2.

$$\begin{tikzcd}
M_1 \arrow[r, "\beta_1"] \arrow[d, "\alpha_1"] & M_2 \arrow[r, "\beta_2"] \arrow[d, "\alpha_2"] & M_3 \arrow[r, "\beta_3"] \arrow[d, "\alpha_3"] & M_4 \arrow[d, "\alpha_4"] \\
N_1 \arrow[r, "\gamma_1"'] & N_2 \arrow[r, "\gamma_2"'] & N_3 \arrow[r, "\gamma_3"'] & N_4
\end{tikzcd}$$

commutative diagram w/ exact rows

<u>Claim:</u> α_2 & α_4 injective, α_1 surjective $\Rightarrow$ α_3 injective

<u>Proof:</u> Let $x \in M_3$ & suppose $\alpha_3(x) = 0$. Required to prove $x = 0$.

$$\alpha_4 \beta_3(x) = \gamma_3 \alpha_3(x) = 0 \underset{\alpha_4 \text{ inj}}{\Rightarrow} \beta_3(x) = 0 \underset{\substack{\text{exactness of} \\ \text{row 1 at } M_3}}{\Rightarrow} \exists\, y \in M_2 .\ \beta_2(y) = x$$

$$\gamma_2 \alpha_2(y) = \alpha_3 \beta_2(y) = 0 \underset{\substack{\text{exactness of} \\ \text{row 2 at } N_2}}{\Rightarrow} \exists\, w \in N_1 \text{ s.t. } \gamma_1(w) = \alpha_2(y) =: z$$

$$\underset{\alpha_1 \text{ surj}}{\Rightarrow} \exists\, t \in M_1 \text{ s.t. } \alpha_1(t) = w$$

Now $$\alpha_2 \beta_1(t) = \gamma_1 \alpha_1(t) = \cancel{\alpha_2}\, z = \alpha_2(y)$$

$$\underset{\alpha_2 \text{ inj}}{\Rightarrow} \beta_1(t) = y$$

$$\Rightarrow x = \beta_2(y) = \beta_2 \beta_1(t) \underset{\text{exactness of row 1 at } M_2.}{=} 0$$

$\square$.

Picture:

$$\begin{tikzcd}
 & y' \arrow[d, equal] & & \\
t \arrow[ur, mapsto, bend left] \arrow[d, mapsto] & y \arrow[r, mapsto] \arrow[d, mapsto] & x \arrow[r, mapsto] \arrow[d, mapsto] & 0 \arrow[d, mapsto] \\
w \arrow[r, mapsto] & z \arrow[r, mapsto] & 0 \arrow[r, mapsto] & 0
\end{tikzcd}$$

$\rightsquigarrow x = 0.$

3. R PID M f.g. R-module

$\Rightarrow \quad M \simeq R^r \oplus R/(d_1) \oplus \cdots \oplus R/(d_s)$

$d_1, \ldots, d_s \in R$, nonzero, not units, $d_1 | d_2 | \cdots | d_s$

$\Rightarrow \quad \mathrm{Hom}(M, R) \simeq \mathrm{Hom}(R^r, R) \oplus \mathrm{Hom}(R/(d_1), R) \oplus \cdots \oplus \mathrm{Hom}(R/(d_s), R)$

$\simeq R^r \oplus 0 \oplus \cdots \oplus 0$

$\simeq R^r. \quad \square$

(Note: $\mathrm{Hom}(R/I, R) = 0$ for $0 \neq I \subset R$, R integral domain.)

4. $R = \mathbb{Z}, \quad P = \mathbb{Z}/2\mathbb{Z}$

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \cdot)$

$$\begin{array}{ccccccc} 0 \to & \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) & \to & \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) & \to & \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \\ & \| & & \| & & \wr\| \\ & 0 & & 0 & & \mathbb{Z}/2\mathbb{Z} \end{array}$$

not surjective.

$\square$.

5. a. Recall: a subset $I \subset R$ of a ring R is an ideal if $(I, +) \subset (R, +)$ is a subgroup

& $r \cdot x \in I$ for $r \in R, x \in I$.

Equivalently: 1. $0 \in I$, 2. $x, y \in I \Rightarrow x + y \in I$, & 3. $r \in R, x \in I \Rightarrow rx \in I$.

Now, check $(I : J)$ is an ideal :-

1. $0 \cdot I = \{0\} \subset J$ ✓

2. $x \cdot I \subset J$ & $y \cdot I \subset J \Rightarrow (x+y) \cdot I \subset J$ by 2. for J

3. $x \cdot I \subset J$ & $r \in R \Rightarrow (rx) \cdot I \subset J$ by 3. for J.

Also, $r \in J \Rightarrow r \cdot I \subset J$ by 3. for J.

So $J \subset (I:J)$.

b. <u>Claim:</u>

$$\mathrm{Hom}_R(R/I, R/J) \xrightarrow{\sim} (I:J)/J$$
$$\theta \longmapsto \theta(1)$$

<u>Proof:</u> R/I is generated by $1 \in R/I$ as an R-module.

So a hom. $\theta: R/I \to M$ of R-modules is determined by $\theta(1)$.

(via $\theta(\bar{r}) = \theta(r \cdot 1) = r \cdot \theta(1)$)

Necessarily $I \cdot \theta(1) = 0$ (because $x \cdot \theta(1) = \theta(\bar{x}) = \theta(0) = 0$ for $x \in I$)

Conversely, if $m \in M$ satisfies $I \cdot m = 0$ then $\theta: R/I \to M$, $\theta(\bar{r}) = r \cdot m$

is a well defined hom. of R-modules.

Thus $\mathrm{Hom}_R(R/I, M) \xrightarrow{\sim} \{m \in M \mid I \cdot m = 0\} \subset M$.

$$\theta \longmapsto \theta(1).$$

Now if $M = R/J$, $m = \bar{r} \in R/J$, then $I \cdot m = 0 \Leftrightarrow r \cdot I \subset J$

$\Leftrightarrow r \in (I:J)$.

So $\mathrm{Hom}_R(R/I, R/J) \xrightarrow{\sim} (I:J)/J$. $\square$.

c. $R = F[x,y]$, $I = (x,y)$, $J = I^2 = (x^2, xy, y^2)$

$$\mathrm{Hom}_R(R/I, R/J) \overset{b}{\simeq} (I:J)/J = (x,y)/(x^2,xy,y^2) \xleftarrow{\sim} R/I \oplus R/I$$
$$\bar{x} \longleftrightarrow e_1$$
$$\bar{y} \longleftrightarrow e_2$$

Note $R/I \xrightarrow{\sim} F$ as F-vector space. So $\dim_F \mathrm{Hom}_R(R/I, R/J) = 2$. $\square$.

$\overline{f(x,y)} \longmapsto f(0,0)$

6. a. Recall: A $\mathbb{Z}$-module (or abelian group) $(A,+)$ is injective

iff A is divisible i.e. $\forall\, a \in A,\ n \in \mathbb{N},\ \exists\, b \in A$ s.t. $nb = a$.

In our case $A = \mathbb{C}^*$ (written multiplicatively)

so we're required to prove: $\forall\ a \in \mathbb{C}^*,\ n \in \mathbb{N},\ \exists\ b \in \mathbb{C}^*$ s.t. $b^n = a$

For example, writing $a = re^{i\theta}$ $r \in \mathbb{R}_{>0}$, $0 \le \theta < 2\pi$,

we can take $b = r^{1/n} \cdot e^{i\theta/n}$

b.
$$0 \longrightarrow \mathbb{Z} \xrightarrow[\alpha]{\times n} \mathbb{Z} \xrightarrow[\beta]{} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

$\mathrm{Hom}(\cdot, \mathbb{C}^*)$

$$(k \mapsto \theta(k)) \longmapsto (k \mapsto \theta(n \cdot k))$$

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^*) & \xrightarrow{\beta^*} & \mathrm{Hom}(\mathbb{Z}, \mathbb{C}^*) & \xrightarrow{\alpha^*} & \mathrm{Hom}(\mathbb{Z}, \mathbb{C}^*) & \longrightarrow 0 \\
& \wr\!\wr \ \ \theta \mapsto \theta(1) & & \wr\!\wr \ \ \theta \mapsto \theta(1) & & \wr\!\wr & \\
0 \longrightarrow & \mu_n & \xrightarrow{\text{inclusion}} & \mathbb{C}^* & \longrightarrow & \mathbb{C}^* & \longrightarrow 0 \\
& \| & & a & \longmapsto & a^n & \\
& \{z \in \mathbb{C}^* \mid z^n = 1\} & & & & & \\
& \wr\!\wr & & & & & \\
& \mathbb{Z}/n\mathbb{Z} & & & & &
\end{array}$$

n^{th} roots of 1.

exact.

8. Claim A f.g. / R-module P is projective $\iff$ $\exists$ R-module G s.t. $P \oplus G \simeq R^n$, some $n \in \mathbb{N} \cup \{0\}$

Proof: $\Rightarrow$. P f.g. $\Rightarrow$ $\exists$ surjection $R^n \overset{\varphi}{\twoheadrightarrow} P$, some $n \geq 0$.

Let $K = \ker \varphi$. So $0 \longrightarrow K \longrightarrow R^n \xrightarrow{\varphi} P \longrightarrow 0.$ $(*)$ exact

P projective, i.e. $\mathrm{Hom}_R(P, -)$ exact $\Rightarrow$ $\mathrm{Hom}_R(P, R^n) \twoheadrightarrow \mathrm{Hom}_R(P,P)$.

i.e.
$$\begin{array}{ccc}
R^n & \xrightarrow{\varphi} & P \longrightarrow 0 \\
\exists\ \nwarrow \tilde{\theta} & & \uparrow \theta \\
& & P
\end{array}$$

Now let $\theta = id_P$, then $\exists\ s = \tilde{\theta}$ s.t. $\varphi \circ s = id_P$.

$\Rightarrow$ $(*)$ is split exact sequence (cf. class notes)

$\Rightarrow$ $R^n \simeq K \oplus P$.

$\Leftarrow$. If $P \oplus Q \cong R^n$

Then $\mathrm{Hom}(P,-) \oplus \mathrm{Hom}(Q,-) \cong \mathrm{Hom}(R^n,-) \cong (-)^{\oplus n}$ exact

$\Rightarrow$ $\mathrm{Hom}(P,-)$ ($\&$ $\mathrm{Hom}(Q,-)$) exact $\Rightarrow$ P projective. $\square$.

9. $R = R_1 \times R_2$ direct product of rings

$\Rightarrow$ $R = R_1 \oplus R_2$ direct sum of R-modules (where R_1 is an R-module via $(r_1, r_2) \cdot r_1' = r_1 r_1'$; similarly for R_2).

$\overset{\&8}{\Rightarrow}$ R_1 & R_2 are projective R-modules.

10. a). See Hint.

b). $R = R^1$ projective (free!) $\Rightarrow$ exact sequence in (a) is split (cf. &8)

$\Rightarrow$ $I \oplus J \cong IJ \oplus R$

c) IJ principal & R ID $\Rightarrow$ $R \xrightarrow{\sim} I \cdot J = (f)$, $1 \longmapsto f$ iso of R-Modules

$\overset{b}{\Rightarrow}$ $I \oplus J \cong R^2$ $\overset{\&8}{\Rightarrow}$ I, J projective R-Modules.

d.

$I + J \ni 3 - (1+\sqrt{-5}) - (1-\sqrt{-5}) = 1$ $\Rightarrow$ $I + J = R$.

$IJ = (3^2, 3\cdot(1+\sqrt{-5}), 3\cdot(1-\sqrt{-5}), (1+\sqrt{-5})(1-\sqrt{-5})) = (9, 3\cdot(1\pm\sqrt{-5}), 6) = (3)$

I & J are not principal: if $I = (\alpha)$ then, defining $N: R \to \mathbb{Z}_{\geq 0}$, $N(a+b\sqrt{-5}) = a^2+5b^2 = |a+b\sqrt{-5}|^2$ ($\Rightarrow N(\alpha_1\alpha_2) = N(\alpha_1)\cdot N(\alpha_2)$)

have $N(\alpha) \mid \underset{9}{N(3)}$ & $\underset{6}{N(1+\sqrt{-5})}$ $\Rightarrow$ $\underset{a^2+5b^2}{N(\alpha)} \mid 3$ $\Rightarrow$ $\alpha = \pm 1$

$\Rightarrow$ $I = R$ $\cancel{\#}$ e.g. $R/I \cong \mathbb{Z}[x]/(x^2+5, 3, 1+x) \cong \mathbb{Z}/3\mathbb{Z}[x]/(x+1) \cong \mathbb{Z}/3\mathbb{Z}$

$x^2+5 = (x+1)(x+2) \bmod 3$

Similarly, J not principal.

Finally, if R is an integral domain & $I \subset R$ is an ideal that is not principal, then I is not a free R-module – if $d_1, \dots, d_n$ were a basis of I as a free R-module then $n > 1$ (I not principal) but $(d_2) \cdot d_1 + (-d_1) \cdot d_2 = 0$ ⨳ $d_1, \dots, d_n$ not linearly independent over R.