Math 797W Homework 1

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September 19, 2016

We work over an algebraically closed field $k$ (unless explicitly stated otherwise). Questions 1 and 2 are preliminary and will not be graded.

(1) We say a topological space $X$ is \textit{irreducible} if there does not exist a decomposition $X = X_1 \cup X_2$ where $X_1, X_2 \subseteq X$ are closed subsets. Prove the following statements.

(a) $X$ is irreducible iff for all non-empty open sets $U_1, U_2 \subset X$ the intersection $U_1 \cap U_2$ is non-empty.

(b) If $X$ is irreducible and $U \subset X$ is a non-empty open set, then $U$ is dense (that is, $\overline{U} = X$) and $U$ is irreducible.

(c) If $X$ is irreducible and $f : X \to Y$ is continuous then $f(X)$ is irreducible.

(d) If $X$ is irreducible, $Y \subset X$, and $\overline{Y}$ is the closure of $Y$ in $X$, then $\overline{Y}$ is irreducible.

(e) If $X$ has an open covering $X = \bigcup U_i$ where each $U_i$ is irreducible and $U_i \cap U_j \neq \emptyset$ for all $i$ and $j$, then $X$ is irreducible.

(2) Let $f : X \to Y$ be a morphism of affine varieties and

$$f^* : k[Y] \to k[X], \quad g \mapsto g \circ f$$

the corresponding morphism of $k$-algebras. Recall that, for $J \subset k[X]$ an ideal we have $I(V(J)) = \sqrt{J}$, the radical of $J$ (Hilbert’s Nullstellensatz). Verify the following statements.

(a) For $Z \subset X$ a subset, $V(I(Z)) = \overline{Z}$ (the closure of $Z$ in the Zariski topology).
(b) For $J \subset k[Y]$ an ideal, $f^{-1}V(J) = V(f^*(J))$.

[Note: $f^*(J)$ is not necessarily an ideal of $k[X]$. But we can define $V(S)$ for any subset $S$ of $k[X]$, then $V(S) = V(\langle S \rangle)$ where $\langle S \rangle \subset k[X]$ is the ideal generated by $S$.]

(c) For $Z \subset X$ a subset we have $I(f(Z)) = f^{-1}I(Z)$. In particular (using the case $Z$ is a point), the map of sets $f : X \to Y$ corresponds to the map $m \mapsto f^{-1}(m)$ from maximal ideals of $k[X]$ to maximal ideals of $k[Y]$.

(d) For $J \subset k[X]$ an ideal, $f(V(J)) = V(f^*(J))$. In particular, $f(X) = Y$ iff $f^*$ is injective.

(3) Let $X$ be the union of the coordinate axes in $\mathbb{A}^3$.

(a) Compute the ideal $I(X) \subset k[x, y, z]$.

(b) Prove that $I(X)$ cannot be generated by 2 elements.

(c) Let $J = (xy, (x - y)z) \subset k[x, y, z]$. Show that $V(J) = X$. What is $\sqrt{J}$?

(4) Let $J = (x^2 + y^2 + z^2, xy + yz + xz) \subset k[x, y, z]$ and $X = V(J) \subset \mathbb{A}^3$. Determine the irreducible components of $X$. What is $\sqrt{J}$?

(5) Let $X$ be an affine variety and $f \in k(X)$ a rational function on $X$. Define

$$\text{domain}(f) = \{ p \in X \mid f \in \mathcal{O}_{X,p} \}.$$ 

(a) Prove $\text{domain}(f) \subset X$ is an open subset.

(b) Let $p \in X$. Suppose $f = g/h$, where $g, h \in k[X]$, and $g(p) \neq 0$, $h(p) = 0$. Show that $p \notin \text{domain}(f)$.

(c) Compute $\text{domain}(f)$ in the following cases:

i. $X = V(x_1^2 + x_2^2 - 1) \subset \mathbb{A}^2$, $f = (1 - x_2)/x_1$.

ii. $X = V(x_1x_3 - x_2^2) \subset \mathbb{A}^3$, $f = x_1/x_2$.

(6) Consider the morphism

$$f : \mathbb{A}^1 \to \mathbb{A}^2, \quad t \mapsto (t^2, t^3).$$

(a) Show that $X := f(\mathbb{A}^1) \subset \mathbb{A}^2$ is closed and find its ideal $I(X) \subset k[x, y]$.
(b) Draw a sketch of $X$ in the case $k = \mathbb{R}$, and observe that the origin is a singular point of $X$ (there is no well-defined tangent line).

[WARNING: In general we don’t allow non-algebraically closed fields, but it is sometimes useful for visualization to consider $k = \mathbb{R}$.

(c) One can also try to draw a (partial) sketch in the case $k = \mathbb{C}$ as follows. Consider the intersection of $X$ with a small sphere $S^3 \subset \mathbb{C}^2$ with center the origin. Show that the intersection $X \cap S^3$ is a trefoil knot in $S^3 = \mathbb{R}^3 \cup \{\infty\}$. This shows in particular that the origin is a singular point of $X$ (otherwise $X \cap S^3 \subset S^3$ would be an unknotted $S^1$).

[Hint: The intersection $X \cap S^3$ lies on one of the tori $S^1 \times S^1$ in $S^3$ defined by $|x| = a, |y| = \sqrt{r^2 - a^2}$ for some $0 < a < r$, where $r$ is the radius of the sphere.]

(d) Show that the map $\mathbb{A}^1 \to X$ is a homeomorphism of topological spaces (for the Zariski topology).

(e) Show that, via $f^*$, the coordinate ring $k[X]$ is identified with the subring of the polynomial ring $k[\mathbb{A}^1] = k[t]$ consisting of polynomials $g(t)$ such that $g'(0) = 0$.

(f) Show that $f^*$ defines an isomorphism of the function fields $k(X) \sim \to k(\mathbb{A}^1) = k(t)$.

(g) Using (e) and (f) or otherwise, determine the integral closure of $k[X]$.

(7) Assume $\text{char}(k) \neq 2$. Consider the morphism

$$f: \mathbb{A}^2 \to \mathbb{A}^3, \quad (x_1, x_2) \mapsto (x_1^2, x_1x_2, x_2^2).$$

(a) Prove that $X := f(\mathbb{A}^2) \subset \mathbb{A}^3$ is closed and find its ideal $I(X) \subset k[y_1, y_2, y_3]$.

(b) Show that, as a topological space (for the Zariski topology), $X$ is the quotient of $\mathbb{A}^2$ by the action of $\mathbb{Z}/2\mathbb{Z}$ given by $(x_1, x_2) \mapsto (-x_1, -x_2)$.

(c) Show that, via $f^*$, the coordinate ring $k[X]$ is identified with the invariant ring $k[x_1, x_2]^{\mathbb{Z}/2\mathbb{Z}}$ of the group action on the coordinate ring $k[x_1, x_2]$ of $\mathbb{A}^2$. [Here for a group $G$ acting on a ring $R$ the
invariant ring $R^G$ is the subring of $R$ consisting of elements $r$ such that $g \cdot r = r$ for all $g \in G$. This implies that $X$ is the quotient of $\mathbb{A}^2$ by the group action as an algebraic variety.

[Hint / Remark: If $f : X \to Y$ is a morphism of affine varieties, we say that $f$ is a finite morphism if the corresponding homomorphism of $k$-algebras $f^* : k[Y] \to k[X]$ gives $k[X]$ the structure of a finitely generated $k[Y]$-module. In this case, it follows from the “going up theorem” (cf. 612) that the morphism $f$ is closed, that is, if $Z \subset X$ is closed then $f(Z) \subset Y$ is closed. Moreover, if $f : X \to Y$ is a finite morphism then $f^{-1}(p)$ is a finite set for all $p \in Y$. The morphisms $f$ in questions 6 and 7 above are examples of finite morphisms.]

(8) For each of the following morphisms $f : X \to Y$, compute the image $f(X) \subset Y$ of $f$. Show that $f(X)$ is neither open nor closed in $Y$, and $f(X) = Y$. Describe the fiber $f^{-1}(p)$ of $f$ over each point $p \in Y$.

(a) $f : \mathbb{A}^2 \to \mathbb{A}^2$, $(x, y) \mapsto (x, xy)$.
(b) $f : \mathbb{A}^3 \to \mathbb{A}^3$, $(x, y, z) \mapsto (x, xy, xyz)$.

(9) (a) Let $J \subset S = k[X_0, \ldots, X_n]$ be a homogeneous ideal. Show that if $J$ is not prime then there exist homogeneous elements $a, b \in S$ such that $ab \in J$ and $a, b \notin J$.

(b) Let $X \subset \mathbb{P}^n$ be an algebraic set. Show that $X$ is irreducible iff $I(X) \subset S$ is prime.

(10) Let

$$X = V(x_1^3 + x_1x_2^2 + x_1^2 + x_2 + 1) \subset \mathbb{A}^2.$$ 

Let $\overline{X}$ denote the closure of $X$ in

$$\mathbb{P}^2 = (X_0 \neq 0) \sqcup (X_0 = 0) = \mathbb{A}^2 \sqcup \mathbb{P}^1.$$ 

(a) Write down the homogeneous equation of $\overline{X}$ and identify the set $\overline{X} \setminus X = \overline{X} \cap \mathbb{P}^1$.

(b) Find another affine chart $Y \subset \mathbb{A}^2$ for $\overline{X}$ such that $\overline{X} = X \cup Y$, write down the equation of $Y \subset \mathbb{A}^2$, and describe the transition map between the two charts explicitly.
(11) Let \( F \in k[X_0, X_1, X_2] \) be an irreducible homogeneous polynomial of degree \( d \). Let \( X = V(F) \subset \mathbb{P}^2 \) be the associated projective variety, a projective plane curve of degree \( d \). Let \( L \subset \mathbb{P}^2 \) be a line (that is, \( L = V(a_0X_0 + a_1X_1 + a_2X_2) \subset \mathbb{P}^2 \) is the zero locus of a linear form). Show that \( X \cap L \) consists of exactly \( d \) points counting multiplicities (unless \( d = 1 \) and \( X = L \)).

(12) Show directly using the standard affine charts that \( \mathcal{O}_X(X) = k \) for \( X = \mathbb{P}^1 \).

(13) Let \( X = V(f) \subset \mathbb{A}^2 \). Suppose

\[
f = a_1x_1 + a_2x_2 + \cdots
\]

where \( \cdots \) denotes higher order terms in \( x_1, x_2 \), and \( (a_1, a_2) \neq (0, 0) \).

(Geometrically, we have \((0, 0) \in X \), and \( X \) is smooth at \((0, 0) \) with tangent line \( V(a_1x_1 + a_2x_2) \subset \mathbb{A}^2 \).

Consider the morphism

\[
q : \mathbb{A}^2 \setminus \{(0, 0)\} \to \mathbb{P}^1, \quad (x_1, x_2) \mapsto (x_1 : x_2).
\]

(a) Show that the restriction of \( q \) to \( X \setminus \{(0, 0)\} \) extends to a morphism \( g : X \to \mathbb{P}^1 \).

(b) What is the geometric interpretation of the point \( g(0, 0) \in \mathbb{P}^1 \)?

(14) Let \( n \in \mathbb{Z} \). Let \( X = X(n) = U_1 \cup U_2 \) where \( U_1 = \mathbb{A}^2_{x_1,y_1}, U_2 = \mathbb{A}^2_{x_2,y_2}, \) and the glueing is given by

\[
U_1 \supset (x_1 \neq 0) \sim (x_2 \neq 0) \subset U_2, \quad (x_1, y_1) \mapsto (x_1^{-1}, x_1^n y_1).
\]

(a) Show that \( C \subset X \) defined by \( C \cap U_i = V(y_i) \) for \( i = 1, 2 \) is a closed subvariety isomorphic to \( \mathbb{P}^1 \).

(b) Show that the morphisms

\[
p_i : U_i \to \mathbb{A}^1, \quad (x_i, y_i) \mapsto x_i
\]

patch to give a morphism \( p : X \to \mathbb{P}^1 \). Moreover there is a morphism \( s : \mathbb{P}^1 \to X \) such that \( p \circ s = \text{id}_{\mathbb{P}^1} \) and \( s(\mathbb{P}^1) = C \).

(c) Compute \( \mathcal{O}_X(X) \) as a subring of \( k[x_1, y_1] \). For \( n < 0 \), show that \( \mathcal{O}_X(X) = k \). For \( n \geq 0 \), find an explicit set of \( n + 1 \) generators for \( \mathcal{O}_X(X) \) as a \( k \)-algebra.
(d) Let \( f: X \to \mathbb{A}^{n+1} \) be the morphism defined by the generators for \( \mathcal{O}_X(X) \) found in (c). Show that \( f(X) \) is closed, \( f(C) \) is a point, and the restriction of \( f \) to \( X \setminus C \) is an isomorphism.

[Hint: If you are stuck, try \( n = 1 \) and \( n = 2 \) first.]