Effective upper bound of analytic torsion under Arakelov metric

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Abstract

Given a choice of metric on the Riemann surface, the regularized determinant of Laplacian (analytic torsion) is defined via the complex power of elliptic operators:

$$\det(\Delta) = \exp(-\zeta'(0))$$

In this paper we gave an asymptotic effective estimate of analytic torsion under Arakelov metric. In particular, after taking the logarithm it is asymptotically upper bounded by g for $g > 1$. The construction of a cohomology theory for arithmetic surfaces in Arakelov theory has long been an open problem. In particular, it is not known if $h^1(X, L) \geq 0$. We view this as an indirect piece of evidence that if such a cohomology theory exists, the h^1 term may be effectively estimated.

Introduction

Let (X_σ, g) be a compact connected smooth Riemann surface without boundary (which we henceforth abbreviate as *compact Riemann surface*). The metric Laplacian is defined to be

$$\Delta_g(f) = \frac{1}{\sqrt{\det(g)}}\partial_i(\sqrt{\det(g)}g^{ij}\partial_j f)$$

The regularized determinant of the metric Laplacian (which we henceforth abbreviate as *analytic torsion*) is defined to be

$$\det(\Delta_g) = \exp(-\zeta'(0)), \zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\mathrm{Tr}(e^{-t\Delta} - P)dt$$

where P is the projection operator onto the kernel of the Laplacian. This may be viewed as a 'secondary global invariant' that gives spectral properties of X_σ missing from the information of kernel and cokernel of an elliptic operator. Unlike the index, it is well known that there is no formula expresses analytic torsion as the integral of a local quantity. As a result, estimation and evaluation of analytic torsion is difficult in general. In particular, for compact Riemann surface of $g > 1$ with fixed area equal to 1, Osgood, Sanark and Phillips proved there is an upper bound associated to metric of constant negative curvature.

Motivation

Our motivation of the present work stems from number theory. From the point of view of Arakelov theory, X_σ corresponds to the information from archimedean places: If we consider $C = X_\eta$ as semi-stable algebraic curve over a number field K, then we can extend it to a scheme X over $B = \mathrm{Spec}(\mathcal{O}_K)$. The theory of arithmetic surfaces can then be used to prove results about C. However, such construction would necessarily incorporate information from archimedean places of K. If we use $S = S_f \cup S_{\mathrm{inf}}$ to denote the different places of K, then $X_\sigma, \sigma \in S_{inf}$ corresponds to compatification of C after base change to $\mathbb{C}$. What had been missing is a cohomology theory associated to a metrized line bundle on X. We have the following conjecture due to Bost:

Method of the proof

Our proof of this result heavily used previous results by Jorgenson and Kramer, Wentworth and Wilms. The main idea behind the paper is that with Richard Wentworth's proof of the correct bosonization formula, we may write the scalar analytic torsion in terms of Faltings' delta function and area of the compact Riemann surface under Arakelov metric. With Robert Wilms's result, we may directly estimate Faltings' delta function. Thus to estimate the scalar analytic torsion, it is enough to bound the area of the surface. This was done essentially by Jorgenson and Kramer in their paper.

Bost's conjecture: Positivity of h^1

Let $\pi : X \to B = \mathrm{Spec}(\mathcal{O}_K)$ be an integral, flat, projective scheme of dimension 2. Let L be a metrized line bundle on X. Then for a potential reasonable cohomology theory associated to X, $h^1(X, L)$ is non-negative. Further, h^1 should be related to $\det(\Delta_{\overline{\partial}^*\overline{\partial}})$ or Quillen metric in one way or the other. Here $\overline{\partial}^*\overline{\partial}$ stands for the Dolbeault Laplacian associated to the metrized line bundle.

Main Theorem

Let $\pi : X \to B = \mathrm{Spec}(\mathcal{O}_K)$ be an integral, flat, projective scheme of dimension 2. Let $\sigma : K \to$ be a fixed archimedean place of K.

(i) (Effective upper bound) We have the following effective estimate of the analytic torsion under Arakelov metric: For g large enough:

$$-\infty < \log(\det(\Delta_{Ar})) < g$$

In particular, for $g > 1$ the analytic torsion under Arakelov metric is always bounded from above.

(ii) (Comparison of two metrics) The difference of Faltings metric and Quillen metric's logarithm has an asymptotic lower bound by a constant. The constant only depends on g.

Additional remarks

We note that it is crucial that we restrict ourselves to metric (scalar) Laplacian under Arakelov metric. As we mentioned earlier, the paper [**?**] showed it would be hard to estimate the upper bound even for hyperbolic metric. On the other hand, for the set up of Conjecture 1.1, we are really concerned with the Dolbeault Laplacian instead of the metric Laplacian. For the general situation, the two metrics are related by WeitzenbÃűck formula. For spin line bundle L, the analytic torsion of two Laplacians are related by the 'spin $\frac{1}{2}$ formula':

$$\det(\overline{\partial}^*\overline{\partial}) = C_g\left(\frac{\det'(\Delta)}{A\det(\mathrm{Im}\Omega)}\right)^{-1/2}|\theta(0,\Omega)|^2$$

where the constant C_g is only dependent upon g and A is the area of the Riemann surface. Ideally we would like to obtain an upper bound of the type

$$|\theta(0,\Omega)|^4\det(\mathrm{Im}\Omega) < D_g$$

where D_g is certain constant only depend upon g.

Further work to be done

It would be interesting to know whether we can give an effective version of Bismut-Vasserot type formula for the analytic torsion part of Quillen metric. This may be useful for number theory purposes.

References

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