# Compact moduli spaces of surfaces and exceptional vector bundles

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# Chapter 1

# Compact moduli spaces of surfaces and exceptional vector bundles

The moduli space of surfaces of general type has a natural compactification due to Kollár and Shepherd-Barron [KSB88] which is analogous to the Deligne-Mumford compactification of the moduli space of curves [DM69]. However, very little is known about this moduli space or its compactification in general (for example it can have many irreducible components [C86] and be highly singular [V06]). A key question is to enumerate the boundary divisors in cases where the moduli space is well behaved. The most basic boundary divisors are those given by degenerations of the smooth surface to a surface with a cyclic quotient singularity of a special type, first studied by J. Wahl [W81]. We describe a construction which relates these boundary divisors to the classification of stable vector bundles on the smooth surface in the case $H^{2,0} = H^1 = 0$. In particular we connect with the theory of exceptional collections of vector bundles used in the study of the derived category of coherent sheaves.

We review the necessary background material and put a strong emphasis on examples. In particular we discuss the examples of del Pezzo surfaces and surfaces of general type with $K^2 = 1$ (based on work by Anna Kazanova).

*Notation and Background.*

We work throughout over the field $k = \mathbb{C}$ of complex numbers. We write $\mathbb{G}_m = \mathbb{C}^*$ and $\mu_n \subset \mathbb{G}_m$ for the group of $n$th roots of unity.

For $X$ a variety and $P \in X$ a point we write $(P \in X)$ to denote a small complex analytic neighbourhood of $P \in X$ or an étale neighbourhood of $P \in X$.

If $D$ is a Weil divisor on a normal variety $X$, we say $D$ is $\mathbb{Q}$-*Cartier* if $mD$ is Cartier for some $m \in \mathbb{N}$. If $D_1, \ldots, D_n$ are $\mathbb{Q}$-Cartier divisors on a proper normal variety $X$ of dimension $n$, let $m_i \in \mathbb{N}$ be such that $m_i D_i$ is Cartier. We define the intersection number

$$D_1 D_2 \cdots D_n := ((m_1 D_1)(m_2 D_2) \cdots (m_n D_n))/m_1 m_2 \cdots m_n \in \mathbb{Q}.$$

Let $r \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathbb{Z}/r\mathbb{Z}$. We write $\mathbb{A}^n/\frac{1}{r}(a_1, \dots, a_n)$, or just $\frac{1}{r}(a_1, \dots, a_n)$, for the quotient

$$\mathbb{A}^n/\mu_r, \quad \mu_r \ni \zeta \colon (x_1, \dots, x_n) \mapsto (\zeta^{a_1} x_1, \dots, \zeta^{a_n} x_n).$$

We always assume that $\gcd(a_1, \dots, \widehat{a_i}, \dots, a_n, r) = 1$ for each $i$ so that the $\mu_r$ action is free in codimension 1. In the case $n = 2$, the weights $a_1$ and $a_2$ are coprime to $r$, so, composing the action with an automorphism $\mu_r \to \mu_r$, $\zeta \mapsto \zeta^b$ we may assume $a_1 = 1$. The singularity $(P \in X) = (0 \in \mathbb{A}^2/\frac{1}{r}(1, a))$ has resolution

$$\pi \colon (E \subset \tilde{X}) \to (P \in X)$$

with exceptional locus $E = \pi^{-1}(P)$ a nodal chain of smooth rational curves with self-intersection numbers $-b_1, \dots, -b_r$, where $b_i \geq 2$ for each $i$ and

$$r/a = [b_1, \dots, b_r] := b_1 - 1/(b_2 - 1/(b_3 - 1/(b_4 \cdots - 1/b_r) \cdots))$$

is the expansion of $r/a$ as a Hirzebruch–Jung continued fraction [F93], 2.6.

Let $n \in \mathbb{N}$ and $a_0, \dots, a_n \in \mathbb{N}$. We write $\mathbb{P} = \mathbb{P}(a_0, \dots, a_n)$ for the weighted projective space

$$\mathbb{P}(a_0, \dots, a_n) = (\mathbb{A}^{n+1} \backslash \{0\})/\mathbb{G}_m, \quad \mathbb{G}_m \ni \lambda \colon (X_0, \dots, X_n) \mapsto (\lambda^{a_0} X_0, \dots, \lambda^{a_n} X_n).$$

We always assume that $\gcd(a_0, \dots, \widehat{a_i}, \dots, a_n) = 1$ for all $i$. Then $\mathbb{P}(a_0, \dots, a_n)$ is a normal projective variety covered by affine charts

$$(X_i \neq 0) = \mathbb{A}^n/\tfrac{1}{a_i}(a_0, \dots, \widehat{a_i}, \dots, a_n)$$

where the affine orbifold coordinates are given by $x_{ji} = X_j/X_i^{a_j/a_i}$ for $j \neq i$. We have

$$\mathbb{P}(a_0, \dots, a_n) = \operatorname{Proj} k[X_0, \dots, X_n]$$

where the grading of the polynomial ring is given by $\deg X_i = a_i$. The sheaf $\mathcal{O}_{\mathbb{P}}(1) = \mathcal{O}_{\mathbb{P}}(H)$ is a rank one reflexive sheaf corresponding to a Weil divisor class $H$. The global sections of $\mathcal{O}_{\mathbb{P}}(n) = \mathcal{O}_{\mathbb{P}}(nH)$ are the homogeneous polynomials of (weighted) degree $n$. The divisor class group $\operatorname{Cl}(\mathbb{P})$ is isomorphic to $\mathbb{Z}$, generated by $H$. The divisor $H$ is $\mathbb{Q}$-Cartier, and satisfies

$$H^n = 1/(a_0 \cdots a_n).$$

The canonical divisor class $K_{\mathbb{P}}$ is given by

$$K_{\mathbb{P}} = -(a_0 + a_1 + \cdots + a_n)H.$$

The variety $\mathbb{P}$ is the toric variety associated to the fan $\Sigma$ in the lattice $N = \mathbb{Z}^{n+1}/\mathbb{Z}(a_0, \dots, a_n)$ consisting of cones generated by proper subsets of the standard basis of $\mathbb{Z}^{n+1}$. See [F93] for background on toric varieties.

## 1.1 Moduli spaces of surfaces of general type

### 1.1.1 Surfaces of general type

Let $X$ be a smooth projective complex surface. We say $X$ is of *general type* if for $n \gg 0$ the rational map defined by the linear system $|nK_X|$ is birational onto its image. Equivalently, $h^0(nK_X) \sim cn^2$ for some $c > 0$ as $n \to \infty$.

**Example 1.1.1.** If $X \subset \mathbb{P}^3$ is a smooth hypersurface of degree $d$ then $K_X = \mathcal{O}_X(d-4)$ by the adjunction formula. Hence $X$ is a surface of general type if and only if $d \geq 5$ (and in this case $K_X$ is very ample).

If $X$ is a surface of general type then we have the following more precise statements: There is a birational morphism

$$f\colon X \to X_{\min}$$

from $X$ to a smooth surface $X_{\min}$ such that $K_{X_{\min}}$ is *nef*, that is,

$$K_{X_{\min}} \cdot C \geq 0 \text{ for all curves } C \subset X_{\min}.$$

The morphism $f$ is the composition of a sequence of contractions of $(-1)$-curves, or, equivalently, $X$ is obtained from $X_{\min}$ by a sequence of blowups. The surface $X_{\min}$ is called the *minimal model* of $X$ and is uniquely determined. For a single blowup $\pi\colon \tilde{S} \to S$ of a point of a smooth surface with exceptional curve $E$ we have

$$K_{\tilde{S}} = \pi^* K_S + E,$$

so $K_{\tilde{S}}^2 = K_S^2 - 1$. Thus $K_{X_{\min}}^2 = K_X^2 + N \geq K_X^2$, where $N$ is the number of exceptional curves of $f$.

Moreover, there is a further birational morphism

$$g\colon X_{\min} \to X_{\mathrm{can}}$$

to a normal surface $X_{\mathrm{can}}$ such that $K_{X_{\mathrm{can}}}$ is ample. The morphism $g$ is given by contracting all the $(-2)$-curves on $X_{\min}$. Each connected component of the union of $(-2)$-curves is necessarily a nodal curve with dual graph one of the $A$,$D$,$E$ Dynkin diagrams, and is contracted to a Du Val singularity $P \in X_{\mathrm{can}}$. Here a *Du Val singularity* is a quotient singularity $(0 \in \mathbb{A}^2/G)$ where $G \subset \mathrm{SL}(2,\mathbb{C})$. The surface $X_{\mathrm{can}}$ is called the *canonical model* of $X$. The morphism $g\colon X_{\min} \to X_{\mathrm{can}}$ is the *minimal resolution* of $X_{\mathrm{can}}$. The Weil divisor $K_{X_{\mathrm{can}}}$ is Cartier and $K_{X_{\min}} = g^* K_{X_{\mathrm{can}}}$.

Returning to the definition of surfaces of general type, the rational map defined by $|nK_X|$ for $n \gg 0$ is the morphism $g \circ f\colon X \to X_{\mathrm{can}}$ (and $n \geq 5$ suffices [BHPV04], VII.5, p. 279). The constant $c$ such that $h^0(nK_X) \sim cn^2$ is given by $c = \frac{1}{2}K_{X_{\min}}^2 = \frac{1}{2}K_{X_{\mathrm{can}}}^2$ (by the Riemann–Roch formula and Kodaira vanishing).

If $X$ is a minimal surface of general type then it has finite automorphism group. Indeed, we have an embedding

$$\varphi_{|nK_{X_{\mathrm{can}}}|}\colon X_{\mathrm{can}} \hookrightarrow \mathbb{P}^m$$

for some $n \gg 0$ and $m = m(n)$, and so an injective homomorphism

$$\mathrm{Aut}(X) = \mathrm{Aut}(X_{\mathrm{can}}) \hookrightarrow \mathrm{PGL}(m+1, \mathbb{C}).$$

which realizes $\mathrm{Aut}(X)$ as a quasiprojective scheme. Now the tangent space to $\mathrm{Aut}(X)$ at the identity equals $H^0(T_X)$, and we have

$$H^0(T_X) = H^2(\Omega_X \otimes \omega_X)^* = 0$$

by Serre duality and Kodaira–Nakano vanishing ([GH78], p. 154). Thus $\mathrm{Aut}(X)$ is discrete and hence finite.

### 1.1.2 Simultaneous resolution of Du Val singularities

If $\mathcal{X} \to S$ is a flat family of surfaces with Du Val singularities, then there exists a finite surjective base change $S' \to S$ such that the pullback $\mathcal{X}' \to S'$ admits a simultaneous resolution [KM98], 4.28. That is, there exists a flat family $\mathcal{Y} \to S'$ and a birational morphism $\mathcal{Y} \to \mathcal{X}'$ over $S'$ such that the induced morphism $\mathcal{Y}_s \to \mathcal{X}'_s$ is the minimal resolution of $\mathcal{X}'_s$ for each $s \in S'$.

**Example 1.1.2.** Let

$$\mathcal{X} = (x^2 + y^2 + z^2 + t = 0) \subset \mathbb{A}^3_{x,y,z} \times \mathbb{A}^1_t \to S = \mathbb{A}^1_t.$$

Thus $\mathcal{X}$ is a smooth 3-fold, the fiber $\mathcal{X}_t$ is smooth for $t \neq 0$, and the fiber $\mathcal{X}_0$ has an $A_1$ singularity (or ordinary double point). Consider the base change

$$S' = \mathbb{A}^1_s \to S = \mathbb{A}^1_t, \quad s \mapsto t = s^2.$$

Then

$$\mathcal{X}' = (x^2 + y^2 + z^2 + s^2 = 0) \subset \mathbb{A}^3_{x,y,z} \times \mathbb{A}^1_s \to S' = \mathbb{A}^1_s$$

Thus $\mathcal{X}'$ has a 3-fold ordinary double point singularity, and there are two small resolutions $f_j\colon \mathcal{Y}_j \to \mathcal{X}'$, $j = 1, 2$, of $\mathcal{X}'$ given by blowing up the loci

$$Z_j = (x + iy = z + (-1)^j is = 0).$$

In each case the exceptional locus $C_j$ of $f_j$ is a copy of $\mathbb{P}^1$ with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. The morphism $f_j$ defines a simultaneous resolution of the family $\mathcal{X}/S$, the curve $C_j$ being identified with the unique exceptional $(-2)$-curve of the minimal resolution of the $A_1$ singularity $(0 \in \mathcal{X}_0)$.

An alternative construction of the resolutions $f_j$ is as follows: we can blowup the point $P \in \mathcal{X}'$ to obtain a resolution $\mathcal{Z} \to \mathcal{X}'$ with exceptional locus $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$

with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1)$. Then there exist birational morphisms $g_j \colon \mathcal{Z} \to \mathcal{Y}_j$, $j = 1, 2$ with exceptional locus $E$ such that $E$ is contracted to $C_j$ via one of the projections $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. The picture can be also described torically: the variety $\mathcal{X}'$ is isomorphic to the affine toric variety associated to the cone $\langle e_1, e_2, e_3, e_4 \rangle_{\mathbb{R}_{\geq 0}}$ in the lattice $\mathbb{Z}^4/\mathbb{Z}(1, 1, -1, -1)$ and the resolutions correspond to subdivisions of this cone.

It follows that, for surfaces of general type, the coarse moduli space of minimal models is identified with the coarse moduli space of canonical models. This is important because on the canonical model the canonical line bundle is ample, and this can be used to construct the moduli space as a quotient of a locally closed subscheme of a Hilbert scheme.

Note also that the condition that a surface of general type $X$ is minimal is both open and closed in families. (Indeed, if $K_X$ is nef then $nK_X$ is basepoint free for $n \gg 0$, and the same is true for nearby fibers of a deformation $\mathcal{X}/S$ of $X$ because we can lift sections of $nK_X$ to sections of $nK_{\mathcal{X}/S}$ for $n \geq 2$ using Kodaira vanishing and cohomology and base change. Thus being minimal is open. Also, $K_X$ is not nef if and only if $X$ contains a $(-1)$-curve, and $(-1)$-curves deform in families. So being minimal is closed.) Thus we can restrict attention to minimal surfaces of general type.

### 1.1.3 Moduli

Let $M = M_{K^2,\chi}$ denote the moduli space of normal projective surfaces $X$ with at worst Du Val singularities such that $K_X$ is ample and $K_X^2 = K^2$, $\chi(\mathcal{O}_X) = \chi$.

**Example 1.1.3.** Let $U \subset \mathbb{P} = \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^3}(d))$ denote the locus of surfaces of degree $d \geq 5$ in $\mathbb{P}^3$ having at worst Du Val singularities. Then $U$ is Zariski open with complement of codimension $\geq 2$. (Indeed if $U' \subset \mathbb{P}$ denotes the locus of smooth surfaces then $D := \mathbb{P} \setminus U'$ is an irreducible divisor (the zero locus of the discriminant), and $U' \subsetneq U$ because the fiber of the universal family over a general point of $D$ has a Du Val singularity of type $A_1$.) Now the quotient $U/\operatorname{PGL}(4, \mathbb{C})$ is a quasiprojective variety with quotient singularities, and is a Zariski open subset of $M$. (The quotient is a quasiprojective variety by a general result of Gieseker on stability of surfaces of general type in the sense of geometric invariant theory (GIT) [G77].)

Note that, unlike the case of plane curves, every deformation of a smooth surface $X \subset \mathbb{P}^3$ of degree $d \geq 5$ is realized in $\mathbb{P}^3$. Indeed it suffices to show that the line bundle $\mathcal{O}_X(1)$ deforms (then sections lift using $H^1(\mathcal{O}_X(1)) = 0$). For $L$ a line bundle on $X$, if $L^{\otimes n}$ deforms for some $n > 0$ then so does $L$ (this follows from the Lefschetz $(1, 1)$ theorem [GH78], p. 163). In our case $K_X = \mathcal{O}_X(d - 4)$, $d - 4 > 0$, and $K_X$ deforms, so $\mathcal{O}_X(1)$ deforms as required. (However, if $d = 4$ then $X$ is a K3 surface and there exist deformations of $X$ which are not projective, so in particular are not realized in $\mathbb{P}^3$.)

### 1.1.4 Expected dimension

We can compute the expected dimension of $M$ using the Hirzebruch–Riemann–Roch formula. Let $X$ be a smooth projective surface. Write

$$c_1 = c_1(T_X) = -K_X$$

and

$$c_2 = c_2(T_X) = e(X) = \sum_i (-1)^i \dim_{\mathbb{R}} H^i(X, \mathbb{R}).$$

For $F$ a vector bundle on $X$ the *Hirzebruch–Riemann–Roch formula* is the equality

$$\chi(F) = (\mathrm{ch}(F) \cdot \mathrm{td}(X))_2$$

where

$$\mathrm{ch}(F) = \mathrm{rk}(F) + c_1(F) + \frac{1}{2}(c_1(F)^2 - 2c_2(F))$$

is the Chern character and

$$\mathrm{td}(X) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2)$$

is the Todd class. The case $F = \mathcal{O}_X$ is *Noether's formula*

$$\chi(\mathcal{O}_X) = \frac{1}{12}(c_1^2 + c_2).$$

Putting $F = T_X$ we obtain

$$\chi(T_X) = (\mathrm{ch}(T_X) \cdot \mathrm{td}(X))_2 = \frac{1}{6}(7c_1^2 - 5c_2) = 2K_X^2 - 10\chi(\mathcal{O}_X)$$

where the last equality is given by Noether's formula. Now suppose that $X$ is of general type. Then $H^0(T_X) = 0$ ($X$ has no infinitesimal automorphisms), $H^1(T_X)$ is the tangent space to $M$ at $X$ (the space of first order infinitesimal deformations of $X$), and obstructions to extending infinitesimal deformations to higher order are contained in $H^2(T_X)$. Thus the expected dimension of the moduli space $M = M_{K^2,\chi}$ equals

$$\exp.\dim(M) = h^1(T_X) - h^2(T_X) = -\chi(T_X) = 10\chi - 2K^2.$$

In general $\dim M \geq \exp.\dim(M)$. If $H^2(T_X) = 0$ then $M$ is smooth of dimension $\exp.\dim(M)$ at $[X] \in M$.

### 1.1.5 Compactification

The moduli space $M_{K^2,\chi}$ has a natural compactification $M_{K^2,\chi} \subset \overline{M}_{K^2,\chi}$ analogous to the Deligne-Mumford compactification $M_g \subset \overline{M}_g$ of the moduli space of curves of genus $g$.

### 1.1.6 Stable surfaces

A *stable surface* is a reduced Cohen-Macaulay projective surface $X$ such that $X$ has semi log canonical (slc) singularities and the dualizing sheaf $\omega_X$ is ample.

### 1.1.7 Semi log canonical singularities

We will not define slc singularities here, but we note that they include quotients of smooth or double normal crossing points

$$(xy = 0) \subset \mathbb{A}^3.$$

Moreover a slc surface has only double normal crossing singularities away from a finite set.

### 1.1.8 Dualizing sheaf

The dualizing sheaf plays the role of the canonical line bundle for an slc surface $X$.

Let $X$ be an slc surface. If $X$ is normal (equivalently, $X$ has isolated singularities), then the dualizing sheaf is the push forward of the canonical line bundle from the smooth locus $i \colon U \subset X$,

$$\omega_X = i_*\omega_U = i_*\mathcal{O}_U(K_U).$$

Thus the sheaf $\omega_X$ is the rank one reflexive sheaf $\mathcal{O}_X(K_X)$ corresponding to the Weil divisor class $K_X$ given by the closure of the divisor $(\Omega)$ of zeroes and poles of a meromorphic section $\Omega$ of $\omega_U$.

In general, let $U \subset X$ denote the open locus of smooth and double normal crossing points (then the complement $X \setminus U$ is finite). Let $\nu_U \colon U^\nu \to U$ denote the normalization of $U$, $\Delta_U \subset U$ the singular locus of $U$, and $\Delta_U^\nu \subset U^\nu$ the inverse image of $\Delta_U$. Thus $U^\nu$ is smooth, the restriction $U^\nu \setminus \Delta_U^\nu \to U \setminus \Delta_U$ is an isomorphism, and $\Delta_U^\nu \to \Delta_U$ is a finite étale morphism of degree 2. Then $\omega_U$ is a line bundle and is given by the exact sequence

$$0 \to \omega_U \to \nu_{U*}\omega_{U^\nu}(\Delta_U^\nu) \to \omega_{\Delta_U}$$

where the last map is given by the Poincaré residue map

$$\omega_{U^\nu}(\Delta_U^\nu)|_{\Delta_U^\nu} \to \omega_{\Delta_U^\nu}$$

and the norm

$$\nu_{U*}\omega_{\Delta_U^\nu} \to \omega_{\Delta_U}.$$

(This description of the dualizing sheaf $\omega_U$ is a straightforward generalization of the description for nodal curves.) Writing $\nu \colon X^\nu \to X$ for the normalization of $X$, $\Delta \subset X$ for the closure of $\Delta_U$, and $\Delta^\nu \subset X^\nu$ for the inverse image of $\Delta$, we have

$$\omega_X = i_*\omega_U = \mathcal{O}_X(K_X)$$

where $K_X$ is a Weil divisor class on $X$ such that its restriction to $X^\nu$ equals $K_{X^\nu} + \Delta^\nu$.

**Example 1.1.4.** Suppose $P \in X$ is a double normal crossing point, that is, $(P \in X)$ is locally analytically isomorphic to $(0 \in (xy = 0) \subset \mathbb{A}^3)$. Then, working locally analytically at $P \in X$, we have $\Delta = (x = y = 0) \subset X$, $X^\nu = \mathbb{A}^2_{x,z} \sqcup \mathbb{A}^2_{y,z}$ (disjoint union), and

$$\omega_X = \left\{ \left( f(x,z)\frac{dx}{x} \wedge dz,\ g(y,z)\frac{dy}{y} \wedge dz \right) \;\middle|\; f, g \text{ holomorphic, } f(0,z) + g(0,z) = 0 \right\}.$$

### 1.1.9 The index of an slc singularity

Part of the definition of an slc singularity $(P \in X)$ requires that $K_X$ is a $\mathbb{Q}$-Cartier divisor. That is, there exists $N \in \mathbb{N}$ such that $NK_X$ is Cartier, or equivalently

$$\omega_X^{[N]} := i_*(\omega_U^{\otimes N}) = \mathcal{O}_X(NK_X)$$

is a line bundle. The least such $N$ is called the *index* of $(P \in X)$.

**Example 1.1.5.** Suppose $(P \in X)$ is a quotient singularity $(0 \in \mathbb{A}^2/G)$, $G \subset \mathrm{GL}(2,\mathbb{C})$. We may assume that $G$ acts freely on $\mathbb{A}^2 \setminus \{0\}$ by Chevalley's theorem. Then, working locally analytically at $P \in X$, we have the quotient map

$$q\colon (0 \in \mathbb{A}^2) \to (P \in X),$$

and $\omega_X = (q_*\omega_{\mathbb{A}^2})^G$. Now $\omega_{\mathbb{A}^2} = \mathcal{O}_{\mathbb{A}^2} \cdot dx \wedge dy$, and for $g \in G$ we have

$$g^*(dx \wedge dy) = \det(g) dx \wedge dy.$$

The subgroup $\det(G) \subset \mathbb{C}^*$ is the group $\mu_N$ of $N$th roots of unity for some $N \in \mathbb{N}$. So $(dx \wedge dy)^{\otimes N}$ is $G$-invariant and we have

$$\omega_X^{[N]} = (q_*\mathcal{O}_{\mathbb{A}^2})^G \cdot (dx \wedge dy)^{\otimes N} = \mathcal{O}_X \cdot (dx \wedge dy)^{\otimes N}.$$

Thus $\omega_X^{[N]}$ is a line bundle. The number $N$ is the index of $(P \in X)$.

### 1.1.10 The index one cover

Let $P \in X$ be a slc surface singularity of index $N$. Working locally analytically at $P \in X$, there is a canonically defined covering

$$q\colon (Q \in Z) \to (P \in X)$$

such that

(1) $q$ is a finite Galois covering with group $\mu_N \simeq \mathbb{Z}/N\mathbb{Z}$.

(2) $q$ is étale outside $P \in X$, and $q^{-1}(P) = Q$.

(3) $Z$ is a slc singularity and $\omega_Z$ is a line bundle.

Explicitly,

$$Z = \underline{\operatorname{Spec}}_X \left( \bigoplus_{j=0}^{N-1} \mathcal{O}_X(jK_X) \right)$$

where the multiplication is given by fixing an isomorphism $\theta\colon \mathcal{O}_X(NK_X) \to \mathcal{O}_X$, and the $\mu_N$ action is given by

$$\mu_N \ni \zeta\colon \mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X), \quad \Omega \mapsto \zeta \cdot \Omega.$$

(Note: If we work locally analytically at $P \in X$ then the isomorphism type of the covering is independent of the choice of $\theta$.) The covering $q$ is called the *index one cover* of $P \in X$.

**Example 1.1.6.** Let $(P \in X)$ be a quotient singularity $(0 \in \mathbb{A}^2/G)$, where $G \subset \mathrm{GL}(2,\mathbb{C})$ acts freely on $\mathbb{A}^2 \setminus \{0\}$. Then, writing $H = \ker(\det\colon G \to \mathbb{C}^*)$, we have index one cover

$$q\colon (Q \in Z) = (0 \in \mathbb{A}^2/H) \to (P \in X) = (0 \in \mathbb{A}^2/G)$$

with group

$$G/H = \det(G) = \mu_N \subset \mathbb{C}^*$$

for some $N \in \mathbb{N}$. Note that $H \subset \mathrm{SL}(2,\mathbb{C})$ so the index one cover is a Du Val singularity.

### 1.1.11 $\mathbb{Q}$-Gorenstein families of stable surfaces

**Definition 1.1.7.** For a scheme $S$ of finite type over $\mathbb{C}$, a *$\mathbb{Q}$-Gorenstein family of stable surfaces over $S$* is a flat morphism $\mathcal{X} \to S$ with the following properties.

(1) For each (closed) point $s \in S$ the fiber $\mathcal{X}_s$ is a stable surface, that is, $\mathcal{X}_s$ is a projective surface with slc singularities such that the dualizing sheaf $\omega_{\mathcal{X}_s}$ is ample.

(2) For each point $s \in S$ and $P \in \mathcal{X}_s$, the deformation $(P \in \mathcal{X})/(s \in S)$ of the singularity $P \in \mathcal{X}_s$ is induced by a deformation of the index one cover. That is, writing $N$ for the index of the singularity $(P \in \mathcal{X}_s)$ and $q\colon (Q \in Z) \to (P \in \mathcal{X}_s)$ for its index one cover, there is a $\mu_N$-invariant deformation $(Q \in \mathcal{Z})/(s \in S)$ of $(Q \in Z)$ such that $(P \in \mathcal{X}) = (Q \in \mathcal{Z})/\mu_N$.

**Example 1.1.8.** Let $P \in X$ be the quotient singularity $\frac{1}{dn^2}(1, dna-1)$ for some $d, n, a \in \mathbb{N}$ with $\gcd(a,n) = 1$. Then $(P \in X)$ has index $n$ and the index one cover $q\colon (Q \in Z) \to (P \in X)$ is given by $(Q \in Z) = \frac{1}{dn}(1,-1)$. In the notation of Example 1.1.6 we have $H = \mu_{dn} \subset G = \mu_{dn^2}$. The germ $(Q \in Z)$ is a Du Val singularity of type $A_{dn-1}$. There is an identification

$$(0 \in \mathbb{A}^2_{u,v}/\tfrac{1}{dn}(1,-1)) \xrightarrow{\sim} (0 \in (xy = z^{dn}) \subset \mathbb{A}^3_{x,y,z}), \quad (u,v) \mapsto (u^{dn}, v^{dn}, uv)$$

given by writing down generators for the invariant ring $k[u,v]^{\mu_{dn}}$ (the coordinate ring of the affine variety $\mathbb{A}^2/\mu_{dn}$).

The deformation $(Q \in \mathcal{Z})/(0 \in \mathbb{A}^{dn-1})$ of the hypersurface

$$(Q \in Z) = (0 \in (xy = z^{dn}) \subset \mathbb{A}^3)$$

given by

$$(Q \in \mathcal{Z}) = (0 \in (xy = z^{dn} + a_{dn-2}z^{dn-2} + \cdots + a_1 z + a_0) \subset \mathbb{A}^3_{x,y,z} \times \mathbb{A}^{dn-1}_{a_0,\ldots,a_{dn-2}})$$

is *versal*, that is, every deformation of $(Q \in Z)$ is obtained from it by pullback. In general, if $(Q \in Z)$ is an isolated hypersurface singularity $(0 \in (f = 0) \subset \mathbb{A}^3_{x,y,z})$, then the $\mathbb{C}$-vector space

$$T^1 := \mathbb{C}[[x,y,z]] \Bigg/ \left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

is finite dimensional. Letting $g_1, \ldots, g_\tau \in \mathbb{C}[x,y,z]$ be a lift of a basis of $T^1$, the deformation $(Q \in \mathcal{Z}) \to (0 \in \mathbb{A}^r)$ of $(Q \in Z)$ given by

$$(Q \in \mathcal{Z}) = (0 \in (f + t_1 g_1 + \cdots + t_\tau g_\tau = 0) \subset \mathbb{A}^3_{x,y,z} \times \mathbb{A}^r_{t_1,\ldots,t_\tau})$$

is versal [KM98], 4.61.

The action of $\mu_n = \mu_{dn^2}/\mu_{dn}$ on $(Q \in Z)$ is given by

$$\mu_n \ni \zeta \colon (x,y,z) \to (\zeta x, \zeta^{-1} y, \zeta^a z). \tag{1.1.1}$$

It lifts to an action on $(Q \in \mathcal{Z})/(0 \in \mathbb{A}^{dn-1})$ given by

$$\mu_n \ni \zeta \colon ((x,y,z),(a_j)) \to ((\zeta x, \zeta^{-1} y, \zeta^a z), (\zeta^{-ja} a_j)).$$

The versal $\mu_n$-invariant deformation of $(Q \in Z)$ is obtained as the restriction of $(Q \in \mathcal{Z})/(0 \in \mathbb{A}^{dn-1})$ to the fixed locus of the action of $\mu_n$ on the base. Assuming $n > 1$, we obtain

$$(Q \in \mathcal{Z}') = (xy = z^{dn} + a_{(d-1)n} z^{(d-1)n} + \cdots + a_n z + a_0) \subset \mathbb{A}^3_{x,y,z} \times \mathbb{A}^d_{a_0, a_n, \ldots, a_{(d-1)n}}.$$

The versal $\mathbb{Q}$-Gorenstein deformation of $(P \in X)$ is given by the quotient

$$(P \in \mathcal{X}) = (xy = z^{dn} + a_{(d-1)n} z^{(d-1)n} + \cdots + a_n z + a_0) \subset (\mathbb{A}^3_{x,y,z}/\tfrac{1}{n}(1,-1,a)) \times \mathbb{A}^d.$$

### 1.1.12 The relative dualizing sheaf

If $\mathcal{X}/S$ is a $\mathbb{Q}$-Gorenstein family of stable surfaces then the relative dualizing sheaf $\omega_{\mathcal{X}/S}$ is a $\mathbb{Q}$-line bundle on $\mathcal{X}$ which is relatively ample over $S$. This is the reason we require the condition (2) in Definition 1.1.7. To explain, note first that it is a general fact that for a flat morphism $\mathcal{X} \to S$ with Cohen-Macaulay fibers the dualizing sheaf $\omega_{\mathcal{X}/S}$ is defined and commutes with base change [C00], 3.6.1. In particular, if $s \in S$ is a point then the natural map

$$\omega_{\mathcal{X}/S}|_{\mathcal{X}_s} \to \omega_{\mathcal{X}_s}$$

is an isomorphism. In particular, $\omega_{\mathcal{X}_s}$ is a line bundle near $P \in \mathcal{X}_s$ if and only if $\omega_{\mathcal{X}/S}$ is a line bundle near $P \in \mathcal{X}$ (by Nakayama's lemma). Let $i : \mathcal{U} \subset \mathcal{X}$ denote the open locus where $\omega_{\mathcal{X}/S}$ is a line bundle, then its complement $\mathcal{X} \setminus \mathcal{U}$ has finite fibers over $S$, and we define $\omega_{\mathcal{X}/S}^{[N]} = i_* \omega_{\mathcal{U}/S}^{\otimes N}$.

Now, given $s \in S$ and $P \in \mathcal{X}_s$, let $N$ be the index of $(P \in \mathcal{X}_s)$ and $(Q \in Z) \to (P \in \mathcal{X}_s)$ the index one cover. Then the deformation $(P \in \mathcal{X})/(s \in S)$ is obtained as the quotient of a $\mu_N$-invariant deformation $(Q \in \mathcal{Z})/(s \in S)$ of $(Q \in Z)$. Now $\omega_Z$ is a line bundle by construction, so $\omega_{\mathcal{Z}/S}$ is also a line bundle by the base change property. So, working locally analytically at $P \in \mathcal{X}$, we have

$$\omega_{\mathcal{X}/S}^{[N]} = (q_* \omega_{\mathcal{Z}/S}^{\otimes N})^{\mu_N} \simeq (q_* \mathcal{O}_{\mathcal{Z}})^{\mu_N} = \mathcal{O}_X,$$

that is, $\omega_{\mathcal{X}/S}^{[N]}$ is a line bundle near $P \in \mathcal{X}$. Moreover, we have a natural isomorphism

$$\omega_{\mathcal{X}/S}^{[N]}|_{\mathcal{X}_s} \to \omega_{\mathcal{X}_s}^{[N]} = \mathcal{O}_{\mathcal{X}_s}(NK_{\mathcal{X}_s})$$

(Indeed, we have a natural isomorphism over the open set $\mathcal{U}_s$ by the base change property for $\omega_{\mathcal{U}/S}$, $\mathcal{U}_s \subset \mathcal{X}_s$ has finite complement, and both sheaves are line bundles, so the isomorphism extends over $\mathcal{X}_s$.)

As a consequence, for a $\mathbb{Q}$-Gorenstein family of slc surfaces $\mathcal{X}/S$ the numerical invariant $K_{\mathcal{X}_s}^2 \in \mathbb{Q}$ is independent of $s \in S$. This property fails in general in the absence of the $\mathbb{Q}$-Gorenstein condition.

**Example 1.1.9.** Let $\mathbb{F}_n$ denote the $n$th Hirzebruch surface, $n \geq 0$. That is, $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$ is the $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ with a section $B \subset \mathbb{F}_n$ such that $B^2 = -n$. One can compute that $h^1(T_{\mathbb{F}_n}) = \max(n-1, 0)$ and $h^2(T_{\mathbb{F}_n}) = 0$. So the versal deformation space of $\mathbb{F}_n$ is smooth of dimension $\max(n-1, 0)$.

The fibers of the versal deformation are $\mathbb{F}_m$ for $m \leq n$ and $m \equiv n \bmod 2$. This can be seen as follows: writing $\mathbb{F}_n = \mathbb{P}(E)$, $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$ we consider deformations of the (trivial) extension

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-n) \to E \to \mathcal{O}_{\mathbb{P}^1} \to 0. \tag{1.1.2}$$

These are parametrized by

$$\operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-n)) = H^1(\mathcal{O}_{\mathbb{P}^1}(-n)) = H^0(\mathcal{O}_{\mathbb{P}^1}(n-2))^* \simeq \mathbb{C}^{\max(n-1,0)}.$$

The versal deformation of $\mathbb{F}_n$ is the projectivization of the versal deformation of the extension (1.1.2).

Now let $n \geq 2$ and consider a non-trivial one parameter deformation $\mathcal{X}/\mathbb{A}^1$ of $X = \mathbb{F}_n$. Then the general fiber $\mathcal{X}_t$ is isomorphic to $\mathbb{F}_m$ for some $m < n$, $m \equiv n \bmod 2$, and the negative section $B \subset \mathbb{F}_n$ does not deform to the general fiber. There is a birational morphism $f\colon \mathcal{X} \to \mathcal{Y}$ over $\mathbb{A}^1$ with exceptional locus $B$. (The morphism $f$ can be defined explicitly by the line bundle $L$ on $\mathcal{X}$ such that $L|_{\mathbb{F}_n} = \mathcal{O}_{\mathbb{F}_n}(B + nA)$, where $A$ denotes a fiber of the morphism $\mathbb{F}_n \to \mathbb{P}^1$.) The special fiber $Y = \mathcal{Y}_0$ is the contraction of the curve $B \subset \mathbb{F}_n$. This is the weighted projective plane $Y = \mathbb{P}(1, 1, n)$ (or, equivalently, the cone over the rational normal curve $\mathbb{P}^1 \hookrightarrow \mathbb{P}^n$ of degree $n$). The exceptional curve $B \subset X$ is contracted to the point $P = (0 : 0 : 1) \in Y$, which is a cyclic quotient singularity of type $\frac{1}{n}(1, 1)$. We have $K^2_{\mathbb{F}_m} = 8$ for all $m$, whereas

$$K_Y^2 = (-(1 + 1 + n)H)^2 = \frac{(n + 2)^2}{n}.$$

Thus $K^2_{\mathcal{Y}_t}$ is not constant for $t \in \mathbb{A}^1$, and so the family $\mathcal{Y}/\mathbb{A}^1$ is not $\mathbb{Q}$-Gorenstein, unless $n = 2$. If $n = 2$ then $Y = \mathbb{P}(1, 1, 2)$ is the quadric cone, and $P \in Y$ is an $A_1$ singularity. Thus $P \in Y$ has index 1 ($\omega_Y$ is a line bundle) and any deformation is automatically $\mathbb{Q}$-Gorenstein. If $n > 2$ then the index of $P \in Y$ is $n/\gcd(n, 2)$.

If $n = 4$ then $K_Y^2 = 9 \in \mathbb{Z}$. In this case the surface $Y = \mathbb{P}(1, 1, 4)$ admits a one parameter $\mathbb{Q}$-Gorenstein deformation $\mathcal{Z}/\mathbb{A}^1$ with general fiber the projective plane $\mathbb{P}^2$. It may be constructed explicitly as follows. Let $\mathcal{W} = \mathbb{P}^2 \times \mathbb{A}^1$ be the trivial family over $\mathbb{A}^1$ with fiber $\mathbb{P}^2$. Let $Q \subset W = \mathcal{W}_0 = \mathbb{P}^2$ be a smooth conic in the special fiber. Let $\tilde{\mathcal{W}} \to \mathcal{W}$ be the blowup of the conic $Q$. The special fiber $\tilde{\mathcal{W}}_0$ is a (reduced) normal crossing divisor with irreducible components the exceptional divisor $E$ and the strict transform $W'$ of the special fiber $W$. Here $W' \to W$ is an isomorphism and $E \simeq \mathbb{F}_4$ (because the normal bundle of $\mathbb{P}^1 \simeq Q \subset \mathcal{W}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(4) \oplus \mathcal{O}_{\mathbb{P}^1}$). The intersection $W' \cap E$ is given by $Q \subset W$ and the negative section $B \subset \mathbb{F}_4$. The normal bundle $\mathcal{N}_{W'/\tilde{\mathcal{W}}}$ of $W' \simeq \mathbb{P}^2$ in $\tilde{\mathcal{W}}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(-2)$. (Indeed $W' + E = \tilde{\mathcal{W}}_0 \sim 0$, so $W'|_{W'} \sim -E|_{W'} = -Q$.) It follows that there is a birational contraction $\tilde{\mathcal{W}} \to \mathcal{Z}$ over $\mathbb{A}^1$ with exceptional locus $W' \simeq \mathbb{P}^2$, with $W'$ being contracted to a singular point $P \in \mathcal{Z}$ which is a cyclic quotient singularity of type $\frac{1}{2}(1, 1, 1)$. The special fiber $Z = \mathcal{Z}_0$ is isomorphic to the contraction of $B \subset \mathbb{F}_4$, that is, $Z \simeq Y = \mathbb{P}(1, 1, 4)$. For $t \neq 0$ the fiber $\mathcal{Z}_t = \mathcal{W}_t = \mathbb{P}^2$ is unchanged. The family $\mathcal{Z}/\mathbb{A}^1$ is $\mathbb{Q}$-Gorenstein because the index one cover of the 3-fold quotient singularity $P \in \mathcal{Z}$ defines a equivariant deformation of the index one cover of $P \in Z$. The local deformation $(P \in \mathcal{Z})/(0 \in \mathbb{A}^1_t)$ is isomorphic to the versal $\mathbb{Q}$-Gorenstein deformation of Example 1.1.8 for $d = n = a = 1$.

The deformation $\mathcal{Z}/\mathbb{A}^1$ may also be described explicitly by equations as follows: The surface $\mathbb{P}(1, 1, 4)$ may be embedded in the weighted projective space $\mathbb{P}(1, 1, 1, 2)$ by the 2-uple embedding

$$\mathbb{P}(1, 1, 4) \xrightarrow{\sim} (X_0X_2 = X_1^2) \subset \mathbb{P}(1, 1, 1, 2)$$

$$(U_0, U_1, V) \mapsto (X_0, X_1, X_2, Y) = (U_0^2, U_0U_1, U_1^2, V).$$

Then the family $\mathcal{Z}/\mathbb{A}^1_t$ is the deformation

$$\mathcal{Z} = (X_0X_2 = X_1^2 + tY) \subset \mathbb{P}(1,1,1,2) \times \mathbb{A}^1_t.$$

### 1.1.13 Definition of the moduli space $\overline{M}_{K^2,\chi}$ of stable surfaces

We can now define the compactification $M_{K^2,\chi} \subset \overline{M}_{K^2,\chi}$ of the moduli space of surfaces of general type. For $S$ a scheme of finite type over $\mathbb{C}$, let $\overline{\mathcal{M}}_{K^2,\chi}(S)$ denote the category with objects $\mathbb{Q}$-Gorenstein families $\mathcal{X}/S$ of stable surfaces over $S$ such that $K^2_{\mathcal{X}_s} = K^2$ and $\chi(\mathcal{O}_{\mathcal{X}_s}) = \chi$ for each $s \in S$, and morphisms isomorphisms $\mathcal{X} \to \mathcal{X}'$ of schemes over $S$. This defines a stack $\overline{\mathcal{M}}_{K^2,\chi}$ over the category of schemes of finite type over $\mathbb{C}$ for the étale topology.

**Theorem 1.1.10.** *[KSB88],[AM04] The stack $\overline{\mathcal{M}}_{K^2,\chi}$ is a proper Deligne–Mumford stack of finite type over $\mathbb{C}$.*

Properness of the moduli stack $\overline{\mathcal{M}}_{K^2,\chi}$ follows from the minimal model program (MMP) for 3-folds. This is the exact analogue of the stable reduction theorem for curves, which uses the classical theory of minimal models of surfaces, cf. [DM69], 1.12. Given a family of smooth surfaces of general type over a punctured disc, results of the MMP produce (after a finite base change) a distinguished extension of the family over the disc (the *relative canonical model* of an extension with special fiber a reduced normal crossing divisor). The definition of stable surface is obtained by characterizing the possible special fibers of relative canonical models, so the moduli stack satisfies the valuative criterion of properness.

The moduli space $\overline{M}_{K^2,\chi}$ is by definition the coarse moduli space of the stack $\overline{\mathcal{M}}_{K^2,\chi}$.

**Theorem 1.1.11.** *[K90] The moduli space $\overline{M}_{K^2,\chi}$ is a projective scheme of finite type over $\mathbb{C}$.*

## 1.2 Wahl singularities

**Definition 1.2.1.** A *Wahl singularity* $(P \in X)$ is a surface cyclic quotient singularity of type $\frac{1}{n^2}(1, na-1)$, for some $n, a$ with $\gcd(a, n) = 1$.

Let $P \in X$ be a Wahl singularity of type $\frac{1}{n^2}(1, na-1)$. Then $P \in X$ has index $n$ and the index one cover $q\colon (Q \in Z) \to (P \in X)$ is a cyclic quotient singularity of type $\frac{1}{n}(1,-1)$, that is, a Du Val singularity of type $A_{n-1}$. We obtain an identification

$$(P \in X) = (Q \in Z)/\mu_n \simeq (0 \in (xy = z^n) \subset \mathbb{A}^3/\tfrac{1}{n}(1,-1,a)).$$

The versal $\mathbb{Q}$-Gorenstein deformation of $(P \in X)$ is given by

$$(P \in \mathcal{X}) = (0 \in (xy = z^n + t) \subset (\mathbb{A}^3/\tfrac{1}{n}(1,-1,a)) \times \mathbb{A}^1_t).$$

This is the special case $d = 1$ of Example 1.1.8. In particular, the versal $\mathbb{Q}$-Gorenstein deformation space of $(P \in X)$ is smooth of dimension 1, and the general fiber of the deformation is smooth.

### 1.2.1 Degenerations with Wahl singularities define boundary divisors of the moduli space $\overline{M}_{K^2,\chi}$

We say that a slc surface $X$ admits a $\mathbb{Q}$-*Gorenstein smoothing* if there exists a one parameter $\mathbb{Q}$-Gorenstein deformation $\mathcal{X}/(0 \in \mathbb{A}^1_t)$ of $X$ with smooth general fiber. For $X$ a variety with isolated singularities, we say a deformation $\mathcal{X}/(0 \in S)$ is *equisingular* if for each singularity $P \in X$ the local deformation $(P \in \mathcal{X})/(0 \in S)$ is trivial.

If $X$ is a normal projective surface with a unique singularity $P \in X$ of Wahl type, $K_X$ is ample, and $X$ admits a $\mathbb{Q}$-Gorenstein smoothing, then the locus of equisingular deformations of $X$ defines a boundary divisor $D \subset \overline{M}_{K^2,\chi}$. Indeed, if $\mathcal{X}/(0 \in S)$ is the versal $\mathbb{Q}$-Gorenstein deformation of $X$, and $(P \in \mathcal{X}^{\mathrm{loc}})/(0 \in \mathbb{A}^1)$ is the versal $\mathbb{Q}$-Gorenstein deformation of the Wahl singularity $P \in X$, then (by the versal property) there is a morphism (not uniquely determined) of local analytic germs

$$F\colon (0 \in S) \to (0 \in \mathbb{A}^1_t)$$

such that the local deformation $(P \in \mathcal{X})/(0 \in S)$ is isomorphic to the pullback of $(P \in \mathcal{X}^{\mathrm{loc}})/(0 \in \mathbb{A}^1)$ under $F$. Since $X$ admits a $\mathbb{Q}$-Gorenstein smoothing, $F$ is not identically zero and the locus $F^{-1}(0) \subset S$ of equisingular deformations is a Cartier divisor. (More carefully, if $S$ is reducible, we require that every component of $S$ contains smoothings, so that $F$ is nonzero on each component.) The moduli space $\overline{M}_{K^2,\chi}$ has local analytic chart

$$([X] \in \overline{M}_{K^2,\chi}) \simeq (0 \in S/\operatorname{Aut}(X))$$

and the locus of equisingular deformations of $X$ is identified with the divisor $D = F^{-1}(0)/\operatorname{Aut}(X)$.

If in addition $H^2(T_X) = 0$ then the morphism $F$ is smooth (there are no local-to-global obstructions), so $S$ is smooth and $F^{-1}(0) \subset S$ is a smooth divisor. Thus $[X] \in D \subset \overline{M}_{K^2,\chi}$ is locally the quotient of a smooth space with smooth divisor by a finite group.

### 1.2.2 Topology of Wahl degenerations

Let $\mathcal{X}/(0 \in \mathbb{A}^1_t)$ be a one parameter $\mathbb{Q}$-Gorenstein smoothing of a projective normal surface $X$ with a unique singularity $P \in X$ of Wahl type $\frac{1}{n^2}(1, na-1)$. Let $Y = \mathcal{X}_t$, $0 < |t| \ll 1$ denote a nearby smooth fiber. We want to understand the topology of the degeneration $Y \rightsquigarrow X$.

### The Milnor fiber of a smoothing and link of a singularity

In general, given a degeneration $\mathcal{X}/(0 \in \mathbb{A}^1_t)$ of a smooth surface $Y$ to a normal surface $X$ with a unique singular point $P \in X$, the change in topology is captured by the *Milnor fiber* of the degeneration. This is defined as follows: Fix an embedding $(P \in X) \subset (0 \in \mathbb{A}^d)$ for some $d$, and lift to an embedding $(P \in \mathcal{X}) \subset \mathbb{A}^d \times \mathbb{A}^1_t$ over $\mathbb{A}^1_t$. Let $B \subset \mathbb{A}^d$ be the closed ball with center the origin and radius $\delta \ll 1$. The *Milnor fiber* $M$ of the smoothing $(P \in \mathcal{X})/(0 \in \mathbb{A}^1_t)$ of $(P \in X)$ is the intersection $\mathcal{X}_t \cap B \subset \mathbb{A}^d$ for $0 < |t| \ll \delta$. The space $M$ is a $C^\infty$ 4-manifold with boundary $\partial M$ (independent of the choice of embedding and $\delta$,$t$). The boundary $\partial M$ is diffeomorphic to the *link* of the singularity, that is, the intersection $L = \partial B \cap X$ of $X$ with a small sphere centered at the singular point in an embedding $X \subset \mathbb{A}^d$.

**Example 1.2.2.** Suppose $(P \in X) \simeq (0 \in (f(x,y,z) = 0) \subset \mathbb{A}^3_{x,y,z})$ is an isolated hypersurface singularity. Then the versal deformation of $(P \in X)$ is given by

$$(P \in \mathcal{X}) = (0 \in (f + t_1 g_1 + \cdots + t_\tau g_\tau = 0) \subset \mathbb{A}^3_{x,y,z} \times \mathbb{A}^r_{t_1,\ldots,t_\tau})$$

where $g_1, \ldots, g_\tau$ is a lift of a basis of the finite dimensional $\mathbb{C}$-vector space

$$T^1 := \mathbb{C}[[x,y,z]] \Big/ \left( f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).$$

In particular, the base of the versal deformation is smooth of dimension $\tau = \dim_{\mathbb{C}} T^1$. The Milnor fiber $M$ of a smoothing of $P \in X$ is homotopy equivalent to a bouquet of $\mu$ copies of $S^2$ [M68], where $\mu$ is the dimension of the finite dimensional $\mathbb{C}$-vector space

$$\mathbb{C}[[x,y,z]]/\left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).$$

In particular $M$ is simply connected and $H_2(M,\mathbb{Z}) \simeq \mathbb{Z}^\mu$.

In general, the Milnor fiber $M$ of the smoothing of a complex surface singularity is a Stein manifold of complex dimension 2. So it has the homotopy type of a CW complex of real dimension 2. In particular $H_2(M,\mathbb{Z})$ is torsion-free and $H_i(M,\mathbb{Z}) = 0$ for $i > 2$. The number $\mu := \operatorname{rk} H_2(M,\mathbb{Z})$ is called the *Milnor number* of the smoothing.

In our case, recall that $P \in X$ is a cyclic quotient singularity of type $\frac{1}{n^2}(1, na-1)$. Thus the link $L$ of $P \in X$ is diffeomorphic to the lens space

$$L \simeq S^3/\mu_{n^2} \quad S^3 = (|u|^2 + |v|^2 = 1) \subset \mathbb{A}^2_{u,v}, \quad \mu_{n^2} \ni \zeta \colon (u,v) \mapsto (\zeta u, \zeta^{na-1} v).$$

In particular $\pi_1(L) = \mu_{n^2} \simeq \mathbb{Z}/n^2\mathbb{Z}$.

The Milnor fiber of the Wahl degeneration can be understood as follows: Recall that the deformation $(P \in \mathcal{X})/(0 \in \mathbb{A}^1)$ is the quotient of the smoothing

$$(Q \in \mathcal{Z}) = (0 \in (xy = z^n + t) \subset \mathbb{A}^3_{x,y,z} \times \mathbb{A}^1_t)$$

of an $A_{n-1}$ singularity $(Q \in Z)$ by the $\mu_n$ action with weights $(1,-1,a)$. Note that the $\mu_n$ action is free on the general fiber. Thus the Milnor fiber $M$ of the smoothing of $(P \in X)$ is the quotient of the Milnor fiber $M_Z$ of the smoothing of $(Q \in Z)$ by a free $\mu_n$ action. Now $M_Z$ is homotopy equivalent to a bouquet of $n-1$ copies of $S^2$ by Example 1.2.2. In particular $M_Z$ is simply connected, so $M_Z$ is the universal cover of $M$ and $\pi_1(M) = \mu_n \simeq \mathbb{Z}/n\mathbb{Z}$. Also, $ne(M) = e(M_Z) = n$, so $e(M) = 1$ and $H_2(M, \mathbb{Z}) = 0$.

One can also give the following more precise topological description of $M$ [K92], 2.1. Let $N_Z$ denote the union of $n$ copies $\overline{\Delta}_j$, $j \in \mathbb{Z}/n\mathbb{Z}$ of the closed disc

$$\overline{\Delta} = \{z \in \mathbb{C} \mid |z| \leq 1\}$$

with their boundaries identified. Let $\mathbb{Z}/n\mathbb{Z}$ act on $N_Z$ via

$$\mathbb{Z}/n\mathbb{Z} \ni 1 \colon \overline{\Delta}_j \to \overline{\Delta}_{j+1}, \quad z \mapsto \zeta z$$

where $\zeta$ is a primitive $n$th root of unity. Then $\mathbb{Z}/n\mathbb{Z}$ acts freely on $N_Z$; let $N$ denote the quotient. Then the Milnor fiber $M$ is homotopy equivalent to $N$. (More precisely, $N_Z$ is a $\mathbb{Z}/n\mathbb{Z}$-equivariant deformation retract of $M_Z$, so $N$ is a deformation retract of $M$.)

Note for future reference that the map $\pi_1(L) \to \pi_1(M)$ given by the inclusion $L = \partial M \subset M$ equals the surjection

$$\mu_{n^2} \to \mu_n, \quad \zeta \mapsto \zeta^n.$$

**Global topology of degenerations of surfaces**

Let $Y \rightsquigarrow X$ be a degeneration of a smooth surface $Y$ to a normal surface $X$ with a unique singularity $P \in X$. Let $M \subset Y$ denote the Milnor fiber and $L$ the link of the singularity $P \in X$. Also let $C = B \cap X$ denote the intersection of $X$ with a small closed ball $B$ centered at the singular point $P \in X$ in some embedding. Then $C$ is homeomorphic to the cone over $L$, in particular, $C$ is contractible. Let $X^o \subset X$ denote the complement of the interior of $C$.

We have the following general result, cf. [SGA7.II], (1.3.6.1).

**Lemma 1.2.3.** *There is a natural exact sequence*

$$\cdots \to H_i(M, \mathbb{Z}) \to H_i(Y, \mathbb{Z}) \to \tilde{H}_i(X, \mathbb{Z}) \to H_{i-1}(M, \mathbb{Z}) \to \cdots$$

*(Here $\tilde{H}_i(X, \mathbb{Z})$ denotes reduced homology, that is, $\tilde{H}_i(X, \mathbb{Z}) = H_i(X, \mathbb{Z})$ for $i > 0$ and $\tilde{H}_0(X, \mathbb{Z}) = \ker(H_0(X, \mathbb{Z}) \to \mathbb{Z}) = 0$.)*

*Proof.* We have $H_i(Y, M, \mathbb{Z}) = H_i(X^o, L, \mathbb{Z}) = H_i(X, C, \mathbb{Z}) = \tilde{H}_i(X, \mathbb{Z})$ by excision and contractibility of $C$. So the long exact sequence of homology for the pair $(Y, M)$ gives the exact sequence in the statement. □

**Corollary 1.2.4.** Let $Y \rightsquigarrow X$ be a degeneration of a smooth surface $Y$ to a normal surface $X$ with a unique singularity $P \in X$ of Wahl type. Then the specialization map $H_i(Y,\mathbb{Q}) \to H_i(X,\mathbb{Q})$ on homology with rational coefficients is an isomorphism for each $i$, and we have an exact sequence of integral homology

$$0 \to H_2(Y,\mathbb{Z}) \to H_2(X,\mathbb{Z}) \to H_1(M,\mathbb{Z}) \to H_1(Y,\mathbb{Z}) \to H_1(X,\mathbb{Z}) \to 0 \quad (1.2.1)$$

Moreover, the following conditions are equivalent.

(1) The map
$$H_2(X,\mathbb{Z}) \to H_1(L,\mathbb{Z}), \quad \alpha \mapsto \alpha \cap [L]$$
is surjective.

(2) The specialization map
$$H_1(Y,\mathbb{Z}) \to H_1(X,\mathbb{Z})$$
is injective.

(3) The exact sequence of Lemma 1.2.3 yields a short exact sequence
$$0 \to H_2(Y,\mathbb{Z}) \to H_2(X,\mathbb{Z}) \to H_1(M,\mathbb{Z}) \to 0.$$

If $H_1(Y,\mathbb{Z})$ is finite of order coprime to the index $n$ of $P \in X$, then the above conditions are satisfied.

*Proof.* The Milnor fiber $M$ is a rational homology ball, that is, $H_i(M,\mathbb{Q}) = 0$ if $i > 0$ and $H_0(M,\mathbb{Q}) = \mathbb{Q}$. So the exact sequence of Lemma 1.2.3 with $\mathbb{Q}$ coefficients shows that the specialization map $H_i(Y,\mathbb{Q}) \to H_i(X,\mathbb{Q})$ is an isomorphism for each $i$. The same exact sequence with $\mathbb{Z}$ coefficients together with the equality $H_2(M,\mathbb{Z}) = 0$ gives the exact sequence of integral homology (1.2.1). The equivalence of conditions (2) and (3) follows immediately.

The map $H_1(L,\mathbb{Z}) \to H_1(M,\mathbb{Z})$ given by the inclusion $L = \partial M \subset M$ is a surjection of the form $\mathbb{Z}/n^2\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. Thus the map $H_2(X,\mathbb{Z}) \to H_1(L,\mathbb{Z})$ of (1) is surjective if and only if the composite map $H_2(X,\mathbb{Z}) \to H_1(M,\mathbb{Z})$ is surjective. So (1) and (2) are equivalent by (1.2.1).

Finally assume $H_1(Y,\mathbb{Z})$ is finite of order coprime to $n$. Then the map $H_1(M,\mathbb{Z}) \to H_1(Y,\mathbb{Z})$ is the zero map (because $H_1(M,\mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}$). Now (1.2.1) implies (2). □

## 1.3 Examples of degenerations of Wahl type

The simplest example of a Wahl degeneration is the family $\mathcal{Z}/(0 \in \mathbb{A}^1)$ of Example 1.1.9 with special fiber $\mathbb{P}(1,1,4)$ and general fiber $\mathbb{P}^2$. Here we describe some more complicated examples.

**Example 1.3.1.** Let $Y = Y_{10} \subset \mathbb{P}(1,1,2,5)$ be a smooth hypersurface of degree 10 in the weighted projective space $\mathbb{P} = \mathbb{P}(1,1,2,5)$. Let $X_0, X_1, U, V$ denote the homogeneous coordinates on $\mathbb{P}$ of degrees $1,1,2,5$. Completing the square we can write

$$Y = (V^2 = F_{10}(X_0, X_1, U)) \subset \mathbb{P}(1,1,2,5).$$

Thus $Y$ is the double cover of the quadric cone $Z = \mathbb{P}(1,1,2)$ branched over a smooth curve $B = (F_{10} = 0) \subset \mathbb{P}(1,1,2)$ of degree 10. (Note also that the cover $Y \to \mathbb{P}(1,1,2)$ is ramified over the singular point $Q = (0:0:1) \in \mathbb{P}(1,1,2)$. This is an $A_1$ singularity and the cover is locally given by the smooth cover $\mathbb{A}^2 \to \mathbb{A}^2/\frac{1}{2}(1,1)$.) We have $K_Y = \mathcal{O}_Y(1)$ by the adjunction formula, $K_Y^2 = 1$, $h^1(\mathcal{O}_Y) = 0$, and $h^2(\mathcal{O}_Y) = h^0(K_Y) = 2$. Thus $Y$ is a surface of general type with invariants $K_Y^2 = 1$ and $\chi(\mathcal{O}_Y) = 3$.

We can also describe $Y$ as a genus 2 fibration: Note that the rational map $\varphi = \varphi_{|K_Y|}$ defined by the linear system $|K_Y|$ is the projection

$$\varphi\colon Y \dashrightarrow \mathbb{P}^1, \quad (X_0 : X_1 : U : V) \mapsto (X_0 : X_1).$$

Moreover $|K_Y|$ has a unique basepoint $P \in Y$. Let $\tilde{Y} \to Y$ denote the blowup of $P$, with exceptional divisor $E$. Then $\varphi$ lifts to a morphism $\tilde{\varphi}\colon \tilde{Y} \to \mathbb{P}^1$ with general fiber a smooth curve of genus 2. Moreover $E$ defines a section of $\tilde{\varphi}$, and for each smooth fiber $F$ the point $W := E \cap F$ is a Weierstrass point of $F$ (because $K_F = (K_{\tilde{Y}} + F)|_F = K_{\tilde{Y}}|_F = (\pi^* K_Y + E)|_F = (F + 2E)|_F = 2W$).

Now let $\mathcal{A} \subset \mathbb{P}(1,1,2) \times \mathbb{A}^1_t$ be a degeneration of $B$ with special fiber $A = \mathcal{A}_0$ a curve containing the singular point $Q \in \mathbb{P}(1,1,2)$ but otherwise general. Let $\mathcal{X}/(0 \in \mathbb{A}^1_t)$ be the associated degeneration of $Y$ (the double cover with branch locus $\mathcal{A}$). Then, writing $x_i = X_i/U^{1/2}$, $i = 0,1$ for the orbifold coordinates at $(Q \in Z) \simeq (0 \in \mathbb{A}^2/\frac{1}{2}(1,1))$ and $g = g(x_0, x_1)$ for the local equation of $A$, we have

$$g(x_0, x_1) = a x_0^2 + b x_0 x_1 + c x_1^2 + \cdots$$

where $a, b, c \in \mathbb{C}$ are general and $\cdots$ denotes higher order terms. By a local analytic change of coordinates we may assume $g(x_0, x_1) = x_0 x_1$. Let $P$ denote the inverse image of $Q$ under the double cover $X \to \mathbb{P}(1,1,2)$. Then

$$(P \in X) \simeq (0 \in (v^2 = x_0 x_1) \subset \mathbb{A}^3/\tfrac{1}{2}(1,1,1))$$

where $v = V/U^{5/2}$. Thus $P \in X$ is a Wahl singularity of type $\frac{1}{4}(1,1)$. Moreover the deformation $(P \in \mathcal{X})/(0 \in \mathbb{A}^1_t)$ is $\mathbb{Q}$-Gorenstein.

One can study the surface $X$ via its minimal resolution $\pi\colon \tilde{X} \to X$. Since $P \in X$ is a $\frac{1}{4}(1,1)$ singularity the exceptional locus of $\pi$ is a $(-4)$-curve $C$ and $K_{\tilde{X}} = \pi^* K_X - \frac{1}{2}C$. Thus $K_{\tilde{X}}^2 = 0$. The rational map $\psi\colon X \dashrightarrow \mathbb{P}^1$ defined by $|K_X|$ lifts to a morphism $\tilde{\psi}\colon \tilde{X} \to \mathbb{P}^1$ which realizes $\tilde{X}$ as a minimal elliptic fibration over $\mathbb{P}^1$. We have $e(\tilde{X}) = 12\chi(\mathcal{O}_{\tilde{X}}) = 12\chi(\mathcal{O}_X) = 36$ (by Noether's formula), so generically $\tilde{\psi}$ has 36 ordinary singular fibers. The exceptional $(-4)$-curve $C \subset \tilde{X}$

has degree 2 over the base $\mathbb{P}^1$. The equisingular deformations of $X$ (or equivalently, the deformations of $\tilde{X}$ such that the $(-4)$-curve $C$ deforms) define a boundary divisor $D \subset \overline{M}_{K^2,\chi}$ for $K^2 = K_Y^2 = 1$ and $\chi = \chi(\mathcal{O}_Y) = 3$.

**Example 1.3.2.** In this example we describe a degeneration of a smooth quintic $Y = Y_5 \subset \mathbb{P}^3$ with a $\frac{1}{4}(1,1)$ singularity due to Julie Rana [R13]. A related example was described by R. Friedman [F83].

We consider a degeneration of quintic surfaces of the form

$$\mathcal{V} = (AU^2 + tBU + t^2C = 0) \subset \mathbb{P}^3 \times \mathbb{A}^1_t$$

where $U$ is a general quadric and $A, B, C$ are general homogeneous forms of degrees $1, 3, 5$ respectively. In particular, the general fiber $Y = \mathcal{V}_t$ is a smooth quintic and $Q := (U = 0) \subset \mathbb{P}^3$ is a smooth quadric, $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Note that the special fiber $\mathcal{V}_0 = (AU^2 = 0) \subset \mathbb{P}^3$ is the union of the smooth quadric $Q$ with multiplicity 2 and the hyperplane $H = (A = 0)$. We perform a birational transformation $\mathcal{V} \dashrightarrow \mathcal{X}$ which is an isomorphism over $\mathbb{A}^1 \setminus \{0\}$ such that the special fiber $X = \mathcal{X}_0$ is a normal surface with a $\frac{1}{4}(1,1)$ singularity and the family $\mathcal{X}/\mathbb{A}^1_t$ is $\mathbb{Q}$-Gorenstein.

Write $\mathcal{W} = \mathbb{P}^3 \times \mathbb{A}^1_t$ and let $\tilde{\mathcal{W}} \to \mathcal{W}$ denote the blowup of $Q \times \{0\} \subset \mathcal{W}$. Then the special fiber of $\tilde{\mathcal{W}}$ is a normal crossing divisor with components the strict transform $W'$ of the special fiber $W = \mathcal{W}_0 = \mathbb{P}^3$ and the exceptional divisor $E$. The induced morphism $W' \to W$ is an isomorphism, the exceptional divisor $E$ is a $\mathbb{P}^1$-bundle over the quadric $Q$, and $W'$ and $E$ meet along the quadric $Q \subset W'$ and a section of the $\mathbb{P}^1$-bundle $E \to Q$. There is a birational contraction $\mathcal{W} \to \mathcal{Z}$ with exceptional locus $W'$, such that $W'$ is contracted to a $\frac{1}{2}(1,1,1,1)$ singularity. The family $\mathcal{Z}/\mathbb{A}^1$ may be described explicitly by

$$\mathcal{Z} = (U = tY) \subset \mathbb{P}(1,1,1,1,2) \times \mathbb{A}^1_t$$

where $Y$ is the homogeneous coordinate of degree 2 on $\mathbb{P}(1,1,1,1,2)$. (The construction of $\mathcal{Z}/\mathbb{A}^1$ here is analogous to the construction of the family with the same name in Example 1.1.9.)

Write $\tilde{\mathcal{X}} \subset \tilde{\mathcal{W}}$ and $\mathcal{X} \subset \mathcal{Z}$ for the strict transforms of $\mathcal{V} \subset \mathcal{W}$. The special fiber $\tilde{\mathcal{X}}_0$ is a normal crossing divisor with components the hyperplane $H \subset W' \simeq \mathbb{P}^3$ and a surface $\tilde{X} \subset E$ meeting $H$ along the smooth conic $Q \cap H \subset H$. The family $\mathcal{X}/\mathbb{A}^1$ may be described explicitly by

$$\mathcal{X} = (AY^2 + BY + C = 0) \subset \mathcal{Z},$$

or equivalently

$$\mathcal{X} = (U = tY,\, AY^2 + BY + C = 0) \subset \mathbb{P}(1,1,1,1,2) \times \mathbb{A}^1_t.$$

Let $P = ((0:0:0:0:1), 0) \in \mathbb{P}(1,1,1,1,2) \times \mathbb{A}^1$. Then, passing to the affine chart $(Y \neq 0)$, we have

$$(P \in \mathcal{X}) \simeq (u = t, a + \cdots = 0) \subset (\mathbb{A}^4_{x_0,\ldots,x_3}/\tfrac{1}{2}(1,1,1,1)) \times \mathbb{A}^1_t$$

where $x_i = X_i/Y^{1/2}$, $u = U/Y$, $a = A/Y^{1/2}$ and $\cdots$ denotes higher order terms. Now since $A$ is a general linear form and $U$ is a general quadric it follows that the special fiber $(P \in X)$ is a $\frac{1}{4}(1,1)$ singularity and $(P \in \mathcal{X})/\mathbb{A}^1_t$ is a $\mathbb{Q}$-Gorenstein smoothing. Moreover the birational morphism $\tilde{X} \to X$ induced by $\tilde{\mathcal{W}} \to \mathcal{Z}$ is the minimal resolution of $P \in X$.

Now we study the singular surface $X$. We have

$$X = (AY^2 + BY + C = U = 0) \subset \mathbb{P}(1,1,1,1,2).$$

So, the projection

$$\mathbb{P}(1,1,1,1,2) \dashrightarrow \mathbb{P}^3$$

defines a degree 2 rational map

$$\varphi\colon X \dashrightarrow Q = (U = 0) \subset \mathbb{P}^3$$

with branch locus $D \subset Q$ given by the discriminant $B^2 - 4AC$, a smooth curve of bidegree $(6,6)$ on $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Consider the minimal resolution $\tilde{X} \to X$, and let $C$ denote the exceptional locus (a $(-4)$-curve). The rational map $\varphi$ lifts to a finite morphism

$$\tilde{\varphi}\colon \tilde{X} \to Q.$$

Thus $\tilde{X}$ is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ with branch locus $D$ of bidegree $(6,6)$ such that $\tilde{X}$ contains a $(-4)$-curve $C$. In terms of the branch locus $D$, the existence of the $(-4)$-curve corresponds to the existence of a smooth curve $B \subset Q$ of bidegree $(1,1)$ such that $B$ is "totally tangent" to $D$, that is, at each intersection point of $B$ and $D$ the curves meet with contact order 2. Then the inverse image of $B$ is the union of two $(-4)$-curves. These surfaces $X$ define a boundary divisor of the moduli space $\overline{M}_{K^2,\chi}$ for $K^2 = K_Y^2 = 5$, $\chi = \chi(\mathcal{O}_Y) = 5$.

**Example 1.3.3.** Consider the action of $\mu_5$ on $\mathbb{P}^3$ given by

$$\mu_5 \ni \zeta\colon (X_0 : X_1 : X_2 : X_3) \mapsto (X_0 : \zeta^1 X_1 : \zeta^2 X_2 : \zeta^3 X_3).$$

Let $F$ be a general quintic form which is $\mu_5$-invariant and write $W = (F = 0) \subset \mathbb{P}^3$. Then $W$ is smooth, $\mu_5$ acts freely on $W$, and the quotient $Y = W/\mu_5$ is a surface of general type with $K_Y^2 = 1$, $\chi(\mathcal{O}_Y) = 1$, and $\pi_1(Y) = \mu_5 \simeq \mathbb{Z}/5\mathbb{Z}$. The surface $W$ is a *classical Godeaux surface*. Every surface of general type with $K^2 = 1$, $\chi = 1$, and $\pi_1 \simeq \mathbb{Z}/5\mathbb{Z}$ arises in this way [R78].

Now consider a family $\mathcal{Z}/(0 \in \mathbb{A}^1_t)$ of $\mu_5$-invariant quintics such that the special fiber $Z = \mathcal{Z}_0$ passes through the fixed point $Q = (1:0:0:0) \in \mathbb{P}^3$ of the $\mu_5$ action, but the family is otherwise general. Let $\mathcal{X}/(0 \in \mathbb{A}^1_t)$ denote the quotient of the family $\mathcal{Z}/(0 \in \mathbb{A}^1_t)$ by the $\mu_5$ action. Let $X = \mathcal{X}_0$ denote the special fiber and $P \in X$ the image of the point $Q \in Z$. Passing to the affine chart $(X_0 \neq 0)$ we have

$$(P \in X) = (0 \in (ax_2x_3 + bx_1^5 + \cdots = 0) \subset \mathbb{A}^3_{x_1,x_2,x_3}/\tfrac{1}{5}(1,2,3))$$

for general $a, b \in \mathbb{C}$, where $\cdots$ denotes the remaining monomials. After a local analytic change of coordinates we find

$$(P \in X) \simeq (0 \in (x_2 x_3 = x_1^5) \subset \mathbb{A}^3_{x_1,x_2,x_3}/\tfrac{1}{5}(1,2,3))$$

Also, note that the composition of the $\mu_5$ action $\frac{1}{5}(1,2,3)$ with the automorphism $\mu_5 \to \mu_5$, $\zeta \mapsto \zeta^2$ gives the $\mu_5$ action $\frac{1}{5}(2,-1,1)$. Thus $(P \in X)$ is a Wahl singularity of type $\frac{1}{25}(1,9)$, and $(P \in \mathcal{X})/(0 \in \mathbb{A}^1_t)$ is a $\mathbb{Q}$-Gorenstein smoothing. One can show that the minimal resolution $\tilde{X}$ of $X$ is a rational surface.

Note also that in this case the specialization map

$$H_1(Y, \mathbb{Z}) \to H_1(X, \mathbb{Z})$$

is the zero map, so the equivalent conditions of Corollary 1.2.4 are not satisfied. (This is the case whenever the index one cover of the singularity $P \in X$ is induced by a global covering of $X$ which is étale over $X \setminus \{P\}$.)

## 1.4 Exceptional vector bundles associated to Wahl degenerations

Let $Y$ be a smooth projective surface. A vector bundle $F$ on $Y$ is called *exceptional* if $\mathrm{Hom}(F,F) = \mathbb{C}$ and $\mathrm{Ext}^1(F,F) = \mathrm{Ext}^2(F,F) = 0$. In particular $F$ is indecomposable ($F$ does not split as a direct sum) and rigid ($F$ has no non-trivial deformations). Moreover, if $\mathcal{Y}/(0 \in S)$ is a deformation of $Y$, then $F$ deforms in a unique way to the nearby fibers.

*Remark* 1.4.1. Note that $\mathrm{Ext}^i(F,F) = H^i(\mathcal{H}om(F,F))$ (because $F$ is a locally free sheaf) and $\mathcal{H}om(F,F)$ contains $\mathcal{O}_Y$ as a direct summand. So a necessary condition for the existence of exceptional bundles is $H^1(\mathcal{O}_Y) = H^2(\mathcal{O}_Y) = 0$.

**Theorem 1.4.2.** *[H13] Let $\mathcal{X}/(0 \in S)$ be a one parameter $\mathbb{Q}$-Gorenstein smoothing of a normal projective surface $X$ with a unique singularity $P \in X$ of Wahl type $\frac{1}{n^2}(1, na-1)$. Let $Y$ denote a general fiber of $\mathcal{X}/(0 \in S)$. Assume that $H^1(\mathcal{O}_Y) = H^2(\mathcal{O}_Y) = 0$ and the map*

$$H_1(Y, \mathbb{Z}) \to H_1(X, \mathbb{Z})$$

*is injective.*

*(1) We have an exact sequence*

$$0 \to H_2(Y,\mathbb{Z}) \stackrel{\mathrm{sp}}{\to} H_2(X,\mathbb{Z}) \to H_1(M,\mathbb{Z}) \to 0$$

*where* sp *is the specialization map and $M$ denotes the Milnor fiber of the smoothing (so $H_1(M,\mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}$).*

*(2) After a finite base change $S' \to S$ there is a rank $n$ reflexive sheaf $\mathcal{E}$ on $\mathcal{X}'$ with the following properties.*

*(a) $F := \mathcal{E}|_Y$ is an exceptional bundle on $Y$.*

*(b) $E := \mathcal{E}|_X$ is a torsion-free sheaf on $X$ and there is an exact sequence*

$$0 \to E \to L^{\oplus n} \to T \to 0$$

*where $L$ is a rank one reflexive sheaf on $X$ and $T$ is a torsion sheaf supported at $P$.*

*Regarding the topological invariants of $F$, we have*

$$\mathrm{rk}(F) = n, \quad c_1(F) \cdot K_Y \equiv \pm a \bmod n, \ \text{and } c_2(F) = \frac{n-1}{2n}(c_1(F)^2 + n + 1).$$

*Also $c_1(F) = nc_1(L) \in H_2(X, \mathbb{Z})$ is divisible by $n$ in $H_2(X, \mathbb{Z})$, and $c_1(L)$ generates the quotient $H_2(X, \mathbb{Z})/H_2(Y, \mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}$.*

*If $\mathcal{H}$ is a ample line bundle on $\mathcal{X}$ over $S$ then $F$ is slope stable with respect to $\mathcal{H}|_Y$.*

*Remark* 1.4.3. (1) Roughly speaking, the sheaf $E$ is a limit of the family of exceptional bundles $F$ over the punctured disc $S \setminus \{0\}$ which is slope semistable.

(2) The vector bundles obtained from $F$ by dualizing or tensoring by a line bundle arise in the same way from the degeneration $\mathcal{X}/(0 \in S)$.

(3) The isomorphism type of the singularity $\frac{1}{n^2}(1, na-1)$ is determined by $n$ and $\pm a \bmod n$. (The sign ambiguity is given by interchanging the orbifold coordinates.) Thus we can recover the type of the singularity from the vector bundle $F$.

Statement 1.4.2(1) is equivalent to our assumption $H_1(Y, \mathbb{Z}) \subset H_1(X, \mathbb{Z})$ by Corollary 1.2.4. We sketch the proof of 1.4.2(2) in the case $n = 2$. The deformation $(P \in \mathcal{X})/(0 \in S)$ of the singularity $(P \in X)$ is pulled back from the versal $\mathbb{Q}$-Gorenstein deformation

$$(0 \in (xy = z^2 + t) \subset \mathbb{A}^3_{x,y,z}/\tfrac{1}{2}(1,1,1) \times \mathbb{A}^1_t) \to (0 \in \mathbb{A}^1_t)$$

We give the construction for the versal case (the general case is obtained by pull back).

The point $P \in \mathcal{X}$ is a $\frac{1}{2}(1,1,1)$ singularity. So the blowup $\pi \colon \tilde{\mathcal{X}} \to \mathcal{X}$ is a resolution of $\mathcal{X}$ with exceptional locus $W \simeq \mathbb{P}^2$, with normal bundle $\mathcal{N}_{W/\tilde{\mathcal{X}}} \simeq \mathcal{O}_{\mathbb{P}^2}(-2)$. The special fiber $\tilde{\mathcal{X}}_0 \subset \tilde{\mathcal{X}}$ is a normal crossing divisor with irreducible components the strict transform $X'$ of $X$ and the exceptional divisor $W$, meeting along a smooth rational curve $C$. The induced morphism $X' \to X$ is the minimal resolution of the $\frac{1}{4}(1,1)$ singularity $P \in X$ with exceptional locus the curve $C \subset X$ (a $(-4)$-curve). The curve $C \subset W$ is a smooth conic in $\mathbb{P}^2$.

By Corollary 1.2.4, our topological assumption $H_1(Y,\mathbb{Z}) \subset H_1(X,\mathbb{Z})$ is equivalent to surjectivity of the map

$$H_2(X,\mathbb{Z}) \to H_1(L,\mathbb{Z}) \simeq \mathbb{Z}/4\mathbb{Z},$$

where $L \simeq S^3/\frac{1}{4}(1,1)$ denotes the link of the singularity $P \in X$. This map is identified with the restriction map

$$\mathrm{Cl}(X) \to \mathrm{Cl}(P \in X)$$

from the class group of $X$ to the class group of the singularity $(P \in X)$. (Here we are using $H^i(\mathcal{O}_X) = H^i(\mathcal{O}_Y) = 0$ for $i = 1, 2$.) So there is an effective Weil divisor $D \subset X$ such that locally analytically at $P \in X$ the divisor $D$ is linearly equivalent to the divisor given by the zero locus of an orbifold coordinate. Let $D' \subset X'$ denote the strict transform of a general such divisor $D$, then $D'$ is Cartier and meets the exceptional curve $C$ transversely in a smooth point. So, writing $L' = \mathcal{O}_{X'}(D')$ for the associated line bundle, $L'$ is a line bundle on $X'$ such that $L'|_C \simeq \mathcal{O}_{\mathbb{P}^1}(1)$.

Now observe that there exists an exceptional bundle $G$ on the exceptional divisor $W \simeq \mathbb{P}^2$ such that $G|_C \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. Indeed, we can take $G \simeq T_{\mathbb{P}^2}(-1)$. Because $\tilde{\mathcal{X}}_0 \subset \tilde{\mathcal{X}}$ is a normal crossing divisor, we have an exact sequence

$$0 \to \mathcal{O}_{\tilde{\mathcal{X}}_0} \to \mathcal{O}_{X'} \oplus \mathcal{O}_W \to \mathcal{O}_C \to 0. \tag{1.4.1}$$

It follows that we can glue $G$ and $(L')^{\oplus 2}$ along $C$ (by identifying their restrictions to $C$) to obtain a vector bundle $\tilde{E}$ on $\tilde{\mathcal{X}}_0$. One can check using the exact sequence (1.4.1) that $\tilde{E}$ is an exceptional vector bundle on the reducible surface $\tilde{\mathcal{X}}_0$. As such it deforms to give a vector bundle $\tilde{\mathcal{E}}$ over $\tilde{\mathcal{X}}$ such that the restriction $F = \tilde{\mathcal{E}}|_Y$ to the general fiber $Y$ is an exceptional bundle. The sheaf $\mathcal{E}$ on $\tilde{\mathcal{X}}$ in the statement is the reflexive hull of the pushforward of $\tilde{\mathcal{E}}$. The rank one reflexive sheaf $L$ on $X$ is the reflexive hull of the pushforward of the line bundle $L'$, or equivalently the sheaf $\mathcal{O}_X(D)$ corresponding to the divisor $D$.

*Remark* 1.4.4. The construction of the vector bundle $F$ can be viewed as an algebraic version of the gluing constructions used in the study of Donaldson invariants of smooth 4-manifolds [DK90]. The surgery of smooth 4-manifolds given by passing from the minimal resolution of $X$ to its smoothing $Y$ is known as a *rational blowdown* [FS97].

## 1.5 Examples

### 1.5.1 del Pezzo surfaces

There is a complete classification of exceptional vector bundles on del Pezzo surfaces [R89], [KO95]. Here we use it to show that every exceptional bundle on a del Pezzo surface arises by the construction of Theorem 1.4.2. This is joint work with Anna Kazanova [HK14].

Let $Y$ be a smooth projective surface. We say a sequence $F_1, \ldots, F_N$ of exceptional bundles on $Y$ is an *exceptional collection* if

$$\operatorname{Ext}^i(F_j, F_k) = 0 \text{ for all } i \text{ and all } j > k.$$

We say the collection is *full* if in addition it generates the bounded derived category $D(Y)$ of coherent sheaves on $Y$ as a triangulated category. We note that if $F_1, \ldots, F_N$ is a full exceptional collection then in particular the Grothendieck group $K(Y)$ of coherent sheaves on $Y$ is a free abelian group with a basis given by the classes $[F_1], \ldots, [F_N]$.

If $Y$ is a del Pezzo surface then any exceptional collection can be extended to a full exceptional collection. In particular, an exceptional collection $F_1, \ldots, F_N$ is full if and only if $N = \operatorname{rk} K(Y) = e(Y)$, the Euler number of $Y$.

**Theorem 1.5.1.** *Let $\mathcal{X}/(0 \in S)$ be a one parameter $\mathbb{Q}$-Gorenstein smoothing of a normal projective surface $X$ with Wahl singularities $P_1, \ldots, P_N$. Suppose that there exists a nodal chain of smooth rational curves $D_1, \ldots, D_{N-1}$ such that $D_i$ passes through the singular points $P_i, P_{i+1}$ and is given by the zero locus of an orbifold coordinate at each point. Let $Y$ denote a general fiber of $\mathcal{X}/(0 \in S)$. Assume that $H^1(\mathcal{O}_Y) = H^2(\mathcal{O}_Y) = 0$ and the map*

$$H_2(X, \mathbb{Z}) \to \bigoplus_{i=1}^{r} H_1(L_i, \mathbb{Z})$$

*is surjective, where $L_i$ denotes the link of the singularity $(P_i \in X)$. Then there exists an exceptional collection of bundles $F_1, \ldots, F_N$ on $Y$ given by the construction of Theorem 1.4.2 such that*

$$\frac{c_1(F_{i+1})}{\operatorname{rk}(F_{i+1})} - \frac{c_1(F_i)}{\operatorname{rk}(F_i)} = [D_i] \in H_2(X, \mathbb{Z}).$$

The proof of the theorem is based on the following elementary result.

**Lemma 1.5.2.** *Let $Y$ be a smooth projective surface such that $H^1(\mathcal{O}_Y) = H^2(\mathcal{O}_Y) = 0$ and $D_1, \ldots, D_{N-1}$ a nodal chain of smooth rational curves on $Y$. Then the sequence of line bundles*

$$\mathcal{O}_Y, \mathcal{O}_Y(D_1), \ldots, \mathcal{O}_Y(D_1 + \cdots + D_{N-1})$$

*is an exceptional collection.*

*Proof.* Exercise. □

We now use Theorem 1.5.1 to reverse engineer a degeneration $Y \rightsquigarrow X$ from a full exceptional collection in the case $Y$ is a del Pezzo surface. A related result was obtained by M. Perling [P13].

**Theorem 1.5.3.** *Let $Y$ be a del Pezzo surface and $F_1, \ldots, F_N$ a full exceptional collection on $Y$. Form the vectors*

$$u_i := \frac{c_1(F_{i+1})}{\mathrm{rk}(F_{i+1})} - \frac{c_1(F_i)}{\mathrm{rk}(F_i)} \in H^2(Y, \mathbb{Q}), \quad i = 1, \ldots, N-1$$

*and define $u_N$ by requiring*

$$u_1 + \ldots + u_N = -K_Y.$$

*Let $M$ denote the kernel of the homomorphism*

$$\mathbb{Z}^N \to H^2(Y, \mathbb{Q}), \quad e_i \mapsto u_i$$

*and let*

$$\mathbb{Z}^N \to L, \quad e_i \mapsto v_i$$

*denote the dual of the inclusion $M \subset \mathbb{Z}^N$. Then $L$ is a free abelian group of rank 2, and the vectors $v_i \in L$ are primitive and generate the rays of a complete fan $\Sigma$ in $L \otimes_{\mathbb{Z}} \mathbb{R}$ in cyclic order.*

*Let $X$ be the normal projective toric surface associated to $\Sigma, L$ and write $D_i$ for the toric boundary divisor corresponding to $v_i$. Then $X$ has Wahl singularities and $H^2(T_X) = 0$ (so there are no local-to-global obstructions to deformations of $X$).*

*Let $\mathcal{X}/(0 \in S)$ be a general $\mathbb{Q}$-Gorenstein smoothing of $X$. Then the general fiber of $\mathcal{X}/(0 \in S)$ is a smooth del Pezzo surface deformation equivalent to $Y$. The construction of Theorem 1.5.1 applied to $\mathcal{X}/(0 \in S)$ and the chain $D_1, \ldots, D_{N-1}$ of smooth rational curves on $X$ produces an exceptional collection deformation equivalent to the original collection $F_1, \ldots, F_N$.*

Any exceptional bundle on $Y$ can be included in a full exceptional collection. This gives the following result.

**Corollary 1.5.4.** Every exceptional bundle on a del Pezzo surface $Y$ arises via the construction of Theorem 1.4.2.

### 1.5.2 Godeaux surfaces

This section describes work of Anna Kazanova [K13].

A *Godeaux surface* $Y$ is a minimal surface of general type such that $K_Y^2 = 1$ and $\chi(\mathcal{O}_Y) = 1$. Such surfaces necessarily satisfy $H^1(\mathcal{O}_Y) = H^2(\mathcal{O}_Y) = 0$. Moreover $H_1(Y, \mathbb{Z})$ is cyclic of order $n \leq 5$, and all cases occur. A complete description of the moduli space is known for $n = 3, 4, 5$ [R78]. It is conjectured by Reid and Catanese that the moduli space of Godeaux surfaces with $|H_1(Y, \mathbb{Z})| = n$ is irreducible for each $n$.

**Theorem 1.5.5.** *Let $Y$ be a Godeaux surface with $|H_1(Y,\mathbb{Z})| = n$. Let $Y \rightsquigarrow X$ be a Wahl degeneration of $Y$ to a normal surface $X$ with a singularity $(P \in X)$ of type $\frac{1}{4}(1,1)$ such that $K_X$ is ample. (So locally trivial deformations of $X$ define a boundary divisor of the compactification of the moduli space of Godeaux surfaces.)*

*Let $\pi\colon \tilde{X} \to X$ be the minimal resolution of $X$. Then $\tilde{X}$ is minimal of Kodaira dimension 1 and admits an elliptic fibration $\tilde{X} \to \mathbb{P}^1$ with two multiple fibers. The possible multiplicites $(m_1, m_2)$ are*

$$(m_1, m_2) = (4,4), (3,3), (2,4), (2,6), (2,3); \quad n = \gcd(m_1, m_2).$$

All the possibilites for $(m_1, m_2)$ occur except possibly $(2,6)$. This is shown by either an explicit construction in weighted projective space or an abstract construction as a partial $\mathbb{Q}$-Gorenstein smoothing of a surface with several Wahl singularities (cf. [LP07]).

The following result gives a complete classification of exceptional bundles $F$ on Godeaux surfaces $Y$ such that $\operatorname{rk}(F) = 2$ and $c_1(F) = K_Y$ modulo torsion.

**Theorem 1.5.6.** *Let $Y$ be a Godeaux surface, $\sigma$ a torsion divisor class on $Y$, and $P$ a base point of the linear system $|2K_Y + \sigma|$. Then there is a unique non-trivial extension*

$$0 \to \mathcal{O}_Y \to F \to \mathcal{O}_Y(K_Y + \sigma) \otimes \mathcal{I}_P \to 0.$$

*The sheaf $F$ is a vector bundle of rank 2 with $c_1(F) = K_Y$ modulo torsion and $c_2(F) = 1$, and is slope stable with respect to $K_Y$. All such bundles are obtained as an extension of this form tensored by a torsion line bundle.*

*The vector bundle $F$ is exceptional if $\sigma \in H_1(Y,\mathbb{Z}) \setminus 2H_1(Y,\mathbb{Z})$ and $P$ is a simple base point.*

If $Y \rightsquigarrow X$ is a degeneration as in Theorem 1.5.5, then $K_X$ is 2-divisible in $H_2(X,\mathbb{Z})$ modulo torsion if and only if $(m_1, m_2) = (4,4)$ or $(2,6)$. In this case the exceptional bundle constructed by Theorem 1.4.2 is as described in Theorem 1.5.6.

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