THE MIRROR OF THE CUBIC SURFACE

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To Miles Reid on the occasion of his 70^{th} *birthday*

Contents

Introduction

A number of years ago, one of us (M.G.) was giving a lecture at the University of Warwick on the material on scattering diagrams from [GPS]. Of course Miles was in the audience, and he asked (paraphrasing as this was many years ago) whether, at some point, the lecturer would come back down to earth. The goal of this note is to show in fact we have not left the planet by considering a particularly beautiful example of the mirror symmetry construction of [GHK11], namely the mirror to a cubic surface.

More precisely, the paper [GHK11], building on [GS11], [GPS] and [CPS], constructs mirrors of rational surfaces equipped with anti-canonical cycles of rational curves. Specifically, one begins with the data of a pair (Y, D), where Y is a non-singular projective rational surface over an algebraically closed field $\Bbbk$ of characteristic 0, and $D \in |-K_Y|$ is an effective reduced anti-canonical divisor with at least one node, necessarily then forming a wheel of projective lines. Choose in addition a finitely generated, saturated sub-monoid $P \subset H_2(Y, \mathbb{Z})$ whose only invertible element is 0, such that P contains the class of every effective curve on Y. Let $\mathfrak{m}$ denote the maximal monomial ideal of the monoid ring $\Bbbk[P]$ and $\widehat{\Bbbk[P]}$ denote the completion of $\Bbbk[P]$ with respect to $\mathfrak{m}$. Then the main construction of [GHK11] produces a family of formal schemes $\mathfrak{X} \to \operatorname{Spf} \widehat{\Bbbk[P]}$ which is interpreted as the mirror family to the pair (Y, D). In the more pleasant case when D supports an ample divisor, the construction is in fact algebraic: there is a family $X \to S := \operatorname{Spec} \Bbbk[P]$ of affine surfaces extending the above formal family. In general, if D has $n \geq 3$ components, then X is a closed subscheme of $\mathbb{A}^n_S$, with central fibre a reducible union of n copies of $\mathbb{A}^2$.

[GHK11], Example 6.13, contains the equation[1] for X in the case that Y is a cubic surface in $\mathbb{P}^3$ and $D = D_1 + D_2 + D_3$ is a triangle of lines. The intent was to include a proof of this in [GHKII], which, at the time, was circulated rather narrowly in an extreme rough draft form.

As [GHKII] has seen no change for more than five years, and many pieces of it have been cannibalized for other papers or become out-of-date, it seemed that, in the grand tradition of second parts of papers, this paper is unlikely to ever see the light of day. On the other hand, the full details of the cubic surface have not appeared anywhere else, although Lawrence Barrott [B18] verifies the given equation for the mirror of the cubic surface. The cubic surface is in particular especially attractive. This is unsurprising, given the rich classical geometry of the cubic (see e.g., [Reid]). So we felt that it would be a pity for this construction never to appear. Further, since [GHKII] first began to circulate, the technology for understanding the product rule for theta functions on the mirror, and hence the equations for the mirror, has improved at a theoretical level, see [GS18],[GS19],[KY19]. Thus in particular it will be possible to give a completely enumerative interpretation for the equations to the mirror cubic. This gives us an opportunity to exposit a number of different viewpoints on the construction here.

Without further ado, here is the main result. Describe the pair (Y, D) as follows. First fix the pair $(\mathbb{P}^2, \bar{D} = \bar{D}_1 + \bar{D}_2 + \bar{D}_3)$ where $\bar{D}$ is a triangle of lines. Let (Y, D) be obtained as the blow-up of two general distinct points on each of the three lines, with D the strict transform of $\bar{D}$. Let E_{ij}, $i = 1, 2, 3$, $j = 1, 2$ be the exceptional curves, with E_{ij} intersecting D_i. For $i = 1, 2$ or 3, denote by L_{ij}, $1 \leq j \leq 8$, the eight lines on the cubic surface not contained in D but intersecting D_i. We note that $\{E_{i1}, E_{i2}\} \subseteq \{L_{ij} \mid 1 \leq j \leq 8\}$.

Theorem 0.1. *Taking $P = \mathrm{NE}(Y)$, the cone of effective curves of Y, $S = \operatorname{Spec} \Bbbk[P]$, the mirror family defined over S to the cubic surface $(Y, D = D_1 + D_2 + D_3)$ is given by the equation in $\mathbb{A}^3_S$:*

$$\vartheta_1\vartheta_2\vartheta_3 = \sum_i z^{D_i}\vartheta_i^2 + \sum_i \left(\sum_j z^{L_{ij}}\right) z^{D_i}\vartheta_i + \sum_\pi z^{\pi^* H} + 4z^{D_1+D_2+D_3}.$$

Here for a curve class C, z^C denotes the corresponding monomial of $\Bbbk[P]$, and $\vartheta_1, \vartheta_2, \vartheta_3$ are the coordinates on the affine 3-space. The sum over π is the sum over all possible

[1] Unfortunately with a sign error!

birational morphisms $\pi\colon Y \to Y'$ *of* (Y, D) *to a pair* (Y', D') *isomorphic to* $\mathbb{P}^2$ *with its toric boundary, with* $\pi|_D : D \to D'$ *an isomorphism and* H *the class of a line in* $\mathbb{P}^2$*.*

The original guess for the shape of these equations was motivated by the paper [Ob04], which gave a similar equation for a non-commutative cubic surface. Once one knows the shape of the equation, it is not difficult to verify it, as we shall see.

Finally, we note that this paper does not intend to be a complete exposition of the ideas of [GHK11], but rather, we move quickly to discuss the cubic surface. For a more comprehensive expository account, see the forthcoming work of Argüz [Ar19].

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1. The tropicalization of the cubic surface

We explain the basic combinatorial data we associate to the pair (Y, D), namely a pair (B, Σ) where:

- B is an *integral linear manifold with singularities*;
- Σ is a decomposition of B into cones.

First, an integral linear manifold B is a real manifold with coordinate charts $\psi_i : U_i \to \mathbb{R}^n$ (where $\{U_i\}$ is an open covering of B) and transition maps $\psi_i \circ \psi_j^{-1} \in \mathrm{GL}_n(\mathbb{Z})$. An *integral linear manifold with singularities* is a manifold B with an open set $B_0 \subseteq B$ and $\Delta = B \setminus B_0$ of codimension at least 2 such that B_0 carries an integral linear structure.

We build B and Σ by pretending that the pair (Y, D) is a toric variety. If it were, we could reconstruct its fan in $\mathbb{R}^2$ (up to $\mathrm{GL}_2(\mathbb{Z})$) knowing the intersection numbers of the irreducible components D_i of D. So we just start constructing a fan and we will run into trouble when (Y, D) isn't a toric variety. This problem is fixed by introducing a singularity in the linear structure of $\mathbb{R}^2$ at the origin.

Explicitly, for the cubic surface, $D_i^2 = -1$ for $1 \leq i \leq 3$, and we proceed as follows. Take rays in $\mathbb{R}^2$ corresponding to D_1 and D_2 to be $\rho_1 := \mathbb{R}_{\geq 0}(1, 0)$ and $\rho_2 := \mathbb{R}_{\geq 0}(0, 1)$ respectively. See the left-hand picture in Figure 1.1.

Since $D_2^2 = -1$, toric geometry instructs us that the ray corresponding to D_3 would be $\rho_3 := \mathbb{R}_{\geq 0}(-1, 1)$ if (Y, D) were a toric pair. Indeed, if ρ_1, ρ_2, ρ_3 are successive rays in a two-dimensional fan defining a non-singular complete toric surface, and if n_i is the primitive generator of ρ_i and D_i is the divisor corresponding to ρ_i, we have the relation

$$n_1 + D_2^2 n_2 + n_3 = 0.$$

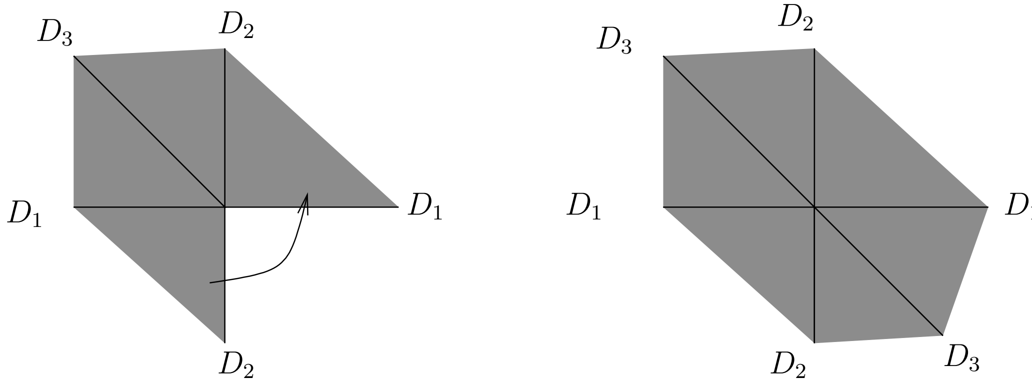

FIGURE 1.1.

Thus with $n_1 = (1,0)$ and $n_2 = (0,1)$, n_3 is determined by D_2^2. As $D_3^2 = -1$, we then need a ray corresponding to D_1 to be $\mathbb{R}_{\geq 0}(-1,0)$, which does not coincide with the ray ρ_1 (telling us that (Y,D) wasn't really a toric pair). If we continue, we obtain a new ray $\mathbb{R}_{\geq 0}(0,-1)$ for D_2 also. Thus we have two cones spanned by the rays corresponding to D_1 and D_2, and there is an integral linear transformation identifying these two cones. In this case, this transformation is $-\mathrm{id}$. After cutting out the fourth quadrant from $\mathbb{R}^2$ and gluing the third and first quadrants via $-\mathrm{id}$, we obtain the integral affine manifold B, along with a decomposition (or fan) Σ into rational polyhedral cones. Here the cones of Σ consist of $\{0\}$, the images of the rays ρ_1, ρ_2, ρ_3, and three two-dimensional cones $\sigma_{i,i+1}$ $i = 1,2,3$, with indices taken mod 3, and $\sigma_{i,i+1}$ having faces ρ_i and ρ_{i+1}. Note the rays correspond to irreducible components of D and the two-dimensional cones to double points of D.

To see the details of this construction in general, and further examples, see [GHK11], §1.2.

We use the convention that $v_i \in B$ is the primitive integral point on the ray ρ_i, so that any element of $\sigma_{i,i+1}$ can be written as $av_i + bv_{i+1}$ for some $a, b \in \mathbb{R}_{\geq 0}$.

While we have just described the general construction for (B, Σ) as applied to our particular case, in fact there is a more elegant description for the cubic surface. By continuing to build the fan, we close up to get a fan $\widetilde{\Sigma}$ in $\mathbb{R}^2$. Then $B = \mathbb{R}^2/\langle -\mathrm{id}\rangle$, and $\widetilde{\Sigma}$ descends to Σ on the quotient. See the right-hand side of Figure 1.1.

Remark 1.1. Note that $\widetilde{\Sigma}$ defines a toric variety $\widetilde{Y}$ which is a del Pezzo surface of degree 6. The automorphism $-\mathrm{id}$ of $\widetilde{\Sigma}$ induces an involution $\iota : \widetilde{Y} \to \widetilde{Y}$, which is given on the dense torus orbit as $(z_1, z_2) \mapsto (z_1^{-1}, z_2^{-1})$. This surface can be embedded in $\mathbb{P}^3$ as follows. First, one maps the quotient of the dense torus orbit of $\widetilde{Y}$ to $\mathbb{A}^3$ using the map

$$(z_1, z_2) \mapsto (z_1 + z_1^{-1}, z_2 + z_2^{-1}, z_1^{-1}z_2 + z_1 z_2^{-1}).$$

The image satisfies the equation $x_1x_2x_3 = x_1^2+x_2^2+x_3^2-4$, which is then projectivized to obtain the Cayley cubic given by the equation

$$x_1x_2x_3 = x_0(x_1^2+x_2^2+x_3^2) - 4x_0^3,$$

the unique cubic surface with four ordinary double points, the images of the fixed points of ι. This is in fact isomorphic to $\widetilde{Y}/\langle\iota\rangle$. □

Remark 1.2. We write $B_0 = B \setminus \{0\}$. Let $B_0(\mathbb{Z})$ be the subset of B_0 of points with integer coordinates with respect to any integral linear chart. Set $B(\mathbb{Z}) = B_0(\mathbb{Z}) \cup \{0\}$.

The set $B(\mathbb{Z})$ has another natural interpretation, as the *tropicalization* of the log Calabi-Yau manifold $U = Y \setminus D$, see [GHK13], Definition 1.7. Here one takes a nowhere vanishing 2-form Ω on U with at worst simple poles along D, and we have

$$B(\mathbb{Z}) = \{\text{divsorial discrete valuations } \nu : k(U)^* \to \mathbb{Z} \,|\, \nu(\Omega) < 0\} \cup \{0\}.$$

The advantage of this description is that an automorphism of U which does not extend to an automorphism of Y still induces an automorphism of $B(\mathbb{Z})$, which in general extends to a piecewise linear automorphism of B. □

It will also be useful to consider piecewise linear functions on B with respect to the fan Σ, i.e., continuous functions $F : B \to \mathbb{R}$ which restrict to linear functions on each $\sigma \in \Sigma$. Just as in the toric case, there is in fact a one-to-one correspondence between such functions with integral slopes and divisors supported on the boundary. Indeed, each boundary divisor D_i defines a piecewise linear function on B, written as $\langle D_i, \cdot\rangle$, uniquely defined by the requirement that

$$\langle D_i, v_j\rangle = \delta_{ij}.$$

Additively, this allows us to obtain a PL function $\langle D', \cdot\rangle$ associated to any divisor D' supported on D. Conversely, given a piecewise linear function $F : B \to \mathbb{R}$ with integral slopes, we obtain a divisor $\sum_i F(v_i)D_i$ supported on D.

2. The scattering diagram associated to the cubic surface

As (B, Σ) only involves purely combinatorial information about (Y, D), it is insufficient to determine an interesting mirror object. We need to include extra data of a *scattering diagram* on (B, Σ).

Before doing so, we need to select some additional auxiliary data, namely a monoid $P \subseteq H_2(Y, \mathbb{Z})$ of the form $\sigma_P \cap H_2(Y, \mathbb{Z})$ for $\sigma_P \subseteq H_2(Y, \mathbb{R})$ a strictly convex rational polyhedral cone which contains all effective curve classes. In the case of the cubic surface, we may take σ_P to be the Mori cone, i.e., the cone generated by all effective curve classes. We write $\Bbbk[P]$ for the corresponding monoid ring, and $\mathfrak{m} \subseteq \Bbbk[P]$ for the maximal monomial ideal, generated by $\{z^p \,|\, p \in P \setminus \{0\}\}$.

Definition 2.1. A *ray* in B is a pair $(\mathfrak{d}, f_{\mathfrak{d}})$ where:

(1) $\mathfrak{d} \subseteq \sigma_{i,i+1}$ for some i is a ray generated by some $av_i + bv_{i+1} \neq 0$, $a, b \in \mathbb{Z}_{\geq 0}$ relatively prime. We call $\mathfrak{d}$ the *support* of the ray.

(2) $f_{\mathfrak{d}} = 1 + \sum_{k\geq 1} c_k X_i^{-ak} X_{i+1}^{-bk} \in \Bbbk[P][\![X_i^{-a}X_{i+1}^{-b}]\!]$ with $c_k \in \mathfrak{m}$ for all k, satisfying the property that for any monomial ideal $I \subseteq \Bbbk[P]$ with $\Bbbk[P]/I$ Artinian, (i.e., I is co-Artinian), $f_{\mathfrak{d}} \mod I$ is a finite sum.

Definition 2.2. A *scattering diagram* $\mathfrak{D}$ for B is a collection of rays with the property that for each co-Artinian monomial ideal $I \subseteq \Bbbk[P]$,

$$\mathfrak{D}_I := \{(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D} \,|\, f_{\mathfrak{d}} \not\equiv 1 \mod I\}$$

is finite. We assume further that $\mathfrak{D}$ contains at most one ray with a given support.

The purpose of a scattering diagram is to give a way of building a flat family over $\operatorname{Spec} A_I$, where $A_I := \Bbbk[P]/I$. Explicitly, suppose given a scattering diagram $\mathfrak{D}$ and a co-Artinian ideal I. Assume further that each ρ_i is the support of a ray $(\rho_i, f_i) \in \mathfrak{D}$. This ray is allowed to be trivial, i.e., $f_i = 1$. Now define rings

$$R_{i,I} := A_I[X_{i-1}, X_i^{\pm 1}, X_{i+1}]/(X_{i-1}X_{i+1} - z^{[D_i]} X_i^{-D_i^2} f_i)$$
$$R_{i,i+1,I} := A_I[X_i^{\pm 1}, X_{i+1}^{\pm 1}].$$

Here $z^{[D_i]}$ is the monomial in $\Bbbk[P]$ corresponding to the class of the boundary curve D_i, necessarily lying in P by the assumption that P contains all effective curve classes.

Localizing, note we have canonical isomorphisms

$$(R_{i,I})_{X_{i+1}} \cong R_{i,i+1,I} \text{ and } (R_{i,I})_{X_{i-1}} \cong R_{i-1,i,I}.$$

Set

$$U_{i,I} := \operatorname{Spec} R_{i,I} \text{ and } U_{i,i+1,I} := \operatorname{Spec} R_{i,i+1,I}.$$

Note that if $I = \mathfrak{m}$, then $U_{i,I}$ is the reducible variety defined by $X_{i-1}X_{i+1} = 0$ in $\mathbb{A}^2_{X_{i-1},X_{i+1}} \times (\mathbb{G}_m)_{X_i}$, where the subscripts denote the coordinates on the respective factors. On the other hand, $U_{i,i+1,I} = (\mathbb{G}_m^2)_{X_i,X_{i+1}}$. For more general I, we instead obtain thickenings of these schemes just described.

For any I, we have canonical open immersions $U_{i-1,i,I}, U_{i,i+1,I} \hookrightarrow U_{i,I}$. As $U_{i,I}$ and $U_{i,\mathfrak{m}}$ have the same underlying topological space, we can describe the underlying open sets in $U_{i,\mathfrak{m}}$ of these two open immersions as subsets of $V(X_{i-1}X_{i+1}) \subseteq \mathbb{A}^2 \times \mathbb{G}_m$ as follows. We have $U_{i-1,i,I}$ is given by the open set where $X_{i-1} \neq 0$ (hence $X_{i+1} = 0$) and $U_{i,i+1,I}$ is given by the open set where $X_{i+1} \neq 0$ (hence $X_{i-1} = 0$). Thus in particular the images of these immersions are disjoint. Thus, if for all i we glue $U_{i,I}$ and $U_{i+1,I}$ via the canonically identified copies of $U_{i,i+1,I}$, there is no cocycle gluing condition to check and we obtain a scheme X_I° flat over $\operatorname{Spec} A_I$.

It is easy to describe this if we take $I = \mathfrak{m}$. One obtains in this case that $X_I^\circ = \mathbb{V}_n \setminus \{0\}$, where n is the number of irreducible components of D and, assuming $n \geq 3$,

$$\mathbb{V}_n = \mathbb{A}^2_{x_1,x_2} \cup \cdots \cup \mathbb{A}^2_{x_{n-1},x_n} \cup \mathbb{A}^2_{x_n,x_1} \subseteq \mathbb{A}^n = \operatorname{Spec} \Bbbk[x_1, \ldots, x_n],$$

where $\mathbb{A}^2_{x_i,x_{i+1}}$ denotes the affine coordinate plane in $\mathbb{A}^n$ for which all coordinates but x_i, x_{i+1} are zero. Here, $\mathbb{V}_n$ is called the *n-vertex*.

The problem is that for I general, X_I° may be insufficiently well-behaved to extend to a flat deformation of $\mathbb{V}_n$. To do so, we need to perturb the gluings we made above, and the role of the scattering diagram is to provide a data structure for doing so.

Let $\gamma : [0,1] \to \operatorname{Int}(\sigma_{i,i+1})$ be a path. We define an automorphism of $R_{i,i+1,I}$ called the *path ordered product*. Assume that whenever γ crosses a ray in $\mathfrak{D}_I$ it passes from one side of the ray to the other. In particular, suppose γ crosses a given ray

$$(\mathfrak{d} = \mathbb{R}_{\geq 0}(av_i + bv_{i+1}), f_\mathfrak{d}) \in \mathfrak{D}_I$$

with a, b relatively prime. Define the A_I-algebra homomorphism $\theta_{\gamma,\mathfrak{d}} : R_{i,i+1,I} \to R_{i,i+1,I}$ by

$$\theta_{\gamma,\mathfrak{d}}(X_i) = X_i f_\mathfrak{d}^{\mp b}$$
$$\theta_{\gamma,\mathfrak{d}}(X_{i+1}) = X_{i+1} f_\mathfrak{d}^{\pm a}$$

where the signs are $-b, +a$ if γ passes from the ρ_{i+1} side of $\mathfrak{d}$ to the ρ_i side of $\mathfrak{d}$, and $+b, -a$ if γ crosses in the opposite direction. Note these two choices are inverse automorphisms of $R_{i,i+1,I}$, and $f_\mathfrak{d}$ is invertible because $f_\mathfrak{d} \equiv 1 \mod \mathfrak{m}$ from Definition 2.1, (2).

If γ crosses precisely the rays $(\mathfrak{d}_1, f_{\mathfrak{d}_1}), \ldots, (\mathfrak{d}_s, f_{\mathfrak{d}_s}) \in \mathfrak{D}_I$, in that order, then we define the *path ordered product*

$$\theta_{\gamma,\mathfrak{D}} := \theta_{\gamma,\mathfrak{d}_s} \circ \cdots \circ \theta_{\gamma,\mathfrak{d}_1}.$$

Now, for each i, choose a path γ inside $\sigma_{i,i+1}$ which starts near ρ_{i+1} and ends near ρ_i so that it crosses all rays of $\mathfrak{D}_I$ intersecting the interior of $\sigma_{i,i+1}$. Then $\theta_{\gamma,\mathfrak{D}}$ induces an automorphism $\theta_{\gamma,\mathfrak{D}} : U_{i,i+1,I} \to U_{i,i+1,I}$, and we can use this to modify our gluing via

$$U_{i,I} \hookleftarrow U_{i,i+1,I} \xrightarrow{\theta_{\gamma,\mathfrak{D}}} U_{i,i+1,I} \hookrightarrow U_{i+1,I}.$$

This produces a new scheme $X_{I,\mathfrak{D}}^\circ$, still a flat deformation of $\mathbb{V}_n \setminus \{0\}$ over $\operatorname{Spec} A_I$.

Now comes the key point: we need to make a good choice of $\mathfrak{D}$ in order to be able to construct a partial compactification $X_{I,\mathfrak{D}}$ of $X_{I,\mathfrak{D}}^\circ$ such that $X_{I,\mathfrak{D}} \to \operatorname{Spec} A_I$ is a flat deformation of $\mathbb{V}_n$. One of the main ideas of [GHK11] is the use of results of [GPS] to write down a good choice of scattering diagram, the *canonical scattering diagram*, in terms of relative Gromov-Witten invariants of the pair (Y, D).

We first discuss the nature of these invariants. Choose a curve class β and a point $v \in B_0(\mathbb{Z})$, say $v = av_i + bv_{i+1}$. We sketch the construction of a Gromov-Witten type invariant N_v^β counting what we call $\mathbb{A}^1$-*curves*. Roughly speaking, these are one-pointed stable maps of genus 0, $f : (C, p) \to Y$, representing the class β, with $f^{-1}(D) = \{p\}$. Further, f has contact order $\langle D_i, v\rangle$ with D_i at p. Roughly, this contact order is the order of vanishing of the regular function $f^*(t)$ at p, for t a local defining equation for D_i at $f(p)$. However, as stated, this isn't quite right because of standard issues of compactness in relative Gromov-Witten theory. In [GHK11], these numbers are defined rigorously following [GPS] by peforming a weighted blow-up of (Y, D) at $D_i \cap D_{i+1}$ determined by $\mathfrak{d}$ and then using relative Gromov-Witten theory. As relative Gromov-Witten theory only works relative to a smooth divisor, one removes all double points of the proper transform of D under this blow-up, and then shows that this doesn't interfere with compactness of the moduli space. We refer to [GHK11], §3.1 for the precise definition, as we will not need here the subtleties of the general definition. However, we note that in order for such a map to exist, and hence possibly have $N_v^\beta \neq 0$, we must have β an effective curve class and

$$\beta \cdot D_j = \langle D_j, v\rangle.$$

A more modern definition of these invariants is via logarithmic Gromov-Witten theory, as developed by [GS11],[AC14],[C14]. Using that theory, one can allow contact orders with multiple divisors simultaneously, and thus do not need to perform the weighted blow-up. It follows from invariance of logarithmic Gromov-Witten theory under toric blow-ups [AW18] and the comparison theorem of relative and logarithmic invariants [AMW] that these two definitions agree.

Definition 2.3. The *canonical scattering diagram* $\mathfrak{D}_{\mathrm{can}}$ of (Y, D) consists of rays $(\mathfrak{d}, f_\mathfrak{d})$ ranging over all possible supports $\mathfrak{d} \subseteq B$ where, if $\mathfrak{d} \subseteq \sigma_{i,i+1}$ with $\mathfrak{d} = \mathbb{R}_{\geq 0}(av_i + bv_{i+1})$ and a, b relatively prime, then

$$f_\mathfrak{d} = \exp\left(\sum_{k\geq 1}\sum_{\beta\in H_2(Y,\mathbb{Z})} kN^\beta_{akv_i+bkv_{i+1}} z^\beta (X_i^{-a}X_{i+1}^{-b})^k\right).$$

We now return to the cubic, where $\mathfrak{D}_{\mathrm{can}}$ is particularly interesting. One might also consider higher degree del Pezzo surfaces. However, del Pezzo surfaces of degree $6, 7, 8$ and 9 are all toric, assuming one takes as D the toric boundary, and they have a trivial scattering diagram (i.e., all $f_{\mathfrak{d}} = 1$ as the invariants N_v^β are always zero). The case of a degree 5 del Pezzo surface was considered as a running example in [GHK11], see e.g., Example 3.7 there. A degree 4 surface is not that much more complicated, see [B18] for details. On the other hand, for the cubic surface, no $f_{\mathfrak{d}}$ is 1, but nevertheless we can essentially determine $f_{\mathfrak{d}}$. On the other hand, the degree 2 del Pezzo surface requires use of a computer to analyze, see [B18].

To describe curve classes on the cubic surface Y, we use the description of Y as a blow-up of $\mathbb{P}^2$ given in the introduction, so that $H_2(Y,\mathbb{Z})$ is generated by the classes of the exceptional divisors E_{ij}, $1 \le i \le 3$, $1 \le j \le 2$, and the class L of a pull-back of a line in $\mathbb{P}^2$.

With this notation, we have:

Proposition 2.4. *The ray (ρ_i, f_{ρ_i}) satisfies*

$$f_{\rho_i} = \frac{\prod_{j=1}^{8}(1 + z^{L_{ij}} X_i^{-1})}{(1 - z^{D_k + D_\ell} X_i^{-2})^4},$$

where the L_{ij} as in the introduction are the lines not contained in D but meet D_i, and $\{i, k, \ell\} = \{1, 2, 3\}$.

Proof. We take $i = 1$, the other cases following from symmetry. We need to calculate the numbers $N_{kv_1}^\beta$. In particular, for β to be represented by an $\mathbb{A}^1$-curve contributing to $N_{kv_1}^\beta$, we must have $\beta \cdot D_1 = k$ and $\beta \cdot D_i = 0$ for $i \neq 1$.

We will first consider those curve classes β which may be the curve class of a generically injective map $f : \mathbb{P}^1 \to Y$ with the above intersection numbers with the D_i. Write

$$\beta = aL - \sum_{i,j} b_{ij} E_{ij}.$$

Then

$$k = \beta \cdot D_1 = a - b_{11} - b_{12}, \quad 0 = \beta \cdot D_2 = a - b_{21} - b_{22}, \quad 0 = \beta \cdot D_3 = a - b_{31} - b_{32}.$$

Thus

$$a = b_{21} + b_{22} = b_{31} + b_{32}.$$

Further, we must have $p_a(f(C)) \ge 0$, so by adjunction and the fact that $K_Y = -D$,

$$-2 \le 2p_a(f(C)) - 2 = \beta \cdot (\beta + K_Y) = a^2 - \sum_{i,j} b_{ij}^2 - k. \tag{2.1}$$

Now of course the curve classes E_{11}, E_{12} satisfy the above equalities and inequality, with $k = 1$, while E_{ij}, $i \neq 1$ do not. Then any other class of an irreducible curve which may contribute necessarily has $a > 0$ and $b_{ij} \ge 0$. Let us fix a and k and try to maximize the right-hand side of (2.1) in the hopes that we can make it at least -2. This means in particular that we should try to minimize $b_{i1}^2 + b_{i2}^2$ for $i = 2, 3$.

We split the analysis into two cases. If a is even, then this sum of squares is minimized by taking $b_{i1} = b_{i2} = a/2$. Thus we see that

$$-2 \le 2p_a(f(C)) - 2 \le a^2 - k - b_{11}^2 - b_{12}^2 - 4(a^2/4) = -k - b_{11}^2 - b_{12}^2.$$

Since $k \ge 1$, we see we immediately get three possibilities:

(1) $k = 1$, $b_{11} = 1$, $b_{12} = 0$, in which case $a = 2$ and the only possible curve class is $\beta = 2L - E_{11} - E_{21} - \cdots - E_{32}$.
(2) $k = 1$, $b_{11} = 0$, $b_{12} = 1$, in which case $a = 2$ and the only possible curve class is $\beta = 2L - E_{12} - E_{21} - \cdots - E_{32}$.
(3) $k = 2$, $b_{11} = b_{12} = 0$, in which case $a = 2$ and the only possible curve class is $2L - E_{21} - \cdots - E_{32}$.

If a is odd, then we minimize $b_{i1}^2 + b_{i2}^2$ by taking $b_{i1} = (a-1)/2$, $b_{i2} = (a+1)/2$ or vice versa. Thus

$$a^2 - b_{21}^2 - \cdots - b_{32}^2 \le a^2 - 2\left(\frac{(a-1)^2}{4} + \frac{(a+1)^2}{4}\right) = -1.$$

Again, since $k \ge 1$, the only possibility is $k = 1$, $b_{11} = b_{12} = 0$, and hence $a = 1$, giving the following possible choices for β:

$$L - E_{21} - E_{31}, \quad L - E_{21} - E_{32}, \quad L - E_{22} - E_{31}, \quad L - E_{22} - E_{32}.$$

Note that these four classes, along with E_{11}, E_{12}, and cases (1) and (2) in the a even case, represent the 8 (-1)-curves in Y which meet D_1 transversally, i.e., the curves L_{1j}. Each of these curve classes is then represented by a unique $\mathbb{A}^1$-curve, and $N_{v_1}^{\beta} = 1$ in these cases.

In the case $a = k = 2$, we consider the curve class $\beta = 2L - E_{21} - \cdots - E_{32} \sim D_2 + D_3$. Note that the linear system $|D_2 + D_3|$ induces a conic bundle $g : Y \to \mathbb{P}^1$, and D_1 is a 2-section of g, i.e., $g|_{D_1} : D_1 \to \mathbb{P}^1$ is a double cover, necessarily branched over two points $p_1, p_2 \in \mathbb{P}^1$. Thus the conics $f^{-1}(p_1)$, $f^{-1}(p_2)$ are also $\mathbb{A}^1$-curves, now with contact order 2 with D_1. So $N_{2v_1}^{\beta} = 2$.[2]

Unfortunately, these are not the only $\mathbb{A}^1$-curves, as there may be stable maps $f : (C, p) \to Y$ which are either not generically injective or don't have irreducible image. Indeed, one may have multiple covers of one of the above $\mathbb{A}^1$-curves already considered,

[2]In general, in Gromov-Witten theory, it is not enough to just count the stable maps, as there may be a virtual count. However, in all the cases just considered, the stable map $f : C \to Y$ in question is a closed immersion, and hence has no automorphisms as a stable map. Further, the obstruction space to the moduli space of stable maps at the point $[f]$ is $H^1(C, f^*T_Y(-\log D))$, which is seen without much difficulty to vanish. Hence each curve in fact contributes 1 to the Gromov-Witten number.

provided the cover is totally ramified at the point p of contact with D. However, for general choice of (Y, D), we will now show that there is no possibility of reducible images.

As argued in [GP], Lemma 4.2, the image of any $\mathbb{A}^1$-curve must be a union of irreducible curves each of which intersect the boundary at the same point. In particular, if $f(C) = C_1 \cup \cdots \cup C_n$ is the irreducible decomposition, then $D \cap C_1 \cap \cdots \cap C_n$ consists of one point, necessarily contained in D_1.

However, since $C_1, \ldots, C_n$ must be a subset of the 10 curves identified above, the possibilities are as follows. The first is that two of these curves are lines on the cubic surface, and hence we must have three lines (including D_1) intersecting in a common point. Such a point on a cubic surface is called an *Eckardt point*, see [Do], §9.1.4. However, the set of cubic surfaces containing Eckardt points is codimension one in the moduli of all cubic surfaces. Since we may assume (Y, D) is general in moduli (as the Gromov-Witten invariants being calculated are deformation invariant), we may thus assume Y has no Eckardt points, so this doesn't occur.

On the other hand, one or both of C_1, C_2 could be fibres of the conic bundle induced by $|D_2 + D_3|$. Since two distinct fibres are disjoint, they can't both be fibres of the conic bundle. Further, any line E of the cubic surface intersecting D_1 at one point has $E \cdot (D_2 + D_3) = 0$, and hence is contained in a fibre of the conic bundle g, and thus is again disjoint from a different fibre of g.

We thus come to the conclusion that any stable map contributing to the $\mathbb{A}^1$-curve count must have irreducible image, and hence be a multiple cover of one of the curves discussed above. The moduli space of such multiple covers is always positive dimensional, but happily the virtual count has been calculated in [GPS], Proposition 6.1. Degree d covers of a non-singular rational curve which meets D transversally contributes $(-1)^{d-1}/d^2$, whilst degree d covers of a non-singular rational curve which is simply tangent to D is $1/d^2$. Note

$$\exp\left(\sum_{d\geq 1} d \cdot \frac{(-1)^{d-1}}{d^2} z^{d\beta} X_1^{-d}\right) = 1 + z^\beta X_1^{-1}$$

and

$$\exp\left(\sum_{d\geq 1} 2d \cdot \frac{1}{d^2} z^{d\beta} X_1^{-2d}\right) = \frac{1}{(1 - z^\beta X_1^{-2})^2}.$$

From this the result follows. □

We now observe that the cubic surface carries sufficient symmetry so that the above computation determines the scattering diagram completely.

Noting that the group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{R}^2$ and $-\mathrm{id}$ lies in the centre of $\mathrm{SL}_2(\mathbb{Z})$, we obtain an action of $\mathrm{PGL}_2(\mathbb{Z})$ on $B = \mathbb{R}^2/\langle -\mathrm{id}\rangle$. Of course, this action acts transitively on all the rays of rational slope in B, so if we can show that this action preserves the scattering diagram in a certain sense, we will have completely determined the scattering diagram.

We first observe that there is a rotational symmetry. For example, the calculation of f_{ρ_1} equally applies to f_{ρ_2} and f_{ρ_3}, subject to a change of relevant curve classes. More generally, if we know $f_{\mathfrak{d}}$ for $\mathfrak{d} = \mathbb{R}_{\geq 0}(av_i + bv_{i+1})$, then we know it for $S(\mathfrak{d})$, where $S(av_i + bv_{i+1}) = av_{i+1} + bv_{i+2}$, with indices taken modulo 3. Here S is an automorphism of B which lifts to an automorphism of the cover $\mathbb{R}^2$, with $S(1,0) = (0,1)$ and $S(0,1) = (-1,1)$ (so that $S(-1,1) = (-1,0)$, completing the rotation). Thus on the cover, S is represented by $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. We also have an action S^* on $H_2(Y,\mathbb{Z})$ given by $S^*(L) = L$, $S^*(E_{ij}) = E_{i+1,j}$. Then we can write $f_{S(\mathfrak{d})} = S^*(f_{\mathfrak{d}})$, where the action of S^* on $f_{\mathfrak{d}}$ is given by $X_i^{-ka}X_{i+1}^{-kb} \mapsto X_{i+1}^{-ka}X_{i+2}^{-kb}$ and $z^\beta \mapsto z^{S^*(\beta)}$.

The second symmetry arises from a birational change to the boundary. We may blow-up the point of intersection of D_1 and D_2, and blow-down D_3, to obtain a surface (Y', D') with $Y' \setminus D' = Y \setminus D$. We use the convention that $D' = D_1' + D_2' + D_3'$ with D_1' the strict transform of D_1, D_2' the exceptional curve of the blow-up, and D_3' the strict transform of D_2.

But in fact Y' is still a cubic surface, and hence we may apply the calculation of Proposition 2.4 with respect to the new divisor D_2'. Because of the way $\mathbb{A}^1$-curve counts are defined, these counts do not depend on toric blow-ups and blow-downs of the boundary. Thus if we know a ray in the scattering diagram for (Y', D'), we have a corresponding ray in the scattering diagram for (Y, D). For example, it is not difficult to check that for $1 \leq j \leq 8$, the curve in the pencil $|D_3 + L_{3j}|$ passing through $D_1 \cap D_2$ has strict transform in (Y', D') a line meeting D_2'. On the other hand, the strict transform of a curve of class $D_1 + D_2 + 2D_3$ on (Y, D) which is cuspidal at $D_1 \cap D_2$ is a conic on Y' which meets D_2' tangentially.

To see this as an action on B, let B' be the integral linear manifold with singularities corresponding to (Y', D'). Then there is a canonical piecewise linear identification of B' with B arising from the description of the tropicalization of Remark 1.2. In particular, this identification sends v_1' to v_1, v_2' with $v_1 + v_2$, and v_3' with v_2. Thus if we know a ray $(\mathfrak{d}', f_{\mathfrak{d}'})$ for (Y', D'), we obtain a ray $(\mathfrak{d}, f_{\mathfrak{d}})$ for (Y, D) under this identification. Instead, we can view this identification as giving an automorphism of B, i.e., consider the automorphism T given by $v_1 \mapsto v_1$, $v_2 \mapsto v_1 + v_2$ and $v_3 \mapsto v_2$. Note this is induced by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

It is not difficult to work out the action[3]

T^* on $H_2(Y, \mathbb{Z})$. It is

$$\begin{aligned} L &\mapsto 2L - E_{31} - E_{32} \\ E_{1j} &\mapsto E_{1j} \\ E_{2j} &\mapsto L - E_{3j} \\ E_{3j} &\mapsto E_{2j} \end{aligned}$$

for $j = 1, 2$. Then the symmetry T takes a ray $(\mathfrak{d}, f_\mathfrak{d})$ to a ray $(T(\mathfrak{d}), T^*(f_\mathfrak{d}))$, where $T^*(f_\mathfrak{d})$ does the obvious thing. In particular, one ray in $\mathfrak{D}_{\mathrm{can}}$ is $\mathfrak{d} = \mathbb{R}_{\geq 0}(v_1 + v_2)$ with

$$f_\mathfrak{d} := \frac{\prod_{j=1}^{8}(1 + z^{D_3 + L_{3j}} X_1^{-1} X_2^{-1})}{(1 - z^{D_1 + D_2 + 2D_3} X_1^{-2} X_2^{-2})^4} \tag{2.2}$$

Happily, this is the only additional ray we will need to understand other than ρ_1, ρ_2 and ρ_3.

Since S and T generate $\mathrm{SL}_2(\mathbb{Z})$, we have now proved:

Theorem 2.5. *Let $\mathfrak{d} = \mathbb{R}_{\geq 0}(av_i + bv_{i+1})$ for $a, b \in \mathbb{Z}_{\geq 0}$ relatively prime. Then there exists curve classes $\beta_1, \ldots, \beta_9 \in H_2(Y, \mathbb{Z})$ such that*

$$f_\mathfrak{d} = \frac{\prod_{j=1}^{8}(1 + z^{\beta_j} X_i^{-a} X_{i+1}^{-b})}{(1 - z^{\beta_9} X_i^{-2a} X_{i+1}^{-2b})^4}.$$

We note that this $\mathrm{SL}_2(\mathbb{Z})$-action has a beautiful explanation in terms of work of Cantat and Loray [CL]. They describe the $\mathrm{SL}_2(\mathbb{C})$ character variety of the four-punctured sphere $S^2_4 = S^2 \setminus \{p_1, \ldots, p_4\}$, i.e., the variety of $\mathrm{SL}_2(\mathbb{C})$ representations of the fundamental group $\pi_1(S^2_4)$, up to conjugation by elements of $\mathrm{SL}_2(\mathbb{C})$. This character variety is naturally embedded in $\mathbb{A}^7$ with coordinates x, y, z, A, B, C, D and has equation

$$xyz + x^2 + y^2 + z^2 = Ax + By + CZ + D,$$

i.e., is a family of affine cubic surfaces whose natural compactifications in $\mathbb{P}^3$ are then precisely of the form we are considering.

Now S^2 can be viewed as a quotient of a torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ by negation, $S^2 = T^2/\langle -\mathrm{id} \rangle$, and the map $T^2 \to S^2$ has four branch points, the two-torsion points of T^2. We take the image of these branch points to be $p_1, \ldots, p_4$, so that any element

[3] In fact this action is not unique: it can always be composed with an automorphism of $H^2(Y, \mathbb{Z})$ preserving the intersection form, permuting the (-1)-curves, and keeping the boundary divisors D_1, D_2, D_3 fixed. We give one possible action.

of $\mathrm{SL}_2(\mathbb{Z})$ acting on T^2 then induces an automorphism of S^2_4, possibly permuting the punctures. Thus we obtain a $\mathrm{PGL}_2(\mathbb{Z})$ action on S^2_4, and hence a $\mathrm{PGL}_2(\mathbb{Z})$ action on the character variety, which in fact is compatible with the projection to $\mathbb{A}^4$ with coordinates A, B, C, D. An element of $\mathrm{PGL}_2(\mathbb{Z})$ permutes fibres if it permutes two-torsion points. Thus, we obtain an action of $\mathrm{PGL}_2(\mathbb{Z})$ on the "relative" tropicalization of this family of log Calabi-Yau manifolds, i.e., the set of valuations with centers surjecting onto $\mathbb{A}^4$ and with simple poles of the relative holomorphic 2-form. This can be shown to be the same action considered above generated by S and T. We omit the details.

3. Broken lines, theta functions and the derivation of the equation

We now explain how to construct theta functions, and what is special about the canonical scattering diagram. We first recall the notion of broken line, fixing here a scattering diagram $\mathfrak{D}$ and a co-Artinian ideal $I \subseteq \Bbbk[P]$.

Definition 3.1. A *broken line* γ in (B, Σ) for $q \in B_0(\mathbb{Z})$ and endpoint $Q \in B_0$ is a proper continuous piecewise integral affine map $\gamma : (-\infty, 0] \to B_0$, real numbers $t_0 = -\infty < t_1 < \cdots < t_n = 0$, and monomials m_i, $1 \le i \le n$, satisfying the following properties:

(1) $\gamma(0) = Q$.
(2) $\gamma|_{[t_{i-1},t_i]}$ is affine linear for all i, and $\gamma([t_{i-1}, t_i])$ is contained in some two-dimensional cone $\sigma_{j,j+1} \in \Sigma$, where j depends on i. Further, $m_i = c_i X_j^a X_{j+1}^b$ for some $a, b \in \mathbb{Z}$, a, b not both zero, $c_i \in \Bbbk[P]/I$, and $\gamma'(t) = -av_j - bv_{j+1}$ for any $t \in (t_{i-1}, t_i)$.
(3) If $q \in \mathrm{Int}(\sigma_{j,j+1})$ for some j, we can write $q = av_j + bv_{j+1}$ for some $a, b \in \mathbb{Z}_{\ge 0}$, and then $\gamma((-\infty, t_1]) \subset \sigma_{j,j+1}$ and $m_1 = X_j^a X_{j+1}^b$. If $q \in \rho_j$ for some j, then $\gamma((-\infty, t_1])$ is either contained in $\sigma_{j-1,j}$ or $\sigma_{j,j+1}$, and writing $q = av_j$, we have $m_1 = X_j^a$.
(4) If $\gamma(t_i)$ lies in the interior of a maximal cone of Σ then $\gamma(t_i)$ lies in the support of a ray $(\mathfrak{d}, f_{\mathfrak{d}})$ and γ passes from one side of $\mathfrak{d}$ to the other, so that $\theta_{\gamma,\mathfrak{d}}$ is defined. Then m_{i+1} is a monomial in $\theta_{\gamma,\mathfrak{d}}(m_i)$. In other words, we expand the expression $\theta_{\gamma,\mathfrak{d}}(m_i)$ into a sum of monomials, and choose m_{i+1} to be one of the terms of this sum.
(5) If $\gamma(t_i) \in \rho_j$ for some j, γ passes from $\sigma_{j-1,j}$ to $\sigma_{j,j+1}$, and $m_i = c_i X_{j-1}^a X_j^b$, then m_{i+1} is a monomial in the expression

$$c_i(z^{[D_j]} X_j^{-D_j^2} f_{\rho_j} X_{j+1}^{-1})^a X_j^b.$$

If, on the other hand, γ passes from $\sigma_{j,j+1}$ to $\sigma_{j-1,j}$, then, with $m_i = c_i X_j^a X_{j+1}^b$, m_{i+1} is a monomial in the expression

$$c_i X_j^a (z^{[D_j]} X_j^{-D_j^2} f_{\rho_j} X_{j-1}^{-1})^b.$$

In other words, in the first case, the monomial m_i, written in the variables X_{j-1}, X_j, is rewritten, using the defining equation of the ring $R_{j,I}$, in the variables X_j, X_{j+1}. The second case is similar.

Definition 3.2. Let $q \in B_0(\mathbb{Z})$ and $Q \in B_0$ be a point with irrational coordinates. Then we define

$$\vartheta_{q,Q} = \sum_{\gamma} \mathrm{Mono}(\gamma)$$

where the sum is over all broken lines for q with endpoint Q, and $\mathrm{Mono}(\gamma)$ denotes the last monomial attached to γ.

We extend this definition to $q = 0 \in B(\mathbb{Z}) \setminus B_0(\mathbb{Z})$ by setting

$$\vartheta_{0,Q} = \vartheta_0 = 1.$$

It follows from the definition of broken line that if $Q \in \sigma_{i,i+1}$ then $\vartheta_{q,Q} \in R_{i,i+1,I}$.

Definition 3.3. We say $\mathfrak{D}$ is *consistent* if for all $q \in B_0(\mathbb{Z})$ and co-Artinian ideals I,

(1) If $Q, Q' \in \sigma_{i,i+1}$ are points with irrational coordinates and γ is a path in $\sigma_{i,i+1}$ joining Q to Q', then
$$\theta_{\gamma,\mathfrak{D}}(\vartheta_{q,Q}) = \vartheta_{q,Q'}.$$
(2) If $Q \in \sigma_{i-1,i}$, $Q' \in \sigma_{i,i+1}$ are chosen sufficiently close to ρ_i such that there is no non-trivial ray of $\mathfrak{D}_I$ between Q and ρ_i or between Q' and ρ_i, then there exists an element $\vartheta_{q,\rho_i} \in R_{i,I}$ whose images in $R_{i-1,i,I}$ and $R_{i,i+1,I}$ are $\vartheta_{q,Q}$ and $\vartheta_{q,Q'}$ respectively.

One of the main theorems of [GHK11], namely Theorem 3.8, states that $\mathfrak{D}_{\mathrm{can}}$ is a consistent scattering diagram.

The benefit of a consistent scattering diagram is that the $\vartheta_{q,Q}$ for various Q can then be glued to give a global function $\vartheta_q \in \Gamma(X^\circ_{I,\mathfrak{D}}, \mathcal{O}_{X^\circ_{I,\mathfrak{D}}})$. This allows us to construct a partial compactification $X_{I,\mathfrak{D}}$ of $X^\circ_{I,\mathfrak{D}}$ by setting

$$X_{I,\mathfrak{D}} := \mathrm{Spec}\,\Gamma(X^\circ_{I,\mathfrak{D}}, \mathcal{O}_{X^\circ_{I,\mathfrak{D}}}),$$

and the existence of the theta functions ϑ_q guarantees that this produces a flat deformation of $\mathbb{V}_n$ over $\mathrm{Spec}\, A_I$, see [GHK11], §2.3 for details.

Morally, another way to think about this is that we are embedding $X^\circ_{I,\mathfrak{D}}$ in $\mathbb{A}^n_{A_I}$ using the theta functions $\vartheta_{v_1}, \ldots, \vartheta_{v_n}$, and then taking the closure.

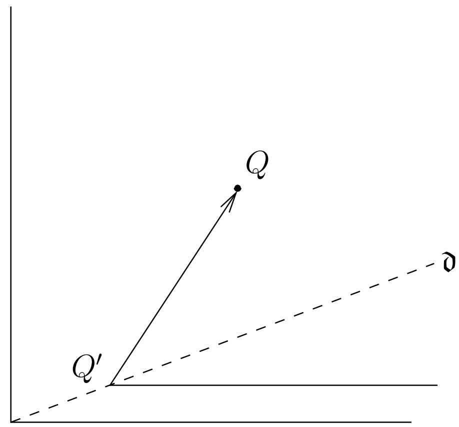

FIGURE 3.1.

Example 3.4. Unfortunately, in the cubic surface example, it is very difficult to write down expressions for theta functions. While for any fixed ideal I, $\theta_{q,Q}$ is a finite sum of monomials, in fact if we take the limit over all I we obtain an infinite sum. Here are some very simple examples of this. Take $Q = \alpha v_1 + \beta v_2$ for some irrational $\alpha, \beta \in \mathbb{R}_{>0}$, and take $q = v_1$. We give examples of broken lines for q ending at Q. Consider a ray $\mathfrak{d} = \mathbb{R}_{\geq 0}(av_1 + bv_2)$ with $(a-1)/b < \alpha/\beta < a/b$. Note that given the choice of Q, there are an infinite number of choices of relatively prime a, b satisfying this condition.

We will construct a broken line as depicted in Figure 3.1. The monomial attached to the segment coming in from infinity is X_1. If we bend along the ray $\mathfrak{d}$, we apply $\theta_{\gamma,\mathfrak{d}}$ to X_1. By Theorem 2.5, $f_{\mathfrak{d}}$ contains a non-zero term $cX_1^{-a}X_2^{-b}$ for some $c \in \Bbbk[P]$, so $\theta_{\gamma,\mathfrak{d}}(X_1)$ contains a term $c'X_1^{1-a}X_2^{-b}$. Choosing this monomial, we now proceed in the direction $(a-1, b)$. In particular, take the bending point to be

$$Q' := \left(\frac{a}{b}((1-a)\beta + b\alpha), (1-a)\beta + b\alpha\right),$$

which lies on $\mathfrak{d}$ because $(1-a)\beta + b\alpha > 0$ by the assumption that $(a-1)/b < \alpha/\beta$. Then

$$Q - Q' = (\frac{a}{b}\beta - \alpha)(a-1, b).$$

Thus the broken line reaches Q, as depicted. So we have indeed constructed a broken line, and there are an infinite number of such broken lines (albeit only a finite number modulo any co-Artinian ideal I).

Of course, here we are using only one term from the infinite power series expansion of $f_{\mathfrak{d}}$ and only considering one possible bend, and we already have an infinite number of broken lines. We believe it would be extremely difficult to get a useful description of all broken lines, and hence broken lines provide a useful theoretical, but not practical, description of theta functions.

This shows that if we take the limit over all I and obtain a formal scheme $\widehat{\mathfrak{X}} \to \operatorname{Spf} \widehat{\Bbbk[P]}$, there is no hope to express the theta functions as algebraic expressions. Thus it is perhaps a bit of a surprise that often the relations satisfied by these theta functions are much simpler, so that we can extend the construction over $\operatorname{Spec} \Bbbk[P]$. In the case of the cubic surface, we will see this explicitly by using the product rule for theta functions in Theorem 2.34 of [GHK11]:

Theorem 3.5. *Let $p_1, p_2 \in B(\mathbb{Z})$. In the canonical expansion*

$$\vartheta_{p_1} \cdot \vartheta_{p_2} = \sum_{r \in B(\mathbb{Z})} \alpha_{p_1 p_2 r} \vartheta_r,$$

where $\alpha_{p_1p_2r} \in \Bbbk[P]/I$ for each q, we have

$$\alpha_{p_1p_2r} = \sum_{\gamma_1,\gamma_2} c(\gamma_1)c(\gamma_2),$$

where the sum and notation is as follows. We fix $z \in B_0$ a point very close to r contained in the interior of a cone $\sigma_{i,i+1}$ for some i. We then sum over all broken lines γ_1, γ_2 for p_1, p_2 satisfying: (1) Both broken lines have endpoint z. (2) If $\operatorname{Mono}(\gamma_j) = c(\gamma_j)X_i^{a_j}X_{i+1}^{b_j}$ with $c(\gamma_j) \in \Bbbk[P]/I$, $j = 1, 2$, then $r = (a_1 + a_2)v_i + (b_1 + b_2)v_{i+1}$.

We shall see that in the case of the cubic surface, only a very small part of $\mathfrak{D}_{\text{can}}$ is necessary to find the equation of the mirror.

There is one more ingredient for the calculation of the equation, namely the notion of a *min-convex function* in the context of a scattering diagram. Let $F : B \to \mathbb{R}$ be a piecewise linear function on B. If γ is a broken line, we obtain a (generally discontinuous) function on $(-\infty, 0]$, the domain of γ, written as $t \mapsto dF(\gamma'(t))$. This means that at a time t, provided F is linear at $\gamma(t)$, we evaluate the differential dF at $\gamma(t)$ on the tangent vector $\gamma'(t)$. Thus $dF(\gamma'(\cdot))$ is a piecewise constant function.

We say F is *min-convex* if for any broken line γ, $dF(\gamma'(\cdot))$ is a decreasing function: see [GHKK], Definition 8.2, where the definition is given in a slightly different context. The use of such a function is that [GHKK], Lemma 8.4 applies, so that F is *decreasing* in the sense of [GHKK], Definition 8.3., i.e., if the coefficient $\alpha_{p_1p_2r} \neq 0$, then

$$F(r) \geq F(p_1) + F(p_2). \tag{3.1}$$

Indeed, suppose that γ_1, γ_2 are broken lines for p_1, p_2 respectively contributing to the expression for $\alpha_{p_1p_2r}$. Note that $F(p_i) = -dF(\gamma_i'(t))$ for $t \ll 0$, while $r = -\gamma_1'(0) - \gamma_2'(0)$. Thus for $t \ll 0$,

$$F(r) - F(p_1) - F(p_2) = \sum_{i=1}^{2} (dF(\gamma_i'(t)) - dF(\gamma_i'(0))), \tag{3.2}$$

which is positive under the decreasing assumption.

We note [B18] also makes use of such a function (with the opposite sign convention). Barrott, however, used a computer program to enumerate all contributions to the products, as his main goal was to find the mirror to a degree 2 del Pezzo, which has a considerably more complex equation than the mirror to the cubic. In the case of the cubic surface, the products can be computed by hand.

In our case, we may take $F = \langle K_Y, \cdot\rangle$. Note this pulls back to the PL function on the cover $\mathbb{R}^2 \to B = \mathbb{R}^2/\langle -\mathrm{id}\rangle$ which corresponds to $K_{\widetilde{Y}}$.

It is easy to check that in fact $dF(\gamma'(\cdot))$ decreases whenever a broken line crosses one of the rays ρ_i (this is just local convexity of F) or when a broken line bends, and hence F is decreasing. However, it will be important to quantify by how much $dF(\gamma'(\cdot))$ changes with each of these occurences. For example, suppose a broken line passes from $\sigma_{1,2}$ into $\sigma_{2,3}$ without bending, with tangent direction $av_1 + bv_2$, necessarily with $a < 0$. Then via parallel transport of this tangent vector into $\sigma_{2,3}$, we can rewrite the vector using the relation $v_1 + v_3 = -D_2^2 v_2$, i.e., $av_1 + bv_2$ is rewritten as $(a+b)v_2 - av_3$. Thus $dF(\gamma'(\cdot))$ takes the value $-(a+b)$ before crossing ρ_2, and the value $-(a+b)+a$ after crossing ρ_2, hence decreasing as $a < 0$.

If γ bends in, say, $\sigma_{i,i+1}$, then γ' changes by some $av_i + bv_{i+1} \neq 0$ for $a, b \geq 0$. But $dF(av_i + bv_{i+1}) = -a - b$, so $dF(\gamma'(\cdot))$ changes by $-(a+b) < 0$.

Note that if γ bends when it crosses ρ_i, in fact $dF(\gamma'(t))$ decreases by at least 2. These observations will be crucial for bounding the search for possible broken lines contributing to the product.

We now calculate the key products necessary to prove the main theorem.

Lemma 3.6. *We have the following products:*

$$\vartheta_{v_i}^2 = \vartheta_{2v_i} + 2z^{D_j+D_k}, \quad \{i,j,k\} = \{1,2,3\}$$

$$\vartheta_{v_1}\vartheta_{v_2} = \vartheta_{v_1+v_2} + z^{D_3}\vartheta_{v_3} + \sum_{j=1}^{8} z^{D_3+L_{3j}}$$

$$\vartheta_{v_1+v_2}\vartheta_{v_3} = z^{D_1}\vartheta_{2v_1} + z^{D_2}\vartheta_{2v_2} + \vartheta_{v_1}\sum_j z^{D_1+L_{1j}} + \vartheta_{v_2}\sum_j z^{D_2+L_{2j}} + \sum_\pi z^{\pi^*H} + 8z^{D_1+D_2+D_3}.$$

Proof. We consider first $\vartheta_{v_i}^2$. By symmetry, we can take $i = 1$. Since $F(v_1) = -1$, if ϑ_r contributes to this product, we must have $-2 \leq F(r) \leq 0$, the first inequality from (3.1) and the second since F is non-positive. Let γ_1, γ_2 be broken lines contributing to $\alpha_{v_1 v_1 r}$ as in Theorem 3.5. It follows immediately from (3.2) that if $F(r) = -2$, then γ_i neither bends nor crosses a wall. It is then obvious the only possible r in this case is $r = 2v_1$. Fixing $z \in \mathrm{Int}(\sigma_{1,2})$ near $2v_1$, we obtain the contribution from two broken lines as in the left in Figure 3.2: this is responsible for the ϑ_{2v_1} term.

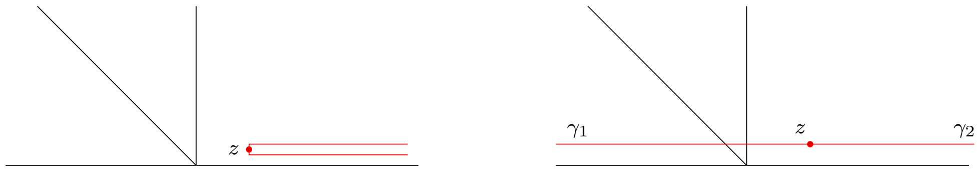

FIGURE 3.2. The contributions to the product $\vartheta_{v_1}^2$. In the left-hand picture, the two broken lines in fact lie on top of each other, but we depict them as distinct lines with endpoint z.

There are only three points $r \in B(\mathbb{Z})$ with $F(r) = -1$, namely $r = v_i$, $i = 1, 2, 3$. Now if a pair of broken lines γ_1, γ_2 contributes to $\alpha_{v_1 v_1 r}$, then one of the γ_j either bends or crosses one of the ρ_k. Such a possibility can be ruled out, however. It is easiest to work on the cover $\mathbb{R}^2 \to B = \mathbb{R}^2/\langle -\mathrm{id}\rangle$, bearing in mind that there are two possible initial directions for the lifting of a broken line γ_i, namely it can come in parallel to $\mathbb{R}_{\geq 0}(1, 0)$ or parallel to $\mathbb{R}_{\geq 0}(-1, 0)$. If $r = v_2$ or v_3, we can fix z in the interior of $\sigma_{2,3}$, and then both broken lines must cross rays to reach z. If $r = v_1$, we may take z in the interior of $\sigma_{1,2}$, and then the only possibility is that one of the γ_i bends. However, if γ_i bends at any ray of $\mathfrak{D}_{\mathrm{can}}$ not supported on one of the ρ_i, then dF decreases by at least 2, ruling out this possibility. Thus we can rule out the case $F(r) = -1$. We shall omit this kind of analysis in the sequel, as it is straightforward.

Finally, if $F(r) = 0$, then $r = 0$. Taking z in $\sigma_{1,2}$ close to the origin, we obtain the possibility shown on the right-hand side of Figure 3.2. This actually represents two possibilities, as the labels γ_1 and γ_2 can be interchanged. Each such pair of broken lines contributes $z^{D_2+D_3}\vartheta_0$, recalling that $\vartheta_0 = 1$. One checks easily that there are no possibilities where one of the broken lines bends. This gives the claimed description of $\vartheta_{v_1}^2$.

Turning to $\vartheta_{v_1} \cdot \vartheta_{v_2}$, if $F(r) = -2$, then again broken lines can't bend or cross walls. In this case, the only possibility is as depicted on the left in Figure 3.3, contributing the term $\vartheta_{v_1+v_2}$.

If $F(r) = -1$, then $r = v_1, v_2$ or v_3. By putting the endpoint z in $\sigma_{1,2}, \sigma_{2,3}$ or $\sigma_{2,3}$ respectively, a quick analysis shows the only possible contribution is from the right-hand picture in Figure 3.3, contributing $z^{D_3}\vartheta_{v_3}$.

Finally, if $F(r) = 0$, again $r = 0$. Taking z near ρ_1 and the origin in the interior of $\sigma_{1,2}$, we now obtain the possibility of γ_2 bending along the ray $\mathfrak{d} = \mathbb{R}_{\geq 0}(1, 1)$ as depicted in Figure 3.4. The bend on γ_2 is calcluated by seeing how $\theta_{\mathfrak{d},\gamma_2}$ acts on the initial monomial X_2, i.e., $X_2 \mapsto X_2 f_{\mathfrak{d}}$. By the form given for $f_{\mathfrak{d}}$ in (2.2), we get the given expression for $\vartheta_{v_1} \cdot \vartheta_{v_2}$.

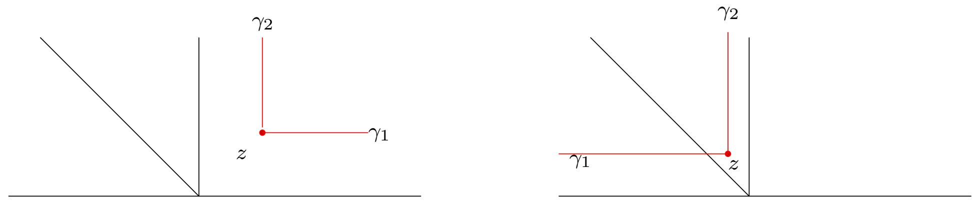

FIGURE 3.3.

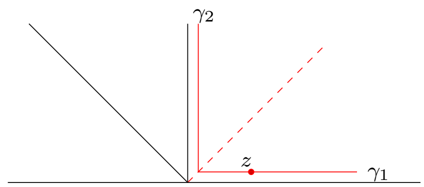

FIGURE 3.4.

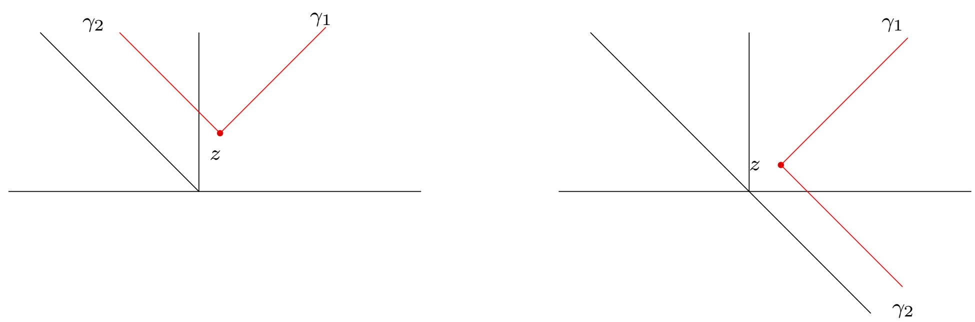

FIGURE 3.5.

Turning to $\vartheta_{v_1+v_2} \cdot \vartheta_{v_3}$, this time we have the possible range $-3 \leq F(r) \leq 0$. However, $F(r) = -3$ is impossible, as this does not allow either γ_i to cross a ray ρ_j, and necessarily γ_1 and γ_2 come in from infinity in different cones.

If $F(r) = -2$, as at least one of the γ_i crosses a ray ρ_j, no bends are possible. One then sees the two possibilities in Figure 3.5. These give rise to the contributions $z^{D_2}\vartheta_{2v_2}$ and $z^{D_1}\vartheta_{2v_1}$ respectively.

If $F(r) = -1$, then a bend is also permitted, and $r = v_1, v_2$ or v_3. By placing z in the interiors of $\sigma_{1,2}$, $\sigma_{1,2}$ or $\sigma_{2,3}$ respectively, near ρ_1, ρ_2, or ρ_3, one rules out v_3 as a possibility and has as remaining possibilities as in Figure 3.6. These contribute $\vartheta_{v_1} \sum_j z^{D_1+L_{1j}}$ and $\vartheta_{v_2} \sum_j z^{D_2+L_{2j}}$ respectively.

Finally we have $F(r) = 0$, i.e., $r = 0$. We put z in the interior of $\sigma_{1,2}$ near the origin, close to ρ_1. Then γ_1 stays in the interior of $\sigma_{1,2}$, and therefore can only bend at a ray of $\mathfrak{D}_{\mathrm{can}}$ intersecting the interior of $\sigma_{1,2}$. However, if it bends at any ray other than $\mathfrak{d} = \mathbb{R}_{\geq 0}(1,1)$, dF decreases by at least 3, while γ_2 crosses some ρ_i, which would

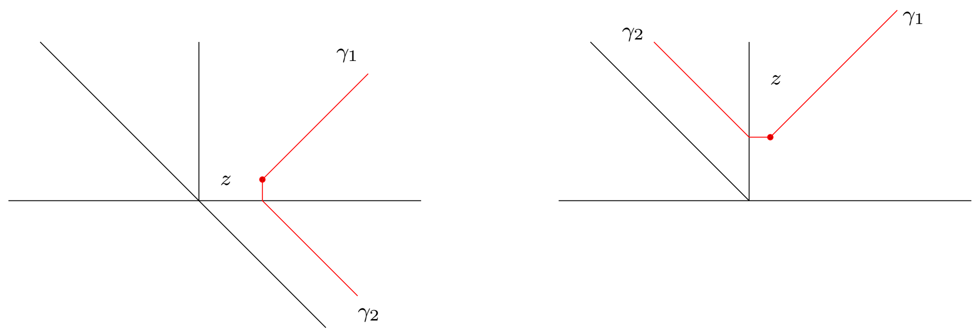

FIGURE 3.6.

require $F(r) \geq 1$. On the other hand, as γ_1 is initially parallel to $\mathfrak{d}$, it can't cross $\mathfrak{d}$, and hence γ_1 doesn't bend. This leaves only the possibility depicted in Figure 3.7. This involves rewriting X_3 using the equation $X_2X_3 = f_{\rho_1} z^{[D_1]} X_1$, i.e., $X_3 = f_{\rho_1} z^{[D_1]} X_1 X_2^{-1}$ and choosing a monomial of the form $cX_1^{-1}X_2^{-1}$ from this expression. We thus need to consider the coefficient of X_1^{-2} in f_{ρ_1}, and this is $\sum_{1\leq j<j'\leq 8} z^{L_{1j}+L_{1j'}} + 4z^{D_2+D_3}$. Thus we get a contribution of

$$\sum_{1\leq j<j'\leq 8} z^{D_1+L_{1j}+L_{1j'}} + 4z^{D_1+D_2+D_3} \tag{3.3}$$

to the product. We can give a clearer description of this expression, however. Consider the class $L_{1j}+L_{1j'}$. If $L_{1j}\cap L_{1j'} = \emptyset$, then L_{1j} and $L_{1j'}$ can be simultaneously contracted. One can easily check that given this choice of j, j', there are unique pairs $L_{2k}, L_{2k'}$ and $L_{3\ell}, L_{3\ell'}$ such that all six of these curves can be simultaneously contracted to give a morphism $\pi : (Y, D) \to (Y', D')$, where $Y' \cong \mathbb{P}^2$ and D' is the image of D. This morphism is in fact induced by the two-dimensional linear system $|D_1 + L_{1j} + L_{1j'}|$. In particular, $D_1 + L_{1j} + L_{1j'} = \pi^* H$ where H is the class of a line on Y'.

On the other hand, a plane in $\mathbb{P}^3$ containing both D_1 and L_{1j} contains a third line $L_{1j'}$ for some j' with $L_{1j}\cap L_{1j'}$ a point. Thus the set $\{L_{1j}\}$ is partitioned into four pairs, with L_{1j}, $L_{1j'}$ in the same pair if $L_{1j}\cap L_{1j'} \neq \emptyset$, in which case $D_1+L_{1j}+L_{1j'} \sim D_1+D_2+D_3$. Thus we can express (3.3) as

$$\sum_{\pi} z^{\pi^* H} + 8z^{D_1+D_2+D_3}.$$

This is responsible for the last contribution to $\vartheta_{v_1+v_2}\vartheta_{v_3}$. □

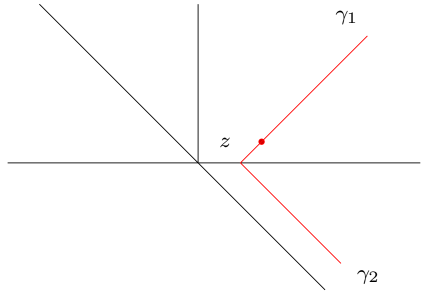

FIGURE 3.7.

Proof of Theorem 0.1. Using the lemma, we calculate

$$\begin{aligned}
\vartheta_{v_1}\vartheta_{v_2}\vartheta_{v_3} &= \left(\vartheta_{v_1+v_2} + z^{D_3}\vartheta_{v_3} + \sum_{j=1}^{8} z^{D_3+L_{3j}}\right)\vartheta_{v_3} \\
&= z^{D_1}\vartheta_{2v_1} + z^{D_2}\vartheta_{2v_2} + z^{D_3}\vartheta_{v_3}^2 + \sum_i \left(\sum_j z^{L_{ij}}\right) z^{D_i}\vartheta_{v_i} + \sum_\pi z^{\pi^* H} + 8z^{D_1+D_2+D_3} \\
&= z^{D_1}\vartheta_{v_1}^2 + z^{D_2}\vartheta_{v_2}^2 + z^{D_3}\vartheta_{v_3}^2 + \sum_i \left(\sum_j z^{L_{ij}}\right) z^{D_i}\vartheta_{v_i} + \sum_\pi z^{\pi^* H} + 4z^{D_1+D_2+D_3},
\end{aligned}$$

as desired. □

Remark 3.7. We have constructed a family of cubic surfaces over $S = \operatorname{Spec} \Bbbk[P]$ where $P = \mathrm{NE}(Y)$, where Y is a non-singular cubic surface. There is an intriguing slice of this family related to the Cayley cubic which has already made its appearance in Remark 1.1.

We may obtain the Cayley cubic as follows. Take four general lines $L_1, \ldots, L_4$ in $\mathbb{P}^2$, giving 6 pairwise intersection points. By blowing up these six points, we obtain a surface Y. Note that the cone of effective curves of Y is different than that of a general cubic surface because it contains some (-2)-curves. However, for this discussion it is convenient to keep P to be the cone of effective curves of a general cubic surface.

The strict transforms of the four lines become disjoint (-2)-curves which may be contracted, giving a cubic surface Y' with four ordinary double points: this is the Cayley cubic.

If we take $\overline{D}_1$ to be the line joining $L_1 \cap L_2$ and $L_3 \cap L_4$, $\overline{D}_2$ the line joining $L_1 \cap L_3$ and $L_2 \cap L_4$, and $\overline{D}_3$ the line joining $L_1 \cap L_4$ and $L_2 \cap L_3$, and let D_i be the strict transform of $\overline{D}_i$ in Y, we obtain a log Calabi-Yau pair $(Y, D = D_1 + D_2 + D_3)$ as usual. With suitable labelling of the exceptional curves, we can write the classes F_i of the strict transforms of the L_i as

$$\begin{aligned} F_1 &:= L - E_{11} - E_{21} - E_{31} \\ F_2 &:= L - E_{11} - E_{22} - E_{32} \\ F_3 &:= L - E_{12} - E_{21} - E_{32} \\ F_4 &:= L - E_{12} - E_{22} - E_{31} \end{aligned}$$

Now consider the big torus $S^\circ = \operatorname{Spec} \Bbbk[P^{\mathrm{gp}}] \subset S = \operatorname{Spec} \Bbbk[P]$, and consider further the subscheme $T \subseteq S^\circ$ defined by the equations $z^{F_i} = 1$, $i = 1, \ldots, 4$. Then $T = \operatorname{Spec} \Bbbk[P^{\mathrm{gp}}/\mathbf{F}]$, where $\mathbf{F}$ is the subgroup of P^{gp} generated by the curve classes F_i. However, it is not difficult to see that this quotient has two-torsion. Indeed,

$$2E_{11} - 2E_{12} = F_3 + F_4 - F_1 - F_2 \in \mathbf{F},$$

while $E_{11} - E_{12} \notin \mathbf{F}$. We also have

$$E_{11} - E_{12} \equiv E_{21} - E_{22} \equiv E_{31} - E_{32} \mod \mathbf{F}.$$

In fact, T has two connected components, one containing the identity element in the torus S°, and the other satisfying the equations $z^{E_{i1}} = -z^{E_{i2}}$ for $i = 1, 2, 3$. Let T' denote this latter component, and restrict the family of cubic surfaces given over S in Theorem 0.1 to T'. One may check that the equation becomes

$$\vartheta_1\vartheta_2\vartheta_3 = \sum_i z^{D_i}\vartheta_i^2 - 4z^{D_1+D_2+D_3} \tag{3.4}$$

and that this is in fact a family of Cayley cubics.

Somewhat more directly, one may also consider the restriction of the scattering diagram to T'. For example, consider the ray (ρ_1, f_{ρ_1}) described in Proposition 2.4. We note that the set of eight lines $\mathbf{L} := \{L_{1i}\}$ split into two groups of four,

$$\begin{aligned} \mathbf{L}_1 &:= \{E_{11}, L - E_{21} - E_{31}, L - E_{22} - E_{32}, 2L - E_{11} - E_{21} - E_{22} - E_{31} - E_{32}\} \\ \mathbf{L}_2 &:= \{E_{12}, L - E_{21} - E_{32}, L - E_{22} - E_{31}, 2L - E_{12} - E_{21} - E_{22} - E_{31} - E_{32}\} \end{aligned}$$

such that if $L_1, L_2 \in \mathbf{L}$ lie in the same $\mathbf{L}_i$ then $L_1 - L_2 \in \mathbf{F}$, so $z^{L_1} = z^{L_2}$ on T'. On the other hand, if they do not lie in the same $\mathbf{L}_i$, then $2(L_1 - L_2) \in \mathbf{F}$ and $z^{L_1} = -z^{L_2}$ on T'. Further, if $L \in \mathbf{L}$, then $2L - D_2 - D_3 \in \mathbf{F}$.

Thus we obtain, after restriction to T', that, for any choice of $L \in \mathbf{L}$,

$$f_{\rho_1} = \frac{[(1 + z^L X_1^{-1})(1 - z^L X_1^{-1})]^4}{(1 - z^{2L} X_1^{-2})^4} = 1.$$

So the ray becomes trivial after restriction to T'. Following the argument of §2, one then sees all rays of $\mathfrak{D}_{\mathrm{can}}$ become trivial after restriction to T'. Thus the equation (3.4) may in fact be obtained using broken lines as carried out in this section, but this time all broken lines involved are straight.

It is unusual that the trivial scattering diagram is consistent when the affine manifold B has a singularity. In the K3 case, there is a similar situation arising when B is an affine two-sphere arising as a quotient of $\mathbb{R}^2/\mathbb{Z}^2$ via negation.

It is also intriguing that the relevant subtorus $T' \subseteq S^\circ$ is translated, i.e., does not pass through the origin; we speculate that this might have an explanation in terms of orbifold Gromov-Witten invariants of the Cayley cubic.

4. The enumerative interpretation of the equation

There is also a much more recent interpretation of the multiplication law of Theorem 3.5, which gives a Gromov-Witten interpretation for the $\alpha_{p_1p_2r} \in \Bbbk[P]/I$, writing

$$\alpha_{p_1p_2r} = \sum_{\beta\in P} N^\beta_{p_1p_2r} z^\beta$$

with $N^\beta_{p_1p_2r} \in \mathbb{Q}$. In [GS18], Gross and Siebert explain how to associate certain Gromov-Witten numbers to the data β, p_1, p_2 and r. The details are given in [GS19]. The paper [KY19] gives a different approach for the same ideas. In general, the construction of these invariants is quite subtle, but happily in the case at hand, all invariants will be easy to calculate. Roughly put, the numbers are defined as follows.

The choice of r defines a choice of stratum $Z_r \subseteq Y$. Indeed, if $\sigma \in \Sigma$ is the minimal cone containing r, then σ corresponds to a stratum of Y, i.e., if $r = 0$, $Z_r = Y$, if $r \in \mathrm{Int}(\rho_i)$ then $Z_r = D_i$, and if $r \in \mathrm{Int}(\sigma_{i,i+1})$ then $Z_r = D_i \cap D_{i+1}$. We choose a general point $z \in Z_r$.

Then $N^\beta_{p_1p_2r}$ is a count of the number of stable logarithmic maps $f : (C, x_1, x_2, x_{\mathrm{out}}) \to Y$ such that:

(1) the order of tangency of f at x_j with D_k is $\langle D_k, p_j\rangle$, $j = 1, 2$.
(2) the order of tangency of f at x_{out} with D_k is $-\langle D_k, r\rangle$ and $f(x_{\mathrm{out}}) = z$.

Note this involves negative orders of tangency at x_{out}, and defining this is subtle. See [GS18] and [ACGS19] for more details for how this notion is defined. Here, usually $r = 0$, so we will only have a couple of cases where we have to accept the possibility of a negative order of tangency. For the complete technically correct definition of the above invariant, see [GS19], §3.

[GS18], Proposition 2.4 is quite useful in telling us when the relevant moduli space is empty. In particular, that proposition tells us that $N^\beta_{p_1p_2r} \neq 0$ implies that if D' is any divisor supported on D, then

$$\beta \cdot D' = \langle D', p_1\rangle + \langle D', p_2\rangle - \langle D', r\rangle. \tag{4.1}$$

In particular, if we take $D' = D = -K_Y$, we obtain

$$\langle D, p_1\rangle + \langle D, p_2\rangle = \beta\cdot D + \langle D, r\rangle.$$

As D is ample in the case of the cubic surface, in particular $\beta\cdot D \ge 0$ for any effective curve class, so we get the stronger result that if $N^{\beta}_{p_1p_2r} \neq 0$, then

$$\langle D, p_1\rangle + \langle D, p_2\rangle \ge \langle D, r\rangle \ge 0,$$

These formulae may be compared with (3.1) and (3.2). In fact, $F = \langle K_Y, \cdot\rangle$, and the above formulae play the same role as those of §3.

We now revisit the calculation of Lemma 3.6. The arguments which follow are necessarily sketchy as we have not given a full definition of the invariants here. We trust the arguments should be sufficiently plausible, however.

For example, let us reconsider the product $\vartheta_{v_1}^2$. We see that if $N^{\beta}_{v_1v_1r} \neq 0$ then $\langle D, r\rangle \le 2$ with equality if and only if $\beta = 0$. Thus if we do have equality, then any map $f : (C, x_1, x_2, x_{\mathrm{out}}) \to Y$ contributing to $N^{\beta}_{v_1v_1r}$ is constant. This is discussed in [GS19], Lemma 1.15, where it is shown that if $N^0_{p_1p_2r} \neq 0$, then p_1, p_2 lie in the same cone and $r = p_1 + p_2$. Further, $N^0_{p_1p_2r} = 1$ in this case. In particular, $N^0_{v_1v_1,2v_1} = 1$. This gives the contribution ϑ_{2v_1} to $\vartheta_{v_1}^2$.

If $\beta\cdot D = 1$, then the only possibilities for r are v_1, v_2 or v_3. Suppose $r = v_1$. As $\beta\cdot D = 1$, β is the class of a line on Y. Since we choose $z \in Z_r$ general, none of the lines L_{ij} pass through z and thus the image of f may not be L_{ij}. If the image of f is D_2, then f has non-trivial contact with D_3, which is not allowed. Similarly, the image of f may not be D_3. Finally, if the image of f is D_1, (4.1) yields a contradiction if one takes $D' = D_1$. Thus we eliminate this case. The cases that $r = v_2, v_3$ are similarly ruled out.

Finally, we have one remaining case, when $\beta\cdot D = 2$ and $r = 0$. Thus we consider conics which meet D_1 transversally at two points (labelled x_1, x_2), are disjoint from D_2 and D_3, and have a third point x_{out} which coincides with a fixed general point $z \in Y$. It is easy to see that any such conic must be in the linear system $|D_2 + D_3|$, and there is one such conic passing through z. However, as the labels of the intersection points of the conic with D_1 can be interchanged, in fact $N^{D_2+D_3}_{v_1v_20} = 2$. This gives the second term in the product $\vartheta_{v_1}^2$.

We now move onto $\vartheta_{v_1}\cdot\vartheta_{v_2}$. A similar analysis with the possible degree of the class β leads to the following choices. First, we may have $\beta = 0$, and so $r = v_1 + v_2$ and $N^0_{v_1,v_2,v_1+v_2} = 1$, giving the first contribution to the product.

Next, if $\beta\cdot D = 1$, then $r = v_1, v_2$ or v_3. As before, β must be the class of a line, and as before, we must have $\beta = D_i$ for some i as otherwise the image of f will not contain z. If $\beta = D_3$, we can identify C with D_3, taking x_1 to be the intersection of D_1 and D_3 and x_2 to be the intersection of D_2 with D_3. Since $\beta \cdot D_3 = -1$, (4.1) tells us that $r = v_3$. After fixing $z \in D_3$, we take $x_{\mathrm{out}} = z$. One can show that $N^{D_3}_{v_1 v_2 v_3} = 1$.[4] This contributes the term $z^{D_3}\vartheta_{v_3}$ to the product. On the other hand, if $\beta = D_1$, taking $D' = D_1$ in (4.1) results in a contradiction regardless of the choice of $r = v_k$, and the same holds if $\beta = D_2$. Thus there are no further choices.

Finally, if $\beta \cdot D = 2$, $r = 0$, we fix $z \in Y$ general. We now need to consider conics which meet both D_1 and D_2 transversally, pass through z, and are disjoint from D_3. There are a total of 27 conic bundles on Y: for E the class of a line on Y, $|D - E|$ is a pencil of conics. Thus one easily checks that only eight of these have the correct intersection properties with D, precisely conics of classes $D_3 + L_{3j}$, $1 \le j \le 8$. For each j, there is precisely one conic in the pencil $|D_3 + L_{3j}|$ passing through z. This is responsible for the last term in the product $\vartheta_{v_1} \cdot \vartheta_{v_2}$.

We now turn to the product $\vartheta_{v_1+v_2} \cdot \vartheta_{v_3}$. As $v_1 + v_2$ and v_3 do not lie in a common cone of Σ, constant maps cannot occur. Thus we are faced with the possibilities $1 \le \beta \cdot D \le 3$.

If $\beta \cdot D = 1$, the same arguments as before reduce to the possibilities that $\beta = D_1$, D_2 or D_3. First $\beta = D_3$ is impossible: any curve with contact order at a point given by $v_1 + v_2$ must pass through $D_1 \cap D_2$. However, in each of the other cases, there is exactly one allowable map. For example, in case $\beta = D_1$, we take $z \in D_1$ general, identify C with D_1, take x_1 to be the intersection point of D_1 and D_2, x_2 the intersection point of D_1 and D_3, take $x_{\mathrm{out}} = z$, and take $r = 2v_1$. Again it is possible to show that these curves exist as punctured logarithmic curves, and $N^{D_i}_{v_1+v_2, v_3, 2v_i} = 1$ for $i = 1, 2$. This gives the first two terms in the product.

If $\beta \cdot D = 2$, then $r = v_i$ for some i, and we must consider conics which pass through $D_1 \cap D_2$, are transversal to D_3, and pass through an additional point $z \in D_i$. We may now show the image of any punctured map contributing to $N^{\beta}_{v_1+v_2, v_3, v_i}$ is reducible. If the image is an irreducible conic, that conic must pass through $D_1 \cap D_2$, intersect D_3 in at least one point, and pass through the generally chosen point $z \in D_i$. This implies that $\beta \cdot (D_1 + D_2 + D_3) \ge 3$. Since $D_1 + D_2 + D_3$ is the class of a hyperplane section of the cubic surface, this contradicts β being a degree 2 class. If, on the other hand, the image of the punctured map is a line (hence the punctured map is a double cover), this

[4] We note that the full verification of this statement is somewhat involved, as one must construct the unique punctured curve in the relevant moduli space and show that it is unobstructed. However, this is fairly routine for those familiar with log Gromov-Witten theory, and we omit the details here as it would involve introducing a lot of additional technology into this survey.

line must be D_1 or D_2, being the only lines passing through $D_1 \cap D_2$. Thus $\beta = 2D_1$ or $2D_2$. However, this case is ruled out via an application of (4.1).

Thus necessarily the image of the punctured map is a union of two lines. The only lines passing through $D_1 \cap D_2$ are D_1 and D_2, and thus $\beta = D_i + L$ for $i = 1$ or 2 and L some other line. As the image of f must be connected, this only leaves the option of $\beta = D_1 + L_{1j}$, $\beta = D_2 + L_{2j}$, or $\beta = D_j + D_k$. The third case can be ruled out from (4.1), and for the first two cases, one can show that $N^{\beta}_{v_1+v_2,v_3,v_i} = 1$. This gives the third and fourth terms in the expression for $\vartheta_{v_1+v_2} \cdot \vartheta_{v_3}$.

Finally, we consider the case of $\beta \cdot D = 3$, so that β is a cubic. There are two choices. Either β is the class of a twisted cubic, i.e., $-2 = \beta \cdot (\beta + K_Y)$, or β is the class of an elliptic curve, i.e., $0 = \beta \cdot (\beta + K_Y)$. Now if β is the class of a twisted cubic, it is easy to see that the linear system $|\beta|$ is two-dimensional and induces a morphism $\pi : Y \to Y' \cong \mathbb{P}^2$. If in addition, $\beta \cdot D_i = 1$ for each i (which follows from (4.1)), π maps D_1, D_2 and D_3 to lines in Y'. Hence there is a one-to-one correspondence between such classes β and morphisms $\pi : Y \to Y'$ as before.

Given such a morphism, $\beta = \pi^* H$, and there is a unique twisted cubic in the linear system $|\beta|$ passing through both z and $D_1 \cap D_2$. Thus $N^{\beta}_{v_1+v_2,v_3,0} = 1$. This gives the fifth term in the expression for $\vartheta_{v_1+v_2} \cdot \vartheta_{v_3}$.

Finally, if β is the class of an elliptic curve of degree 3, it is necessarily planar, and hence $\beta = D$. We now calculate $N^{\beta}_{v_1+v_2,v_3,0}$. First, there is a pencil of plane cubics passing through $D_1 \cap D_2$ and z. If $\ell \subseteq \mathbb{P}^3$ denotes the line joining these points, then each element of the pencil is of the form $H \cap Y$ for $H \subseteq \mathbb{P}^3$ a plane containing ℓ. To study this pencil, we may blow-up its basepoints, which are the three points of $\ell \cap Y$. This gives a rational elliptic surface $g : \tilde{Y} \to \mathbb{P}^1$. Via a standard Euler characteristic computation, such a surface is expected to have 12 singular fibres. However, note that if H contains D_i, $i = 1$ or 2, then $H \cap Y$ is a union $D_i \cup C$ of a line and a conic. In general, C intersects D_i in two points. By normalizing one of these two nodes, we obtain a stable map to Y. However, none of these maps can be equipped with the structure of a stable log map because the point of normalization on the conic maps into D and has non-zero contact order with D, yet it is not a marked point.

Since we have just seen that two of the fibres of this elliptic fibration are of Kodaira type I_2, this leaves 8 additional nodal elliptic curves. By normalizing the node, one obtains a genus zero stable map with the desired intersection behaviour with D. This yields the last term in the description of $\vartheta_{v_1+v_2} \cdot \vartheta_{v_3}$.

We close by noting that the Frobenius structure conjecture (see the first arXiv version of [GHK11], Conjecture 0.9, or [M19] and [KY19]) gives us another explanation for the constant term (i.e., coefficient of ϑ_0,) $\sum_\pi z^{\pi^*H} + 10z^{D_1+D_2+D_3}$ in the equation defining the mirror to the cubic surface. Here we write 10 rather than 4 as we rewrite the equation for the mirror in terms of ϑ_{2v_i} instead of $\vartheta_{v_i}^2$.

Indeed, the Frobenius conjecture implies that we may calculate the constant term in the triple product $\vartheta_{v_1}\vartheta_{v_2}\vartheta_{v_3}$ as $\sum_\beta N^\beta_{v_1v_2v_30} z^\beta$ where, roughly, $N^\beta_{v_1v_2v_30}$ is a count defined as follows. Fix $z \in Y$ general and $\lambda \in \overline{\mathcal{M}}_{0,4}$. Then we count four-pointed stable log maps $f : (C, x_1, x_2, x_3, x_{\mathrm{out}}) \to Y$ such that f meets D_i transversally at x_i, $f(x_{\mathrm{out}}) = z$, and the modulus of the stabilization of C is λ. This can be viewed as fixing the cross-ratio of the four points $x_1, x_2, x_3, x_{\mathrm{out}}$ to be λ. This part of the Frobenius conjecture is shown in [GS19] and [KY19], and see also [M19] for related results.

The class β of such a curve C must satisfy $\beta \cdot D = 3$, so β is either a twisted cubic or a plane cubic. In the former case, one immediately recovers $\sum_\pi z^{\pi^*H}$. Indeed, if one fixes $z \in \mathbb{P}^2$ and a cross-ratio λ, there is a unique line H in $\mathbb{P}^2$ passing through z such that the cross-ratio of z and the three points of intersection of H with the boundary divisor is λ.

The count of plane cubics is more subtle. In this case, it is easiest to fix the modulus of the stabilization of C by insisting the stabilization is a singular curve, with x_2, x_3 on one irreducible component and x_1, x_{out} on the other. There are the following possibilities.

(1) The image of f is a union of three lines. This cannot occur, as such a curve does not pass through a general $z \in Y$.
(2) The image of f is the union of a line and a conic, $E \cup Q$. Suppose $E \neq D_i$ for any i. Then E meets D at one point and is rigid, hence does not pass through z. Thus three of the four marked points of C must lie in Q. This contradicts the choice of modulus. Thus $E = D_i$ for some i, and $Q \in |D - D_i|$. In particular, as Q is irreducible, Q is disjoint from D_j, D_k for $\{i, j, k\} = \{1, 2, 3\}$. Thus D_i must contain those marked points mapping to D_j and D_k, so necessarily $D_i = D_1$. In particular, C is the normalization of $D_1 \cup Q$ at one of the two nodes, and the marked point x_1 is the point of Q mapping to the chosen node. Note that this marking is what allows us to count this curve, as opposed to the same curve considered in the contribution to the constant term in $\vartheta_{v_1+v_2} \cdot \vartheta_{v_3}$. Because of the choice of nodes, this gives two curves of class D.
(3) The image of f is an irreducible nodal cubic. In order for the domain to have the given modulus, x_2 and x_3 must lie on a contracted component of C, i.e., $C = C_1 \cup C_2$ with $x_2, x_3 \in C_1$, $x_1, x_{\mathrm{out}} \in C_2$, $f|_{C_1}$ constant with image $D_2 \cap D_3$, and $f(C_2)$ a nodal cubic. The count is now exactly the same as in the case of the contribution of nodal cubics to $\vartheta_{v_1+v_2} \cdot \vartheta_{v_3}$, and we have 8 such nodal cubics.

This explains the term $10z^{D_1+D_2+D_3}$.

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