Math 621 Homework 3

Paul Hacking

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Reading: Stein and Shakarchi, 3.3, 3.4, 3.5, 3.6.

Justify your answers carefully.

(1) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function which is injective.

  (a) Show that f does not have an essential singularity at ∞.
  [Hint: Use the Casorati–Weierstrass theorem and the open mapping theorem.]
  (b) Deduce that f is a polynomial.
  (c) Deduce that $f(z) = az+b$, for some $a, b \in \mathbb{C}$, $a \neq 0$. (In particular, f is bijective.)

(2) Find the number of zeroes (counted with multiplicities) of the following functions on the indicated domains.

  (a) $f(z) = z^{100} + 8z^{10} - 3z^3 + z^2 + z + 1$ on the unit disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$.
  (b) $f(z) = z^5 + 5z^2 + z + 2$ on the annulus $A = \{z \in \mathbb{C} \mid 1 < |z| < 2\}$.
  (c) $f(z) = e^z + 4z^5 + 1$ on the unit disc.

(3) Show that the equation $ze^{\lambda - z} = 1$ has exactly one solution in the unit disc for $\lambda \in \mathbb{R}$, $\lambda > 1$. Show also that this solution is a real number.

(4) Let $\Omega \subset \mathbb{C}$ be an open set containing the closure of the unit disc D. Let $f : \Omega \to \mathbb{C}$ be a non-constant holomorphic function. Suppose that $|f(z)| = 1$ for $|z| = 1$. Show that $f(D) = D$.

  [Hint: Show that there exists $z \in D$ such that $f(z) = 0$ (argue by contradiction using the maximum principle). Now use Rouché's theorem.]

(5) Let $f \colon \overline{D} \to \mathbb{C}$ be a continuous function on the closure $\overline{D}$ of the unit disc D which is holomorphic on the unit disc.

(a) Show that if $f(z) = 0$ for $|z| = 1$ then $f(z) = 0$ for all $z \in \overline{D}$.

(b) Show that if $f(z) = 0$ for $|z| = 1$ and $\mathrm{Re}(z) \geq 0$ then $f(z) = 0$ for all $z \in \overline{D}$.

(c) (Optional) Let $\theta_1, \theta_2 \in \mathbb{R}$ be such that $\theta_1 < \theta_2$. Show that if $f(z) = 0$ for all $z = e^{i\theta}$ with $\theta_1 \leq \theta \leq \theta_2$ then $f(z) = 0$ for all $z \in \overline{D}$.

[Hint for (b): Consider $g(z) = f(z)f(-z)$.]

(6) Let $\Omega \subset \mathbb{C}$ be a connected open set, $f \colon \Omega \to \mathbb{C}$ a non-constant holomorphic function, and $z_0 \in \Omega$. Suppose that f has a zero of order m at z_0.

(a) Show that for $0 < \epsilon \ll 1$, there exists $\delta > 0$ such that for $0 < |w| < \delta$ the equation $f(z) = w$ has exactly m solutions (each of multiplicity 1) satisfying $|z - z_0| < \epsilon$.
[Hint: Use Rouché's theorem. Recall that a solution $z = \alpha$ of $f(z) = w$ has multiplicity 1 iff $f'(\alpha) \neq 0$.]

(b) Now suppose that $m = 1$. With notation as in part (a), let γ be the circle with center z_0 and radius ϵ oriented counterclockwise. Show that for $|w| < \delta$ the unique solution of $f(z) = w$ inside γ is given by the integral formula

$$z = \frac{1}{2\pi i} \int_\gamma \frac{uf'(u)}{f(u) - w} du.$$

[Hint: Compute the residues of the integrand (compare the proof of the argument principle) using the following fact: if $f(z)$ is holomorphic at $z = \alpha$ and $g(z)$ has a simple pole at $z = \alpha$ then $\mathrm{res}_{z=\alpha} f(z)g(z) = f(\alpha) \cdot \mathrm{res}_{z=\alpha} g(z)$.]

(7) Recall that if $\Omega \subset \mathbb{C}$ is a simply connected open subset and $f \colon \Omega \to \mathbb{C}$ is a holomorphic function such that $f(z) \neq 0$ for all $z \in \Omega$, then we may define a holomorphic function $\log f \colon \Omega \to \mathbb{C}$ by

$$(\log f)(z) = \int_{\gamma_z} \frac{f'(w)}{f(w)} dw + c$$

where γ_z is a path in Ω from a fixed basepoint $z_0 \in \Omega$ to z, and $c \in \mathbb{C}$ is a constant chosen so that $e^c = f(z_0)$ (thus c is determined up to an integer multiple of $2\pi i$). Then $e^{\log f(z)} = f(z)$ for all $z \in \Omega$.

Now let $\Omega = \mathbb{C} \setminus (-\infty, 0]$, $f(z) = z^n$ for some $n \in \mathbb{N}$, $z_0 = 1$, and $c = 0$. Compute $(\log f)(re^{i\theta})$ for $r \in \mathbb{R}$, $r > 0$, and $\theta \in (-\pi, \pi)$.