Math 621 Homework 5

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Reading: Stein and Shakarchi, Chapter 2, Section 5.2; Chapter 9, Section 1.

Justify your answers carefully.

(1) Let $\Omega = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 1\}$. Prove that $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ is holomorphic on Ω and compute its derivative. (Here $n^z := \exp(z \log n)$.)

(2) Let f_n be a sequence of holomorphic functions converging uniformly to a function f on some connected open set $\Omega \subset \mathbb{C}$. Suppose that f vanishes at some $z_0 \in \Omega$ but is not identically zero on Ω. Prove the following statement: given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, f_n has a zero in the disc with center z_0 and radius ϵ.

(3) Define

$$f(z) = \frac{1}{z} + \sum_{0 \neq n \in \mathbb{Z}} \left(\frac{1}{z+n} - \frac{1}{n} \right)$$

for $z \in \mathbb{C} \setminus \mathbb{Z}$. Prove the following assertions.

(a) f is a meromorphic function on $\mathbb{C}$ with a simple pole at each $n \in \mathbb{Z}$.

(b) f is odd (i.e. $f(-z) = -f(z)$ for all $z \in \mathbb{C} \setminus \mathbb{Z}$.)

(c) $f(\frac{1}{2}) = 0$.

(d) $f(z+1) = f(z)$ for all $z \in \mathbb{C} \setminus \mathbb{Z}$.

[Hint: Imitate the analogous proofs for the Weierstrass $\wp$ function. For (c), rearrange the sum (without breaking up the individual terms).]

(4) Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$, let $f\colon \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ be a meromorphic function on $\mathbb{C}$ such that $f(z+\omega_1) = f(z)$ and $f(z+\omega_2) = f(z)$ for all $z \in \mathbb{C}$. Assume for simplicity that f has no zeroes or poles on the boundary of the parallelogram P with vertices $0, \omega_1, \omega_2, \omega_1+\omega_2$, and suppose that on the interior of P the function f has zeroes $a_1, \ldots, a_k$ with multiplicities $m_1, \ldots, m_k$ and poles $b_1, \ldots, b_l$ with multiplicities $n_1, \ldots, n_l$. Prove the following statements.

(a) $\sum_{i=1}^{k} m_i = \sum_{j=1}^{l} n_j$

(b) $\sum_{i=1}^{k} m_i a_i = \sum_{j=1}^{l} n_j b_j + p_1\omega_1 + p_2\omega_2$ for some $p_1, p_2 \in \mathbb{Z}$.

[Hint for (b): Compute a contour integral $\int_\gamma z\frac{f'(z)}{f(z)}dz$ in two ways.]

(5) Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$, and let $\wp$ be the Weierstrass $\wp$-function for the lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$. Let $a_1, a_2 \in \mathbb{C}/\Lambda$ be distinct points such that $a_1,\ a_2,\ a_1+a_2 \neq 0 \in \mathbb{C}/\Lambda$. Let $a, b \in \mathbb{C}$ be such that that $\wp'(a_j) = a\wp(a_j) + b$ for $j = 1$ and 2 (note that $\wp(a_1) \neq \wp(a_2)$ because $a_1 \neq \pm a_2$). Prove that $\wp'(-(a_1+a_2)) = a\wp(-(a_1+a_2)) + b$.

[Hint: Consider the zeroes and poles of the elliptic function $\wp' - a\wp - b$ and apply Q4(b). (Note that the assumption in Q4(b) that there are no zeroes and poles on the boundary of the parallelogram is not essential — we can always translate the parallelogram so that this holds.)]

[Remark: The geometric interpretation of this result is as follows: Let $X = \mathbb{C}/\Lambda$ be the complex torus determined by Λ, and $F\colon X \setminus \{0\} \to \mathbb{C}^2_{z,w}$ the holomorphic map defined by $F(u) = (\wp(u), \wp'(u))$. Then $F(a_1), F(a_2), F(-a_1-a_2)$ are the 3 intersection points of the complex line $w = az + b$ with the complex cubic curve

$$F(X\setminus\{0\}) = (w^2 = 4(z-\wp(\omega_1/2))(z-\wp(\omega_2/2))(z-\wp(\omega_1+\omega_2)/2)) \subset \mathbb{C}^2_{z,w}.$$

It follows that the group law on the complex torus X can be reconstructed from the embedding $F\colon X \setminus \{0\} \to \mathbb{C}^2_{z,w}$. This is used in arithmetic geometry to define the group law without using analysis.]

(6) Let $\Omega = \mathbb{C} \setminus [0,1]$. Show that there exists a holomorphic function $f\colon \Omega \to \mathbb{C}$ such that $f(z)^2 = z(z-1)$ for all $z \in \Omega$.

[Hint: Fix a base point $z_0 \in \Omega$. For $z \in \Omega$ let γ_z be a path in Ω from z_0 to z, and define $g(z) = \exp\left(\frac{1}{2}\int_{\gamma_z} \frac{1}{z} + \frac{1}{z-1} dz\right)$. Prove that $g(z)$ is independent of the choice of path γ_z, g is holomorphic, and $g(z)^2 = cz(z-1)$ for some constant $0 \neq c \in \mathbb{C}$.]

(7) Let $\Omega = \{z \in \mathbb{C} \mid |z| > 3\}$. Consider the function $f\colon \Omega \to \mathbb{C}$ given by $f(z) = \frac{1}{(z-1)(z-2)}$. Show that f has a primitive on Ω.

(8) Let $H = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$ be the upper half plane and $\overline{H} \subset \mathbb{C}$ its closure. Fix $k \in \mathbb{R}$, $0 < k < 1$. For $z \in \overline{H}$, let γ_z be a path in $\overline{H}$ from 0 to z and define $f(z) = \int_{\gamma_z} \frac{1}{\sqrt{(1-w^2)(1-k^2w^2)}} dw$ where we define the square root on $\overline{H}$ so that it is continuous on $\overline{H}$ and real and positive for $w \in (-1, 1)$.

(a) Prove that f restricts to a bijection from the boundary of the upper half plane to the boundary of a rectangle R with one point removed. (Compare Stein and Shakarchi, p. 233–234.)

(b) Using the argument principle or otherwise, show that f defines a holomorphic bijection from H onto the interior of the rectangle R. Thus f gives the explicit solution of the Riemann mapping problem for R.

[Hint for (b): Let $g\colon D \to H$ be a holomorphic bijection from the unit disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$ to H. Then g is the restriction of a Möbius transformation $\tilde{g}\colon \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ which maps the boundary of D to the boundary of H together with the point ∞. Now $f \circ \tilde{g}$ maps the boundary of D to the boundary of the rectangle R.]