Math 621 Homework 2

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February 14, 2018

Reading: Stein and Shakarchi, 2.3, 2.4, 3.1, 3.2.

Justify your answers carefully.

(1) Let γ be the circle with center the origin and radius R with the positive (counterclockwise) orientation. Compute the following integrals directly from the definition of integrals along curves.

(a) $\int_\gamma z^n \, dz$ for each $n \in \mathbb{Z}$.

(b) $\int_\gamma \frac{1}{z-\alpha} \, dz$ for $\alpha \in \mathbb{C}$, $|\alpha| \neq R$.
[Hint: Expand $1/(z-\alpha)$ as a power series in z or z^{-1} and use part (a). The identity

$$1/(1-x) = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \qquad \text{for } |x| < 1$$

can be used to obtain the power series expansions.]

(c) $\int_\gamma \frac{1}{(z-\alpha)(z-\beta)} \, dz$ for $|\alpha| < R < |\beta|$.

(2) Let $n \in \mathbb{N}$. Let $f_n \colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f_n(x) = 0$ for $x \leq 0$ and $f_n(x) = x^n$ for $x > 0$. Show that f_n is differentiable $n-1$ times but not n times.

(3) (Optional) Let $f \colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 0$ for $x \leq 0$ and $f(x) = e^{-1/x}$ for $x > 0$. Show that f is differentiable arbitrarily many times, but f cannot be expressed as a power series on any open neighbourhood of $x = 0$.

[Hint: Prove by induction on $n \in \mathbb{N}$ that the nth derivative $f^{(n)}(x)$ exists and is given by $f^{(n)}(x) = 0$ for $x \leq 0$ and $f^{(n)}(x) = \frac{p_n(x)}{q_n(x)} \cdot e^{-1/x}$ for some polynomials $p_n(x)$ and $q_n(x)$.]

(4) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Assume that there exist $A, B \in \mathbb{R}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq 0}$ such that $|f(z)| \leq A|z|^n + B$ for all $z \in \mathbb{C}$. Prove that f is a polynomial of degree at most n.

(5) Find all the zeroes of $\sin z$ and $\cos z$ and verify that they are simple zeroes.

[Hint: Express $\sin z$ and $\cos z$ in terms of e^{iz} and e^{-iz} and use HW1Q5.]

(6) Find all the zeroes of $f(z) = \sin(\frac{1+z}{1-z})$ and determine any limit points of the set of zeroes. Why does your answer not contradict Theorem 4.8 on p. 52 of Stein and Shakarchi ("Zeroes of holomorphic functions are isolated")?

(7) Consider the power series $f(z) = \sum_{n=0}^{\infty} z^{n!}$.

(a) Show that the radius of convergence of $f(z)$ equals 1.

(b) Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disc. Show that the holomorphic function $f\colon \mathbb{D} \to \mathbb{C}$ does *not* extend to a holomorphic function on a larger open set $\Omega \subset \mathbb{C}$.

[Hint: Write $z = re^{i\theta}$. Show that if $\theta = 2\pi q$ for some rational number q then $f(z) \to \infty$ as $r \to 1$ with θ fixed.]

(8) (Optional) Let $f\colon \Omega \to \mathbb{C}$ be a holomorphic function. Let $z_0 \in \mathbb{C}$ be such that $|z_0| = 1$. Assume that f has a pole at z_0 and f is holomorphic at every point of the closed unit disc $\bar{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ besides z_0.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the power series expansion of f about $z = 0$, valid for $|z| < 1$. Prove that

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = z_0.$$

[Hint: Write $f(z) = b_{-k}(z - z_0)^{-k} + \cdots + b_{-1}(z - z_0)^{-1} + g(z)$ where each $b_j \in \mathbb{C}$ and $g(z)$ is holomorphic at z_0. Now expand each term as a power series in z and compute the limit. Note that g is holomorphic on a disc with center 0 and radius $R > 1$.]

(9) Compute the residues of the following functions.

(a) e^z/z^5 at $z = 0$.

(b) $\cos z/z^6$ at $z = 0$.

(c) $\cot z$ at $z = 0$.

(d) $1/(e^z - 1)$ at $z = 0$.

(e) $1/(\cos z - 1)$ at $z = 0$.

(f) $1/(z^n - 1)$ at $z = 1$ for $n \in \mathbb{N}$ a positive integer.

(g) $1/(z^n - 1)^2$ at $z = 1$ for $n \in \mathbb{N}$ a positive integer.

(10) Use the residue theorem to compute the following integrals.

(a)

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\, dx = \frac{\pi}{\sqrt{2}}.$$

(b)

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\, dx = \frac{\pi e^{-a}}{a} \qquad \text{for } a > 0.$$

(c)

$$\int_{0}^{2\pi} \frac{1}{(a+\cos\theta)^2}\, d\theta = \frac{2\pi a}{(a^2-1)^{3/2}} \qquad \text{for } a > 1.$$

(d)

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}}\, dx = \frac{(2n)!}{(n!)^2 \cdot 2^{2n}} \cdot \pi = \frac{(2n-1)(2n-3)\cdots 1}{2n(2n-2)\cdots 2} \cdot \pi.$$