1.a. $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

$c_A(x) = \det(xI - A) = \det\begin{pmatrix} x-1 & 0 & 1 \\ -1 & x-2 & -1 \\ -1 & 0 & x-3 \end{pmatrix} = (x-1)(x-2)(x-3) + 1\cdot(-(-1)\cdot(x-2))$

char. poly.

$= (x-2)(x^2-4x+3+1)$

$= (x-2)^3.$

$\therefore\ m_A(x) = (x-2)^k, \quad k \le 3.$

min. poly

$$A - 2I = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \neq 0.$$

$$(A-2I)^2 = 0. \quad \Rightarrow \quad m_A(x) = (x-2)^2$$

$\therefore$ invariant factors are $d_1 = (x-2)$, $d_2 = (x-2)^2 = x^2 - 4x + 4$

RCF : $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -4 \\ 0 & 1 & 4 \end{pmatrix}$ JNF : $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$

b. Compute Smith normal form of $x\cdot I - A$.

$$\begin{pmatrix} x-1 & 0 & 1 \\ -1 & x-2 & -1 \\ -1 & 0 & x-3 \end{pmatrix} \underset{-(x-3)R_1}{\overset{+R_1}{\rightsquigarrow}} \begin{pmatrix} 1 & 0 & x-1 \\ -1 & x-2 & -1 \\ x-3 & 0 & -1 \end{pmatrix} \overset{-(x-1)\cdot C_1}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & x-1 \\ 0 & x-2 & x-2 \\ 0 & 0 & -(x-2)^2 \end{pmatrix}$$

$$\rightsquigarrow \underset{-C_2}{\overset{\cancel{+(x-2)}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & x-2 & x-2 \\ 0 & 0 & -(x-2)^2 \end{pmatrix}}} \rightsquigarrow \underset{*-1}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & x-2 & 0 \\ 0 & 0 & -(x-2)^2 \end{pmatrix}} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & x-2 & 0 \\ 0 & 0 & (x-2)^2 \end{pmatrix} =: D$$

$\therefore$ RCF $\quad B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -4 \\ 0 & 1 & 4 \end{pmatrix}$ $\quad$ (checks w/ (a)).

$$\begin{tikzcd}
F[x]^3 \arrow[r, "xI-A"] & F[x]^3 \arrow[r] & F^3 \arrow[r] & 0 & & v_1, v_2, v_3 \\
F[x]^3 \arrow[u, "P", "\wr"'] \arrow[r, "D"] & F[x]^3 \arrow[u, "\wr", "Q"'] \arrow[r] & \cancel{F[x]/(1)} \oplus F[x]/(x-2) \oplus F[x]/(x-2)^2 \arrow[u, "\wr"] \arrow[r] & 0 & & e_2, e_3, x\cdot e_3 \arrow[u]
\end{tikzcd}$$

$Q^{-1}(xI - A)\cdot P = D.$

$Q = E_2 \cdot E_1$ $\quad$ row ops $\quad$ 1. $R2 \rightsquigarrow R2 + R1$

$\quad$ 2. $R3 \rightsquigarrow R3 - (x-3)R1$

$Q^{-1} = E_1^{-1} \cdot E_2^{-1}$ $\quad$: $\quad \begin{matrix} \\ \\ +(x-3)R1 \end{matrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{-R1}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x-3 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ x-3 & 0 & 1 \end{pmatrix}$

$\rightsquigarrow \quad v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_3 = A \cdot v_2 = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$

$R = (v_1\ v_2\ v_3) = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$, $\quad R^{-1}AR = B.$

For JNF $\quad C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ $\quad$ use basis $1, (x-2)$ of $F[x]/(x-2)^2$

i.e. basis $v_1, v_2, -2v_2 + v_3$ of F^3

$S = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $\quad S^{-1}AS = C.$

2. $A^2 = A \Rightarrow A^2 - A = 0$, $m_A(x) \mid x^2 - x = x\cdot(x-1)$

$\Rightarrow A$ diagonalizable (m_A is a product of distinct linear factors in $F[x]$), eigenvalues $\lambda = 0$ or 1.

$$\Rightarrow \text{JNF of } A = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \begin{matrix} \}\, r \\ \\ \}\, n-r \end{matrix} . \quad \square.$$

3. $B = PAP^{-1}$

$\Rightarrow c_B(x) = \det(xI - B) = \det(xI - PAP^{-1}) = \det(P\cdot(xI - A)\cdot P^{-1})$

$= \cancel{\det P}\cdot \det(xI - A)\cdot \cancel{\det P^{-1}} = \det(xI - A) = c_A(x)$.

For $p(x) \in F[x]$, $p(B) = p(PAP^{-1}) = P\,p(A)\,P^{-1}$.

So $p(B) = 0 \iff p(A) = 0$, $\therefore m_B = m_A$. $\square$

b. A diagonalizable, $P^{-1}AP = D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \Rightarrow c_A(x) = (x-\lambda_1)\cdots(x-\lambda_n)$.

So, if A, B diagonalizable & $c_A = c_B$ then $A \sim D \sim B$ (where $\sim$ denotes "is similar to"), where $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ & $\lambda_1, \dots, \lambda_n$ are the roots of $c_A = c_B$ (w/ multiplicities).

$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, $B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, then $c_A = c_B = (x-\lambda)^2$,

but $A \not\sim B$ because $\dim\ker(B - \lambda I) = 1 < \dim\ker(A - \lambda I) = 2$.

c. Recall $c_A = d_1 \cdots d_s$ where $d_1 \mid d_2 \mid \cdots \mid d_s$ are (non constant) invariant factors of $xI - A$ & $m_A = d_s$.

$n = \deg c_A = \sum \deg d_i$

$\underline{n \leq 3}$. If $m_A = d_s$ is linear, $d_s = x - \lambda$, then $A = \lambda I$

Otherwise $s \leq 2$, so $(c_A, m_A) = (c_B, m_B)$

$\Rightarrow$ $xI - A$ & $xI - B$ have same invariant factors

$\Rightarrow$ A & B have same RCF

$\Rightarrow$ $A \sim B$.

$\underline{n=4}$: $A = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 1 & \lambda \end{pmatrix} \not\sim B = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 1 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 1 & \lambda \end{pmatrix}$ because $\dim\ker(B - \lambda I) = 2$ $< \dim\ker(A - \lambda I) = 3$

but $c_A = c_B = (x-\lambda)^4$ & $m_A = m_B = (x-\lambda)^2$.

4. A nilpotent, $A^k = 0$ $\Rightarrow$ $m_A(x) \mid x^k$.

Minimal polynomial of A is product of linear factors in $F[x]$

$\Rightarrow$ $\exists\, P \in GL_n(F)$ s.t. $P^{-1}AP$ is in JNF, & eigenvalues are the roots of $m_A(x)$ (namely 0 in this case.)

So $A \sim \begin{pmatrix} \boxed{J_1} & & \\ & \ddots & \\ & & \boxed{J_r} \end{pmatrix}$ $\quad J_i = J(n_i, 0) = \underbrace{\begin{pmatrix} 0 & & 0 \\ 1 & \ddots & \\ 0 & \ddots\; 1 & 0 \end{pmatrix}}_{\leftarrow n_i \rightarrow}\Big]\, n_i$

Jordan block of size $n_i \times n_i$
w/ eigenvalue $\lambda = 0$

$n_1 \leq \cdots \leq n_r$, $\quad \sum n_i = n$.

In general JNF is uniquely determined up to reordering the blocks (when it exists); here we have fixed the ordering by size of block (given $\exists!$ eigenvalue). $\square$.

5. $P^{-1}AP = B = \begin{pmatrix} \boxed{J_1} & & 0 \\ & \ddots & \\ 0 & & \boxed{J_r} \end{pmatrix}$ JNF

$$J_i = \overbrace{\begin{pmatrix} \lambda_i & & 0 \\ 1 & \ddots & \\ 0 & 1 & \lambda_i \end{pmatrix}}^{n_i} \Big\updownarrow n_i$$

Write $B = D_0 + N_0$ where D_0 is diagonal & N_0 has only zero entries off the sub diagonal $i=j+1$ (i.e., write $J_i = \begin{pmatrix} \lambda_i & & 0 \\ & \ddots & \\ 0 & & \lambda_i \end{pmatrix} + \begin{pmatrix} 0 & & 0 \\ 1 & \ddots & \\ 0 & 1 & 0 \end{pmatrix}$ on each block).

Then N_0 is nilpotent. (b/c $\begin{pmatrix} 0 & & 0 \\ 1 & \ddots & \\ 0 & 1 & 0 \end{pmatrix}^k = 0$, where the matrix is $k \times k$)

Also $D_0 N_0 = N_0 D_0$ (b/c can treat each block separately, & blocks of D_0 are scalar matrices so commute with all matrices.)

Now $A = PD_0P^{-1} + PN_0P^{-1} = D + N$

as required. □

6. Write $J = D + N$ as in Q5 & use binomial thm (recall D & N commute)

$$\therefore J^k = (D+N)^k = \sum_{i=0}^{k} D^{k-i} \cdot N^i \cdot \binom{k}{i}$$

$$= \begin{pmatrix} \lambda^k & & \\ & \ddots & \\ & & \lambda^k \end{pmatrix} + \binom{k}{1}\begin{pmatrix} 0 & & 0 \\ \lambda^{k-1} & \ddots & \\ 0 & \lambda^{k-1} & 0 \end{pmatrix} + \binom{k}{2}\begin{pmatrix} 0 & & & 0 \\ 0 & \ddots & & \\ \lambda^{k-2} & \ddots & & \\ 0 & \lambda^{k-2} & 0 & 0 \end{pmatrix}$$

$$+ \dots$$

$$= \begin{pmatrix} \lambda^k & & & 0 \\ \binom{k}{1}\lambda^{k-1} & \lambda^k & & \\ \binom{k}{2}\lambda^{k-2} & \binom{k}{1}\lambda^{k-1} & \lambda^k & \\ \vdots & & & \ddots \\ \binom{k}{n-1}\lambda^{k-n+1} & \dots & & \lambda^k \end{pmatrix}$$

□.

7.

$(J \circlearrowright F^n) \xrightarrow{\simeq} F[x] / (x-\lambda)^n$

$F[x]$-module

$e_i \longmapsto (x-\lambda)^{i-1}$

$W \subset F^n$ s.t. $J \circlearrowright W$ $\iff$ $F[x]$-submodule of $F[x]/(x-\lambda)^n$ $\left(I/(x-\lambda)^n \right.$

$\iff$ $F[x]$-submodule ($=$ideal) of $F[x]$ $\uparrow$

containing $(x-\lambda)^n$, I

i.e. $I = ((x-\lambda)^k)$, $0 \le k \le n$.

So, possible W's are

$\langle e_{k+1}, \dots, e_n \rangle_F$

$0 \le k \le n$. $\square$.

w/ F-basis for $I/(x-\lambda)^n$

$(x-\lambda)^k, \dots, (x-\lambda)^{n-1}$

8. Min poly $m_T(x) \mid x^5 - 1$

Also, 1 is not an eigenvalue of T, so $(x-1) \nmid m_T(x)$. (roots of m_T are evals of T)

Thus $m_T(x) = x^5 - 1 / x-1 = x^4 + x^3 + x^2 + x + 1 =: f$ (irred in $\mathbb{Q}[x]$ by Eisenstein)

& T has primary RCF w/ blocks $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$ the companion matrix of f

($=$ RCF in this case)

So, $4 \mid \dim_{\mathbb{Q}} V$.

9. $m_A \mid x^6 - 1 = (x^3-1)(x^3+1) = (x-1)(x^2+x+1)(x+1)(x^2-x+1)$

irred factorization in $\mathbb{Q}[x]$.

So, A has primary RCF w/ blocks the corresponding companion matrices

$$\begin{bmatrix} 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

of orders 1, 3, 2, 6

$\therefore$ representatives of conjugacy classes of order 6 are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}. \quad \square$$