Math 621 Homework 4

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Reading: Stein and Shakarchi, 8.1, 8.2, 8.3.

Justify your answers carefully.

(1) Recall that a *Möbius transformation* is a holomorphic bijection

$$f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$$

and is given by

$$f(z) = \frac{az+b}{cz+d}$$

for some $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$. Recall that, for points $z_1, z_2, z_3, z_4 \in \mathbb{C} \cup \{\infty\}$ in the extended complex plane such that at most two coincide, the *cross ratio* $\operatorname{CR}(z_1, z_2, z_3, z_4)$ of z_1, z_2, z_3, z_4 is defined by

$$\operatorname{CR}(z_1, z_2, z_3, z_4) := \frac{z_1 - z_3}{z_1 - z_4} \Big/ \frac{z_2 - z_3}{z_2 - z_4}.$$

[WARNING: There are various alternative definitions of the cross ratio in the literature which differ by an element of the symmetric group S_4.]

In class we proved that Möbius transformations map a circle or line to a circle or line by proving the following statements:

(a) Möbius transformations preserve the cross ratio: With notation as above, $\operatorname{CR}(f(z_1), f(z_2), f(z_3), f(z_4)) = \operatorname{CR}(z_1, z_2, z_3, z_4)$.

(b) With notation as above, z_1, z_2, z_3, z_4 lie on a circle or line iff $\operatorname{CR}(z_1, z_2, z_3, z_4) \in \mathbb{R} \cup \{\infty\}$.

Give an alternative proof that Möbius transformations map circles or lines to circles or lines as follows.

(a) First show that any Möbius transformation can be written as a composition of Möbius transformations of the following types:

i. $f(z) = az$, some $0 \neq a \in \mathbb{C}$. (Writing $a = re^{i\theta}$ this transformation is the composition of a rotation through angle θ counterclockwise about the origin and dilation (scaling) by factor r centered at the origin.)

ii. $g(z) = z + b$, some $b \in \mathbb{C}$. (This is a translation)

iii. $h(z) = 1/z$.

[Hint: Recall that, given distinct points $z_1, z_2, z_3 \in \mathbb{C} \cup \{\infty\}$ and $w_1, w_2, w_3 \in \mathbb{C} \cup \{\infty\}$, there is a unique Möbius transformation f such that $f(z_j) = w_j$ for $j \in \{1, 2, 3\}$. So it's enough (why?) to show that we can send any z_1, z_2, z_3 to $0, 1, \infty$ by a composition of Möbius transformations of the above types.]

(b) Second, show that Möbius transformations of the above types maps circles or lines to circles or lines by an explicit computation.

[Hint: It's enough to consider $h(z) = 1/z$. Write the equation of a circle in the form $|z - a|^2 = r^2$ for some $a \in \mathbb{C}$ and $r \in \mathbb{R}_{>0}$, and expand using $|w|^2 = w\bar{w}$. Write the equation of a line in the form $\mathrm{Re}(az) = c$ for some $0 \neq a \in \mathbb{C}$ and $c \in \mathbb{R}$, and use $\mathrm{Re}(w) = (w + \bar{w})/2$.]

(2) Determine the Möbius transformation f that maps the points $-1, 0, 1$ to the points $1, i, -1$ respectively, and determine the image under f of the upper half plane.

(3) Let $\Omega = \{z \in \mathbb{C} \mid |z| < 1 \text{ and } \mathrm{Im}(z) > 0\}$ be the upper half of the unit disc. Find a Möbius transformation f such that $f(\Omega) = \Omega$, $f(i) = 0$ and $f(1) = -1$. Prove that f is unique.

(4) Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$ be distinct points. For $j \in \{1, 2\}$, let C_j be the unique circle or line passing through z_j, z_3, z_4. Show that C_1 and C_2 meet at right angles iff $\mathrm{CR}(z_1, z_2, z_3, z_4) \in i\mathbb{R}$.

(5) In each of the following cases, determine a holomorphic bijection $f \colon \Omega_1 \to \Omega_2$.

(a)

$$\Omega_1 = \{z \in \mathbb{C} \mid |z - 1| > 1 \text{ and } |z| < 2\}$$

and

$$\Omega_2 = \mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}.$$

(b)

$$\Omega_1 = \{z \in \mathbb{C} \mid |z| < 2 \text{ and } \operatorname{Im}(z) > 1\}$$

and

$$\Omega_2 = \{z \in \mathbb{C} \mid |z| < 2 \text{ and } \operatorname{Im}(z) < 1\}.$$

(6) Let $\Omega \subset \mathbb{C}$ be an open set and $z_0 \in \Omega$ a point. Let $f\colon \Omega \setminus \{z_0\} \to \mathbb{C}$ be a holomorphic function. Suppose there exist $r \in \mathbb{R}_{>0}$ and $M \in \mathbb{R}$ such that $\operatorname{Re}(f(z)) < M$ for all $z \in \Omega$ such that $0 < |z - z_0| < r$. Prove that z_0 is a removable singularity of f.

(7) Recall the statement of the *Schwarz Lemma*: Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disc. Let $f\colon D \to D$ be a holomorphic map such that $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in D$ and $|f'(0)| \leq 1$. Moreover, if $|f(z_0)| = |z_0|$ for some $0 \neq z_0 \in D$ or $|f'(0)| = 1$ then $f(z) = e^{i\theta} z$ for some θ.

(a) Use the Schwarz Lemma to prove the *Schwarz–Pick Lemma*: If $f\colon D \to D$ is a holomorphic map then

$$\left|\frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)} f(z_2)}\right| \leq \left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| \text{ for all } z_1, z_2 \in D.$$

[Hint: Recall that, for $\alpha \in D$, the *Blaschke factor* $\psi_\alpha\colon D \to D$ defined by $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}$ is a holomorphic bijection such that $\psi_\alpha \circ \psi_\alpha = \mathrm{id}$ and ψ_α interchanges 0 and α. In particular, the composition $\psi_{f(z_1)} \circ f \circ \psi_{z_1}$ fixes $0 \in D$.]

(b) Now suppose f is a bijection. Prove that the inequality in part (a) is an equality in this case.

[Remark: The disc D has a hyperbolic metric due to Poincaré which may be defined by

$$d(z_1, z_2) = 2 \tanh^{-1} \left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| \text{ for } z_1, z_2 \in D.$$

So the Schwarz–Pick Lemma can be restated as follows: if $f\colon D \to D$ is holomorphic then f is distance-decreasing for the Poincaré metric, that is, $d(f(z_1), f(z_2)) \leq d(z_1, z_2)$ for all $z_1, z_2 \in D$. In particular, as in (b), if f is a bijection then f is an isometry for the Poincaré metric.

Note the contrast with the case of the Riemann sphere $\mathbb{C} \cup \{\infty\}$: there the holomorphic bijections are given by the Möbius transformations, which form a group isomorphic to $\mathrm{PGL}(2, \mathbb{C})$, but only the subgroup $PSU(2) \simeq SO(3)$ preserves the standard metric on the sphere.]

(8) Let D be the unit disc and $\Omega \subset \mathbb{C}$ an open set such that $D \subset \Omega$. Suppose $f\colon D \to \Omega$ is a holomorphic bijection such that $f(0) = 0$. Prove that $|f'(0)| \geq 1$ with equality iff $f(z) = e^{i\theta} z$ for some θ.

(9) Let D be the unit disc, $\mathcal{H}$ the upper half plane, and $f\colon D \to \mathcal{H}$ a holomorphic map such that $f(0) = i$. Prove that

$$|f(z)| \leq \frac{1 + |z|}{1 - |z|} \text{ for all } z \in D.$$

(10) Let $f\colon D \to D$ be a holomorphic map. We say $z_0 \in D$ is a *fixed point* of f if $f(z_0) = z_0$.

(a) Show that if f has two fixed points then f is the identity map.

(b) Does there exist an f as above with no fixed points?
[Hint: Consider the upper half plane $\mathcal{H}$.]