Math 621 Homework 1

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Reading: Stein and Shakarchi, 1.1–2.2.

Justify your answers carefully.

(1) Check the Cauchy–Riemann equations hold for the following functions (a) z^2, (b) z^3, (c) e^z.

(2) Let $f(z) = f(x+iy) = u + iv$.

(a) Show that the Cauchy–Riemann equations in polar coordinates $z = re^{i\theta}$ on the domain are given by

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

(b) Let $\Omega = \mathbb{C} \setminus (-\infty, 0]$ and

$$f\colon \Omega \to \mathbb{C}, \quad f(z) = \log r + i\theta$$

where $z = re^{i\theta}$ and $-\pi < \theta < \pi$. Using part (a) or otherwise show that f is holomorphic and $e^{f(z)} = z$.

(3) Show that if $f(z)$ is a holomorphic function then so is $g(z) = \overline{f(\bar{z})}$ (where the bars denote complex conjugate). Given a power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for f determine a power series expansion for g.

(4) Compute the images of the coordinate lines $x = c$, $y = d$ under the transformation $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$. (Here we write $z = x + iy$ and $w = f(z) = u + iv$.) Check that they intersect at right angles for $(c, d) \neq 0$. Draw a sketch.

[Hint: Note that the images of the two lines $x = \pm c$ are equal and similarly the images of the two lines $y = \pm d$ are equal.]

(5) Consider the complex exponential function $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = e^z$.

(a) What is the image (or range) of f?

(b) Prove that $f(z_1) = f(z_2)$ iff $z_2 = z_1 + (2\pi i)k$ for some $k \in \mathbb{Z}$.

(c) (Optional) Let $S = \mathbb{C}/(2\pi i)\mathbb{Z}$ be the topological quotient space of the complex plane $\mathbb{C}$ by the equivalence relation

$$z_1 \sim z_2 \iff z_2 - z_1 \in (2\pi i)\mathbb{Z}.$$

So S is an infinite cylinder and the map f descends to a bijection g from S to the image of f. Describe the map g geometrically.

(6) Prove that the addition formulae for sine and cosine hold for complex values of the variables. Use this to write $\cos z$ and $\sin z$ in terms of real trigonometric and hyperbolic functions. Deduce a formula for $|\cos z|^2$ and $|\sin z|^2$ and deduce that $\cos z$ and $\sin z$ are *not* bounded. (We say a function $f\colon \Omega \to \mathbb{C}$ is *bounded* if there exists $M \in \mathbb{R}$ such that $|f(z)| < M$ for all $z \in \Omega$.)

(7) Let $(a_n)_{n=0}^{\infty}$ be a sequence of complex numbers. Show that if

$$\lim_{n\to\infty} |a_{n+1}/a_n| = L$$

then $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R = 1/L$ (i.e. the series converges absolutely for $|z| < R$ and diverges for $|z| > R$).

(8) Fix $\alpha, \beta, \gamma \in \mathbb{C}$, $\gamma \notin \{0, -1, -2, \ldots\}$. Compute the radius of convergence of the hypergeometric series

$$F(z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)n!} z^n.$$

(9) Fix $q \in \mathbb{C}$. Compute the radius of convergence of $\sum_{n=0}^{\infty} q^{n^2} z^n$.

[Remark: This series is related to the Riemann theta function.]

(10) (a) Prove that the series $\sum_{n=0}^{\infty} z^n$ has radius of convergence $R = 1$, and diverges when $|z| = 1$.

(b) Prove that the series $\sum_{n=0}^{\infty} \frac{1}{n^2} z^n$ has radius of convergence $R = 1$, and converges when $|z| = 1$.

(c) Prove that the series $\sum_{n=0}^{\infty} \frac{1}{n} z^n$ has radius of convergence $R = 1$, and converges when $|z| = 1$ iff $z \neq 1$.

[Hint for (c): Use summation by parts, see Stein and Shakarchi, p. 28, Exercise 14.]

(11) Let $z_1, z_2, z_3 \in \mathbb{C}$ be non-collinear points in the complex plane, and consider the triangle with vertices z_1, z_2, z_3

(a) Show that the angles at z_1 and z_2 are equal iff

$$\frac{z_3 - z_1}{z_2 - z_1} \cdot \frac{z_3 - z_2}{z_1 - z_2} \in \mathbb{R}_{>0}.$$

(b) Show that the triangle is equilateral iff

$$\frac{z_3 - z_1}{z_2 - z_1} \cdot \frac{z_3 - z_2}{z_1 - z_2} = 1.$$

Now simplify this formula to obtain a polynomial equation in z_1, z_2, z_3 which is visibly symmetric in the variables.

(12) Let $\alpha \in \mathbb{C}$, $|\alpha| < 1$.

(a) Prove that $|\frac{\alpha - z}{1 - \bar{\alpha} z}| \leq 1$ for $|z| \leq 1$ with equality iff $|z| = 1$. [Hint: Use $|w|^2 = w\bar{w}$ to expand the (squared) absolute value.]

(b) Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disc in $\mathbb{C}$. Using part (a), let $f \colon \mathbb{D} \to \mathbb{D}$ be the holomorphic function defined $f(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}$. Prove that $f \circ f(z) = z$. In particular, f is a bijection, and f interchanges 0 and α. (The function f is called a *Blaschke factor*.)

(13) Let $\Omega \subset \mathbb{R}^2$ be an open set and $u \colon \Omega \to \mathbb{R}$ a function. We say u is *harmonic* if it satisfies *Laplace's equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

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(a) Suppose $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is a holomorphic function. Assume that u and v have continuous second partial derivatives (actually this assumption is always satisfied because holomorphic functions have derivatives of all orders). Prove that u and v are harmonic functions.

(b) (Optional). Conversely, suppose $u\colon \Omega \to \mathbb{R}$ is a harmonic function on a simply connected open set $\Omega \subset \mathbb{C}$ with continuous second partial derivatives. Show that there exists a harmonic function v such that $f = u + iv$ is holomorphic. The function v is called a *harmonic conjugate* of u and is uniquely determined up to an additive constant. [Hint: Define v as a path integral and prove independence of path using Green's theorem / Stokes' theorem.]