} \cdot (1, 0), 1 + tz_1)$. Let $v = (0, 1)$. For $p = (a, b) \in N_{\mathbb{R}} = \mathbb{R}^2$, if $b > 0$ there is a unique broken line for v with endpoint p given by $\gamma(t) = (a, b) - t(0, 1)$ with attached monomial z_2. If $b < 0$ there are two broken lines, one described as before and the second given by $\gamma(t) = (a - b, b) - t(0, 1)$ for $t \leq b$ with attached monomial z_2 and $\gamma(t) = (a, b) - t(1, 1)$ for $b \leq t \leq 0$, with attached monomial tz_1z_2. Thus $\vartheta_{v,p} = z_2$ for $b > 0$ and $\vartheta_{v,p} = z_2 + tz_1z_2 = z_2(1 + tz_1)$ for $b < 0$. This is the tropical version of Example 4.6. See [CGMMRSW17] for more examples.

Recall that $\mathcal{V}_0 = T^\vee$, and note that ϑ_v restricts to the character z^v on $\mathcal{V}_0$ (because $M(\gamma) \equiv 0 \bmod \mathfrak{m}$ for any broken line that bends). So the ϑ_v, $v \in U^{\mathrm{trop}}(\mathbb{Z})$ restrict to a basis of $H^0(\mathcal{V}_0, \mathcal{O}_{\mathcal{V}_0})$. It follows that the ϑ_v, $v \in U^{\mathrm{trop}}(\mathbb{Z})$ define a topological basis of $H^0(\hat{\mathcal{V}}, \mathcal{O}_{\hat{\mathcal{V}}})$. That is, for every element $f \in H^0(\hat{\mathcal{V}}, \mathcal{O}_{\hat{\mathcal{V}}})$ there is a unique expression $f = \sum_{v \in U^{\mathrm{trop}}(\mathbb{Z})} a_v \vartheta_v$ where $a_v \in \hat{A}$ for each v and for all $l \in \mathbb{N}$ there are finitely many a_v such that $a_v \not\equiv 0 \bmod \mathfrak{m}^l$.

The formal function ϑ_v defines a function on $\mathcal{V}$ (and so on the fiber V) if and only if the local expressions $\vartheta_{v,p} \in \widehat{A[N]}$ lie in $A[N]$, that is, are Laurent polynomials with coefficients in A. This is not the case in general. However, one can show that if $\vartheta_{v,p}$ is a Laurent polynomial for some $p \in \operatorname{Supp} \Delta^+$, then the same is true for all p, so that ϑ_v lies in $H^0(\mathcal{V}, \mathcal{O}_{\mathcal{V}})$ [GHKK14], Proposition 7.1.

Example 5.8. Let $\mathcal{C}^+$ be a chamber of Δ^+ and $v \in \mathcal{C}^+ \cap U^{\mathrm{trop}}(\mathbb{Z})$ an integral point. Let $p \in \mathcal{C}^+$ be a general point. Then there is a unique broken line for v and p, given by $\gamma \colon (-\infty, 0] \to N_{\mathbb{R}}$, $\gamma(t) = p - tv$, with attached monomial z^v. See [GHKK14], Corollary 3.9. It follows that ϑ_v is a global function on V such that $\vartheta_v|_{T^\vee_{\mathcal{C}^+}} = z^v$. In the terminology of Fomin–Zelevinsky, ϑ_v is a *cluster monomial*.

Example 5.9. For cluster algebras of *finite type* [FZ03], the cluster complex has finitely many cones and is a complete fan, that is, $\operatorname{Supp} \Delta^+ = U^{\mathrm{trop}}(\mathbb{R}) \simeq N_{\mathbb{R}}$. In this case, every theta function is a cluster monomial, and the cluster monomials form a basis of $H^0(V, \mathcal{O}_V)$.

Cluster algebras of finite type correspond to finite root systems [FZ03]. The mirror U of the A_2-cluster variety V is described in Example 5.3. Let $\vartheta_1, \ldots, \vartheta_5$ be the theta functions on $\mathcal{V}/\mathbb{A}^2$ corresponding to the primitive generators of the rays of the cluster complex in cyclic order. We identify V with the fiber of $\mathcal{V}$ over $(1,1) \in \mathbb{A}^2$. The theta function basis of $H^0(V, \mathcal{O}_V)$ is given by the cluster monomials

$$\{\vartheta_i^a \vartheta_{i+1}^b \mid a, b \in \mathbb{Z}_{\geq 0},\, i \in \mathbb{Z}/5\mathbb{Z}\}.$$

The algebra structure is given by

$$V = (\vartheta_{i-1}\vartheta_{i+1} = \vartheta_i + 1,\, i \in \mathbb{Z}/5\mathbb{Z}) \subset \mathbb{A}^5_{\vartheta_1,\ldots,\vartheta_5}.$$

The closure of V in $\mathbb{P}^5$ is the del Pezzo surface of degree 5 (the blowup of 4 points in $\mathbb{P}^2$ in general position) with an anti-canonical cycle of 5 (-1)-curves at infinity. In this case, the mirror V of U is isomorphic to U (since U is rigid this is a special case of Remark 4.12).

Theorem 5.10. *(*[GHKK14]*, Proposition 0.7) Let $U^{\mathrm{trop}}(\mathbb{R}) = N_{\mathbb{R}}$ be the identification associated to some toric model of U. Suppose that the support of the cluster complex Δ^+ is not contained in a half-space in $N_{\mathbb{R}}$ under this identification. Then each ϑ_v defines a global function on $\mathcal{V}$, and the ϑ_v, $v \in U^{\mathrm{trop}}(\mathbb{Z})$ define a $\mathbb{C}$-basis of $H^0(V, \mathcal{O}_V)$.*

See [GHKK14], §8 for the relation between the hypothesis and positivity of U.

Example 5.11. ([GHKK14], Corollary 0.20, [M17], cf. [GS15]). Let $G = \mathrm{SL}_m$. Let $B \subset G$ be a Borel subgroup, $N \subset B$ the maximal unipotent subgroup, and $H \subset B$ a maximal torus. Let $F = G/B$ be the full flag variety, and $\tilde{F} = G/N$ its universal torsor, a principal $H = B/N$-bundle over F. The variety $\tilde{F}$ is called the *base affine space*. By the Borel–Weil–Bott theorem,

$$H^0(\tilde{F}, \mathcal{O}_{\tilde{F}}) = \oplus_{L \in \operatorname{Pic} F} H^0(F, L) = \oplus_\lambda V_\lambda,$$

the direct sum of the irreducible representations of G (where $\lambda \in \mathrm{Lie}(H)^*$ denotes a dominant weight). Cf. [FH91], p. 392–3.

Let $B_- \subset G$ be the opposite Borel subgroup such that $B \cap B_- = H$. Let $V \subset F$ be the open subset of flags transverse to the flags with stabilizers B and B_-. Let $\tilde{V} \subset \tilde{F}$ be its inverse image. Then $\tilde{V}$ is identified with the double Bruhat cell $G^{w_0,e} := Bw_0B \cap B^- \subset G$ where $w_0 \in W = S_m$, $w_0(i) = m+1-i$, is the longest element of the Weyl group. In particular, $\tilde{V}$ is a cluster variety in the sense of Fomin–Zelevinsky [BFZ05]. (From our point of view, there are algebraic tori T_1 and T_2, an action of T_1 on $\tilde{V}$ and a T_1 invariant fibration $\tilde{V} \to T_2$, such that the quotients $\tilde{V}_t/T_1$ of the fibers are cluster varieties in the sense of Definition 3.1. Thus $\tilde{V}$ is given by a combination of the two constructions in §5.4.) We remark that the cluster structure on $\tilde{V}$ is closely related to the Poisson structure on G associated to the choice $H \subset B \subset G$ [GSV10], §1.3, which is the first order term of the non-commutative deformation of G to the quantum group [CP95],§7.3.

One can show using a generalization of Theorem 5.10 that the theta functions give a canonical basis of $H^0(\tilde{V}, \mathcal{O}_{\tilde{V}})$. The variety $\tilde{F}$ is a partial compactification of $\tilde{V}$, such that $\tilde{F} \setminus \tilde{V}$ is a union of $2(m-1)$ boundary divisors along which the holomorphic volume form Ω on $\tilde{V}$ has a pole. Using positivity properties of the theta function basis $\mathcal{B}$ of $H^0(\tilde{V}, \mathcal{O}_{\tilde{V}})$, one can further show that the subalgebra $H^0(\tilde{F}, \mathcal{O}_{\tilde{F}})$ has basis given by the subset of $\mathcal{B}$ consisting of theta functions which are regular on $\tilde{F}$. The set of such functions is indexed by the integral points of a polyhedral cone in $\tilde{U}^{\mathrm{trop}}(\mathbb{R})$ (where $\tilde{U}$ denotes the Fock–Goncharov mirror of $\tilde{V}$). This polyhedral cone C is given by $C = (W^{\mathrm{trop}} \geq 0)$ where $W\colon \tilde{U} \to \mathbb{A}^1$ is the regular function given by the sum of the theta functions on $\tilde{U}$ corresponding to the boundary divisors of the partial compactification $\tilde{V} \subset \tilde{F}$. (Thus, according to general principles, $\tilde{F}$ is mirror to the *Landau–Ginzburg model* $W\colon \tilde{U} \to \mathbb{A}^1$, cf. [AAK16], §2.2.) For a specific choice of toric model of $\tilde{U}$, the cone C is identified with the Gel'fand–Tsetlin cone [GS15].

The theta function basis is equivariant for the action of H on $\tilde{F}$. Thus the theta functions with weight λ give a canonical basis of the irreducible representation V_λ.

Remark 5.12. The construction of theta functions given here appears to depend on the choice of a toric model of U. However, one can show, using the behavior of the scattering diagram under mutation, that the families $\mathcal{V}/\mathbb{A}^r$ together with the theta functions for different toric models are compatible [GHKK14], Theorem 6.8. In the terminology of mirror symmetry, they correspond to different large complex structure limits of the mirror family.

References

[AAK16] M. Abouzaid, D. Auroux, and L. Katzarkov, Lagrangian fibrations on blowups of toric varieties and mirror symmetry for hypersurfaces, Publ. Math. Inst. Hautes Études Sci. 123 (2016), 199–282.

[AKMW02] D. Abramovich, K. Karu, K. Matsuki, and J. Wlodarczyk, Torification and factorization of birational maps, J. Amer. Math. Soc. 15 (2002), no. 3, 531–572.

[A09] D. Auroux, Special Lagrangian fibrations, wall-crossing, and mirror symmetry, Surv. Differ. Geom., 13, 1–47, Int. Press, 2009.

[A14] D. Auroux, A beginner's introduction to Fukaya categories, Contact and symplectic topology, 85–136, Bolyai Soc. Math. Stud. 26, János Bolyai Math. Soc., Budapest, 2014.

[AKO06] D. Auroux, L. Katzarkov, and D. Orlov, Mirror symmetry for del Pezzo surfaces: vanishing cycles and coherent sheaves, Invent. Math. 166 (2006), no. 3, 537–582.

[BFZ05] A. Berenstein, S. Fomin, and A. Zelevinsky, Cluster algebras III: Upper bounds and double Bruhat cells, Duke Math. J. 126 (2005), no. 1, 1–52.

[CPS] M. Carl, M. Pumperla, and B. Siebert, A tropical view of Landau-Ginzburg models, available at www.math.uni-hamburg.de/home/siebert/ .

[CP95] V. Chari and A. Pressley, A guide to quantum groups, C.U.P., 1995.

[CGMMRSW17] M. Cheung, M. Gross, G. Muller, G. Musiker, D. Rupel, S. Stella, H. Williams, The greedy basis equals the theta basis: a rank two haiku, J. Combin. Theory Ser. A 145 (2017), 150–171.

[CO06] C-H. Cho and Y-G. Oh, Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds, Asian J. Math. 10 (2006), no. 4, 773–814.

[dFKX12] T. de Fernex, J. Kollár, and C. Xu, The dual complex of singularities, preprint arXiv:1212.1675v3.

[D71] P. Deligne, Théorie de Hodge. II, Inst. Hautes Études Sci. Publ. Math. 40 (1971), 5–57.

[FG06] V. Fock and A. Goncharov, Moduli spaces of local systems and higher Teichmller theory, Publ. Math. Inst. Hautes Études Sci. 103 (2006), 1–211.

[FZ02] S. Fomin and A. Zelevinsky, Cluster algebras. I. Foundations, J. Amer. Math. Soc. 15 (2002), no. 2, 497–529.

[FZ03] S. Fomin and A. Zelevinsky, Cluster algebras. II. Finite type classification, Invent. Math. 154 (2003), no. 1, 63–121.

[F93] W. Fulton, Introduction to toric varieties, Ann. of Math. Stud. 131, P.U.P., 1993.

[FH91] W. Fulton and J. Harris, Representation theory: A first course, Grad. Texts in Math. 129, Springer, 1991.

[G13] S. Ganatra, Symplectic cohomology and duality for the wrapped Fukaya category, preprint arXiv:1304.7312v1.

[GSV10] M. Gekhtman, M. Shapiro, and A. Vainshtein, Cluster algebras and Poisson geometry, Mathematical Surveys and Monographs 167, A.M.S., 2010.

[GS15] A. Goncharov and L. Shen, Geometry of canonical bases and mirror symmetry, Invent. Math. 202 (2015), no. 2, 487–633.

[GHK15a] M. Gross, P. Hacking, and S. Keel, Mirror symmetry for log Calabi–Yau surfaces I, Publ. Math. Inst. Hautes Études Sci. 122 (2015), 65–168.

[GHK15b] M. Gross, P. Hacking, and S. Keel, Birational geometry of cluster algebras, Algebr. Geom. 2 (2015), no. 2, 137–175.

[GHKK14] M. Gross, P. Hacking, S. Keel, and M. Kontsevich, Canonical bases for cluster algebras, preprint arXiv:1411.1394v2.

[GPS10] M. Gross, R. Pandharipande, and B. Siebert, The tropical vertex, Duke Math. J. 153 (2010), no. 2, 297–362.

[GS11] M. Gross, B. Siebert, From real affine geometry to complex geometry, Ann. of Math. (2) 174 (2011), no. 3, 1301–1428.

[GS16] M. Gross and B. Siebert, Intrinsic mirror symmetry and punctured Gromov-Witten invariants, preprint arXiv:1609.00624v3.

[IM71] V. Iskovskih and Y. Manin, Three-dimensional quartics and counterexamples to the Lüroth problem, Math. USSR-Sb. 15 (1971), 141–166.

[K15] A. Keating, Homological mirror symmetry for hypersurface cusp singularities, preprint arXiv:1510.08911v2.

[KM98] J. Kollár and S. Mori, Birational geometry of algebraic varieties, Cambridge Tracts in Math. 134, C.U.P., 1998.

[KX16] J. Kollár and C. Xu, The dual complex of Calabi-Yau pairs, Invent. Math. 205 (2016), no. 3, 527–557.

[K95] M. Kontsevich, Homological algebra of mirror symmetry, Proceedings of the International Congress of Mathematicians, Vol. 1 (Zürich, 1994), 120–139, Birkhäuser, 1995.

[KS01] M. Kontsevich and Y. Soibelman, Homological mirror symmetry and torus fibrations, Symplectic geometry and mirror symmetry, 203–263, World Sci. Publ., 2001.

[KS06] M. Kontsevich and Y. Soibelman, Affine structures and non-Archimedean analytic spaces, 321–385, Progr. Math. 244, Birkhäuser, 2006.

[M17] T. Magee, GHK mirror symmetry, the Knutson-Tao hive cone, and Littlewood-Richardson coefficients, preprint arXiv:1709.05776v1.

[MFK94] D. Mumford, J. Fogarty, F. Kirwan, Geometric invariant theory, 3rd ed., Ergeb. Math. Grenzgeb. (2) 34, Springer, 1994.

[P13] J. Pascaleff, On the symplectic cohomology of log Calabi-Yau surfaces, preprint arXiv:1304.5298v2.

[P03] A. Polishchuk, Abelian varieties, theta functions and the Fourier transform, Cambridge Tracts in Math. 153, C.U.P., 2003.

[S02] P. Seidel, Fukaya categories and deformations, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 351–360, Higher Ed. Press, Beijing, 2002.

[S08] P. Seidel, A biased view of symplectic cohomology, in Current developments in mathematics 2006, 211–253, Int. Press, 2008.

[S13] P. Seidel, Lectures on Categorical Dynamics and Symplectic Topology, lecture notes from a course at MIT, Spring 2013, available at www-math.mit.edu/∼seidel/ .

[SYZ96] A. Strominger, S-T. Yau, and E. Zaslow, Mirror symmetry is T-duality, Nuclear Phys. B 479 (1996), no. 1–2, 243–259.

Department of Mathematics and Statistics, Lederle Graduate Research Tower, University of Massachusetts, Amherst, MA 01003-9305

E-mail address: `hacking@math.umass.edu`

Department of Mathematics, 1 University Station C1200, Austin, TX 78712-0257

E-mail address: `keel@math.utexas.edu`