Math 612 Homework 2

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Reading: Dummit and Foote, Section 10.5.

Justify your answers carefully (complete proofs are expected). All rings are assumed commutative with 1.

(1) (Optional) Recall the *snake lemma*: Let

$$\begin{array}{ccccccc} & & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \\ 0 & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & & \end{array}$$

be a commutative diagram of R-module homomorphisms with exact rows, then there is an induced exact sequence

$$\ker \alpha_1 \to \ker \alpha_2 \to \ker \alpha_3 \xrightarrow{\delta} \operatorname{coker} \alpha_1 \to \operatorname{coker} \alpha_2 \to \operatorname{coker} \alpha_3.$$

Complete the proof of the snake lemma started in class.

(2) Let

$$\begin{array}{ccccccc} M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} \\ N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 \end{array}$$

be a commutative diagram of R-module homomorphisms with exact rows. Show that if α_2 and α_4 are injective and α_1 is surjective then α_3 is injective.

(3) Let R be a PID and M a finitely generated R-module. Prove that $\operatorname{Hom}_R(M, R)$ is a free R-module.

(4) Give an example of a ring R and an R-module P such that the functor $\mathrm{Hom}(P, \cdot)$ is not exact. That is, there is a short exact sequence of R-modules

$$0 \to L \to M \to N \to 0$$

such that the induced sequence

$$0 \to \mathrm{Hom}(P, L) \to \mathrm{Hom}(P, M) \to \mathrm{Hom}(P, N) \to 0$$

is *not* exact. (Justify your answer carefully.)

(5) Let R be a ring and let $I, J \subset R$ be ideals of R. We define the *colon ideal*

$$(I : J) := \{r \in R \mid r \cdot I \subset J\}.$$

(a) Show that $(I : J)$ is an ideal of R and $J \subset (I : J)$.

(b) Show that the R-module $\mathrm{Hom}_R(R/I, R/J)$ is isomorphic to the quotient module $(I : J)/J$.

(c) Let F be a field, $R = F[x, y]$, $I = (x, y)$, and $J = (x^2, xy, y^2)$. Describe the R-module $\mathrm{Hom}_R(R/I, R/J)$ explicitly. What is the dimension of $\mathrm{Hom}_R(R/I, R/J)$ when regarded as an F-vector space?

(6) (a) Show that the abelian group $\mathbb{C}^\times$ (the non-zero complex numbers with group operation multiplication) is an injective $\mathbb{Z}$-module. [Recall that an abelian group can be regarded as a $\mathbb{Z}$-module. However care is needed here because the group $A = \mathbb{C}^\times$ is written multiplicatively, so that scalar multiplication $n \cdot a$ of an element $a \in A$ by $n \in \mathbb{Z}$ in the $\mathbb{Z}$-module structure is equal to a^n.]

(b) By part (a), the contravariant functor $\mathrm{Hom}(\cdot, \mathbb{C}^\times)$ from the category of abelian groups (or $\mathbb{Z}$-modules) to itself is exact. Check this explicitly for the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

for $n \in \mathbb{N}$ (describe the sequence obtained by applying the functor explicitly and verify that it is exact).

(7) (Optional) Let F be a field, $R = F[x, y, z]$, and $M = (x, y, z) \subset R$. Determine a presentation of the R-module M.

(8) Recall that we say a R-module P is *projective* if the functor $\mathrm{Hom}_R(P, \cdot)$ is exact. Show that a finitely generated R-module is projective iff it is a direct summand of a free module.

[Remark: The same is true for any R-module provided we allow free modules of infinite rank.]

(9) Let $R = R_1 \times R_2$ be the direct product of two rings R_1 and R_2. Show that R_1 and R_2 are projective R-modules.

(10) Let R be a ring and $I, J \subset R$ ideals of R such that $I + J = R$. Then $IJ = I \cap J$ (why?).

(a) Show that we have an exact sequence of R-modules

$$0 \to IJ \to I \oplus J \to R \to 0.$$

(b) Deduce that we have an isomorphism of R-modules

$$I \oplus J \simeq IJ \oplus R.$$

(c) Now suppose that R is an integral domain and IJ is principal. Deduce that I and J are projective R-modules.

(d) Let $R = \mathbb{Z}[\sqrt{-5}]$ and $I = (3, 1 + \sqrt{-5})$, $J = (3, 1 - \sqrt{-5})$. Show that $I + J = R$, and IJ is principal but I and J are not principal. Deduce that I and J are projective R-modules which are not free.

Hints:

1 Check exactness at each term by diagram chasing. Note that by definition $L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is *exact* at M if $\operatorname{im}\alpha = \ker\beta$; equivalently $\beta \circ \alpha = 0$ and $\ker\beta \subset \operatorname{im}\alpha$.

2 Another diagram chase.

3 What is the structure theorem for finitely generated modules over a PID?

4 Consider e.g. $R = \mathbb{Z}$ and $P = \mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{N}$.

5 (a) Recall a subset $I \subset R$ is an ideal iff $0 \in I$ and I is closed under addition and scalar multiplication by elements of R. (b) A R-module homomorphism $\theta\colon R/I \to R/J$ is determined by $\theta(1)$ (because the R-module R/I is generated by $1 \in R/I$). (c) Use part (b) and compute explicitly.

6 (a) We proved in class that an abelian group A is an injective $\mathbb{Z}$-module iff for all $a \in A$ and $n \in \mathbb{N}$ there exists $b \in A$ such that $n \cdot b = a$ (we say A is *divisible*).

7 Consider the surjective R-module homomorphism $\varphi\colon R^3 \to M$ sending the standard basis vectors e_1, e_2, e_3 to the generators x, y, z of M. Write down some elements of $\ker\varphi$ and prove that they generate $\ker\varphi$.

8 If $0 \to L \to M \to N \to 0$ is a short exact sequence of R-modules and N is projective then the exact sequence is split, in particular $M \simeq L \oplus N$. A module M is finitely generated iff there is a surjection $R^n \to M$. For the converse, note that a free module F is projective, and if $F = P \oplus Q$ then $\operatorname{Hom}(F, \cdot) = \operatorname{Hom}(P, \cdot) \oplus \operatorname{Hom}(Q, \cdot)$.

9 Use Q8.

10 We have $I \cap J = IJ$ because $I \cap J = (I \cap J)(I + J) \subset IJ$ (using the assumption $I + J = R$). (a) This is a special case of a (straightforward) result proved in class: if $L_1, L_2 \subset M$ are submodules of a module M, then $L_1 + L_2 \simeq L_1 \oplus L_2/L_1 \cap L_2$, where the inclusion $L_1 \cap L_2 \to L_1 \oplus L_2$ is given by $l \mapsto l \oplus (-l)$. (b) The exact sequence of (a) splits (why?). (c) Use Q8. (d) Use the norm $N(a + b\sqrt{-5}) = a^2 + 5b^2$ to show I and J are not principal (cf. MATH 611). Now use part (c).