Math 611 Homework 3

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Reading: Dummit and Foote, Sections 2.4, 3.3, 4.4, 4.5, 5.1, 5.4, 5.5.

Justify your answers carefully (complete proofs are expected).

(1) (Optional) Let G be a group. The action of G on itself by conjugation induces a homomorphism

$$\varphi\colon G \to \mathrm{Aut}(G), \quad g \mapsto (x \mapsto gxg^{-1}).$$

Show that $\varphi(G) \simeq \mathrm{Aut}(G)/Z(G)$ and $\varphi(G) \lhd \mathrm{Aut}(G)$.

[Remark: We say $\varphi(G) \leq \mathrm{Aut}(G)$ is the group of *inner automorphisms* of G, and $\mathrm{Aut}(G)/\varphi(G)$ is the *outer automorphism group* of G.]

(2) (Optional, but useful below!) Let G be a group such that $|G| = mn$ where $\gcd(m,n) = 1$. Suppose there exists a normal subgroup $H \lhd G$ of order $|H| = m$ and a subgroup $K \leq G$ of order $|K| = n$. Show that G is isomorphic to a semi-direct product of H and K.

(3) Let $n \in \mathbb{N}$.

(a) Show that the map

$$\theta\colon \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R}), \quad \theta(A) = (A^{-1})^T$$

is an automorphism of $\mathrm{GL}_n(\mathbb{R})$. (We use the notation B^T for the transpose of a square matrix B.)

(b) Prove that the automorphism θ is *not* given by conjugation by some element $B \in \mathrm{GL}_n(\mathbb{R})$. That is, there does *not* exist $B \in GL_n(\mathbb{R})$ such that $BAB^{-1} = (A^{-1})^T$ for all $A \in \mathrm{GL}_n(\mathbb{R})$.

(4) Let D_n be the dihedral group of symmetries of the regular n-gon, $n \geq 3$. Show that $\mathrm{Aut}(D_n)$ is isomorphic to the group G of invertible affine linear transformations of $\mathbb{Z}/n\mathbb{Z}$, that is, functions of the form

$$f\colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}, \quad f(x) = cx + d$$

where $c, d \in (\mathbb{Z}/n\mathbb{Z})$ and $\gcd(c, n) = 1$, with the group law being composition of functions. Deduce a description of $\mathrm{Aut}(D_n)$ as a semi-direct product.

(5) Let p be a prime and consider the cyclic subgroup $H = \langle (123 \cdots p) \rangle$ of the symmetric group S_p.

(a) Determine the number of conjugate subgroups of H. Deduce the order of the normalizer $N(H)$ of H in S_p.

(b) Show that the homomorphism

$$\varphi\colon N(H) \to \mathrm{Aut}(H), \quad g \mapsto (h \mapsto ghg^{-1})$$

is surjective with kernel H.

(c) Show that $N(H)$ can be generated by two elements, and describe a set of two generators explicitly for $p = 5$.

(6) Let G be a finite group. Let p be the smallest prime dividing $|G|$. Suppose H is a normal subgroup of G of order p. Show that H is contained in the center of G.

(7) Show that the general linear group $\mathrm{GL}_n(F)$ of invertible $n \times n$ matrices over a field F is a direct product of two non-trivial groups in the following cases.

(a) $F = \mathbb{R}$ and n is odd.

(b) $F = \mathbb{Z}/p\mathbb{Z}$ and $\gcd(n, p-1) = 1$.

(8) Compute the number of Sylow p-subgroups of G in each of the following cases.

(a) $p = 2$ and $G = D_{60}$, the dihedral group of symmetries of a regular 60-gon.

(b) $p = 3$ and $G = S_6$, the symmetric group on 6 objects.

(c) $p = 5$ and $G = \mathrm{GL}_3(\mathbb{Z}/5\mathbb{Z})$, the general linear group of invertible 3×3 matrices over $\mathbb{Z}/5\mathbb{Z}$.

(9) What are the possibilities for the number of elements of order 5 in a group G of order 50? Include examples showing that each case occurs.

(10) Let $G = \mathrm{GL}_n(\mathbb{Z}/p\mathbb{Z})$ and let $H \leq G$ be a subgroup of order a power of p. Prove that there exists $g \in G$ such that ghg^{-1} is upper triangular for all $h \in H$.

(11) Let G be a non-abelian group of order 57. Describe G (a) as a semi-direct product and (b) in terms of generators and relations.

(12) (Optional) Let G be a group of order $|G| = p^a q^b$ where p and q are distinct primes and $a, b \in \mathbb{N}$. Suppose that the order of p in the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^\times$ is greater than a. Show that G is isomorphic to the semi-direct product of two non-trivial groups.

(13) (Optional) Let G be a group of order $|G| = pqr$ where p, q, r are distinct primes. Show that one of the Sylow subgroups of G is normal.

Hints:

4 Use the presentation $D_n = \langle a, b \mid a^n = b^2 = e, \quad ba = a^{-1}b\rangle$. An automorphism θ of D_n is determined by the images of the generators a and b.

5bc What is the kernel of φ? What is the automorphism group of $\mathbb{Z}/p\mathbb{Z}$?

6 Consider the homomorphism $\varphi\colon G \to \operatorname{Aut} H$, $g \mapsto (h \mapsto ghg^{-1})$ given by the action of G on H by conjugation.

7 What is the special linear group $\mathrm{SL}_n(F)$? What is the center of $\mathrm{GL}_n(F)$?

9 What are the possibilities for the number and isomorphism type of Sylow 5-subgroups of G?

10 What are the Sylow theorems?

12 Show that one of the Sylow subgroups is normal.

13 Argue by contradiction and count the elements of prime order.