Math 611 Midterm review problems

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(1) Let G be a group of order 52 and $x \in G$ an element such that the conjugacy class $C(x)$ of x has size $|C(x)| = 4$. What is the order of x?

(2) Let G_1 and G_2 be finite groups and let $\theta\colon G_1 \to G_2$ be a homomorphism. Show that $|\theta(G_1)|$ divides $\gcd(|G_1|, |G_2|)$.

(3) Let G be a group of order 90. Suppose that there is a nontrivial action of G on a set X of size $|X| = 5$. (Here we say an action of G on X is *nontrivial* if $g \cdot x \neq x$ for some $g \in G$ and $x \in X$.) Prove that G is not a simple group.

(4) (a) Show that any non-trivial subgroup of Q_8 contains the element -1.

(b) Show that Q_8 is isomorphic to a subgroup of S_8, but is not isomorphic to a subgroup of S_n for any $n < 8$.

(5) Let G be a finite group and let $\varphi\colon G \to S_G$ be the homomorphism given by the action of G on itself by left multiplication. (Here S_G denotes the symmetric group of permutations of the set G.)

(a) Show that $\varphi(g)$ is an odd permutation iff the order $|g|$ of g is even and $|G|/|g|$ is odd.

(b) Suppose $|G| = 2m$ where m is odd. Prove that G contains a normal subgroup of index 2.

(6) Let p be a prime and G a non-abelian group of order p^3. Determine the class equation of G.

(7) Let G be a group such that $G/Z(G)$ is cyclic. Prove that G is abelian.

(8) Let G be a group of order 60 such that the order of the center of G is divisible by 4. Prove that G is abelian.

(9) Let G be a non-abelian group of order 21.

  (a) Prove that the center of G is trivial.

  (b) Determine the class equation of G.

(10) Let G be a finite group of odd order and $x \in G$ an element. Show that if x and x^{-1} are conjugate then $x = e$.

(11) Let G be a non-abelian group of order 75. Determine the number of elements of order 3 in G.

(12) Let G be a simple group of order 168. Determine the number of elements of order 7 in G.

(13) Determine the number of Sylow 2-subgroups in the alternating group A_5.

(14) Let G be a non-abelian group of order 44 such that G contains an element of order 4.

  (a) Show that G is uniquely determined up to isomorphism and describe G (i) as a semidirect product (ii) in terms of generators and relations.

  (b) Determine the center $Z(G)$ of G.

  (c) Identify the quotient $G/Z(G)$ with a standard group.

(15) Let G be a non-abelian group of order 28 such that G does not contain an element of order 4. Show that G is uniquely determined up to isomorphism and describe G in terms of generators and relations.

(16) Let G be a non-abelian group of order 18 such that G does not contain an element of order 9. Show that there are two possible isomorphism types and describe G by generators and relations in each case.

(17) Let G be a group of order pq^2 where p and q are distinct primes. Show that one of the Sylow subgroups of G is normal.

(18) Let p and q be primes such that $q^2 \equiv 1 \bmod p$. Prove that there exists a non-abelian group of order pq^2.

(19) Show that a group of order (a) 40 (b) 48 is solvable.

[Note: Actually it is a theorem of Burnside that any group of order $p^a q^b$ is solvable. But please prove these special cases without appealing to Burnside's theorem.]

(20) Let G be a finite group and p a prime dividing $|G|$. Suppose that p is the smallest prime dividing $|G|$. By considering the action of G on the set G/H of left cosets of H by left multiplication or otherwise, prove that H is normal.

(21) (Optional) Show that there is no simple group of order 120.

(22) (Optional) Let G be a finite group, N a normal subgroup of G, and p a prime such that p divides the order of G/N. Show that the number of Sylow p-subgroups of G/N is less than or equal to the number of Sylow p-subgroups of G.

Hints:

1 What is the order of the centralizer $Z(x)$ of x?

2 What is Lagrange's theorem? What is the first isomorphism theorem?

3 Consider the associated homomorphism $\varphi\colon G \to S_5$ and use Q2.

4 (b) If Q_8 acts on a set X with $|X| < 8$, what can you say about the stabilizer subgroups? (What is the orbit–stabilizer theorem?)

5 (a) What is the cycle type of $\varphi(g)$? (b) G contains an element of order $p = 2$ by e.g. Sylow theorem 1.

6 Recall that if $|G| = p^n$ then $Z(G) \neq \{e\}$. What are the possibilities for the centralizer $Z(x)$ of an element $x \in G$?

7 Let $g \in G$ map to a generator of $G/Z(G)$, so that any element of G can be written as $g^n \cdot z$ for some $n \in \mathbb{Z}$ and $z \in Z(G)$.

8 Use Q7. What can you say about groups of order pq for distinct primes p and q?

9 (a) Use Q7 (b) The orders of the conjugacy classes divide $|G|$ (it's not necessary to describe G explicitly).

10 If $gxg^{-1} = x^{-1}$ then g lies in the normalizer of $\langle x \rangle$. What can you say about the order $|g|$ of g?

11 What can you say about the Sylow 3-subgroups of G?

12 Similar to Q11.

14 (a) Use Sylow subgroups to express G as a semi-direct product, then deduce a description in terms of generators and relations (b) Writing a and b for the generators of the two factors in the semi-direct product, each element $g \in G$ can be written as $g = a^i b^j$ for some i, j. An element g lies in the center $Z(G)$ iff it commutes with the generators a and b of G. Now compute explicitly using the relations from part (a).

15 Similar to Q14(a).

16 Similar to Q14(a). If a matrix A with entries in a field F satisfies a polynomial equation which is a product of linear factors over F (e.g. $A^2 - I = (A - I)(A + I)$) then A can be diagonalized over F.

17 What is Sylow theorem 3?

18 What is the order of $\mathrm{GL}_2(\mathbb{Z}/q\mathbb{Z})$? If G is a finite group and p is a prime dividing $|G|$ then there is an element $g \in G$ of order p.

19 Recall (i) A subgroup of a solvable group is solvable, (ii) For $H \triangleleft G$, if H and G/H are solvable then G is solvable, (iii) If $|G| = p^n$ for some prime p and $n \in \mathbb{N}$ then G is solvable. (b) Consider the action of G on the set of Sylow p-subgroups by conjugation for some prime p.

20 Use Q2.