Math 611 Midterm, Wednesday 10/23/19, 7:00PM–9:00PM.

Instructions: Exam time is 2 hours. There are 5 questions for a total of 50 points. Calculators, notes, and textbook are not allowed. Justify all your answers carefully (complete proofs are expected). If you use a result proved in the textbook or class notes, state the result precisely.

Q1. (10 points) Let p be a prime and $G = \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$. Let

$$x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in G.$$

Determine the cardinality $|C(x)|$ of the conjugacy class $C(x)$ of x in G.

Q2. (10 points)

(a) (5 points) Let G be a group. Suppose given an action of G on a set X of cardinality $|X| = n$ such that $g \cdot x \neq x$ for some $g \in G$ and $x \in X$. Show that there is a normal subgroup H of G such that $H \neq G$ and the index of H in G is less than or equal to $n!$.

(b) (5 points) Let G be a group of order 108. Using part (a) or otherwise, prove that G is not simple.

Q3. (10 points) Let G be a group of order $|G| = 175$. Prove that G is abelian.

Q4. (10 points) Let G be a non-abelian group of order $|G| = 75$. Prove that G does not contain an element of order 25.

Q5. (10 points) Let p be a prime. Let G be the group of functions

$$f\colon \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z}, \quad f(x) = ax + b$$

where $a, b \in \mathbb{Z}/p^2\mathbb{Z}$ and $a \equiv 1 \bmod p$, with the group operation being composition of functions. Let G' be the *Heisenberg group*, that is, the subgroup of $\mathrm{GL}_3(\mathbb{Z}/p\mathbb{Z})$ consisting of upper triangular matrices with all diagonal entries equal to 1. Prove that G and G' are *not* isomorphic for $p \neq 2$.