Math 611 Homework 8

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Reading: Dummit and Foote, 10.4, 10.5.
All rings are assumed commutative with 1.

(1) (This question clears up some confusion that arose in HW6, Q9 and Q10.) Let R be a ring and I and J ideals of R.

(a) Show that R/I and R/J are isomorphic as R-modules iff $I = J$. [Hint: If M is an R-module then the *annihilator* $\operatorname{Ann}(M)$ of M is the ideal of R defined by

$$\operatorname{Ann}(M) = \{r \in R \mid rm = 0 \text{ for all } m \in M\}.]$$

(b) Prove or give a counterexample: R/I and R/J are isomorphic as rings iff $I = J$.

(c) Suppose $I + J = R$. Show that there is an isomorphism of R-modules

$$R/IJ = R/I \cap J \xrightarrow{\sim} R/I \oplus R/J.$$

(That is, the isomorphism of the Chinese remainder theorem, although originally formulated as an isomorphism of rings, is also an isomorphism of R-modules.) More generally, if M is an R-module, then there is an isomorphism

$$M/(IM \cap JM) \xrightarrow{\sim} M/IM \oplus M/JM.$$

(2) Let R be a UFD and $f, g \in R$ nonzero elements such that $\gcd(f, g) = 1$. Let $I = (f, g) \subset R$ denote the ideal generated by f and g. [WARNING:

We do *not* assume that R is a PID so $I \neq R$ in general.] Prove that the sequence of R-modules

$$0 \to R \xrightarrow{\alpha} R^2 \xrightarrow{\beta} I \to 0$$

given by

$$\alpha(a) = (ag, -af)$$

and

$$\beta(a, b) = af + bg$$

is exact.

(3) Let F be a field and $R = F[x, y, z]$. Let $m = (x, y, z) \subset R$ be the maximal ideal generated by x, y, and z. Consider the sequence of R-modules

$$R^3 \xrightarrow{\beta} R^3 \xrightarrow{\gamma} m \to 0$$

where

$$\gamma(a, b, c) = ax + by + cz$$

and the homomorphism β is given by the matrix

$$\begin{pmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{pmatrix}.$$

(a) Show that the sequence is exact.

(b) Determine the kernel of β and use your result to describe an exact sequence

$$0 \to R \xrightarrow{\alpha} R^3 \xrightarrow{\beta} R^3 \xrightarrow{\gamma} m \to 0.$$

(4) Let

$$0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$$

be a short exact sequence of R-modules. We say that the exact sequence is *split* if there is an isomorphism of R-modules

$$\varphi \colon M \xrightarrow{\sim} L \oplus N$$

such that $\varphi \circ \alpha(l) = (l, 0)$ and $\beta \circ \varphi^{-1}(l, n) = n$.

In class we showed that the exact sequence is split iff there exists an R-module homomorphism $s\colon N \to M$ such that $\beta \circ s = \mathrm{id}_N$.

Show that the exact sequence is split iff there exists an R-module homomorphism $r\colon M \to L$ such that $r \circ \alpha = \mathrm{id}_L$.

(5) Consider a commutative diagram of R-modules

$$\begin{array}{ccccccc}
 & & L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \theta_L} & & \downarrow{\scriptstyle \theta_M} & & \downarrow{\scriptstyle \theta_N} & & \\
0 & \longrightarrow & L' & \xrightarrow{\alpha'} & M' & \xrightarrow{\beta'} & N' & &
\end{array}$$

where the rows are exact. The *snake lemma* states that the induced sequence

$$\ker(\theta_L) \xrightarrow{\bar{\alpha}} \ker(\theta_M) \xrightarrow{\bar{\beta}} \ker(\theta_N) \xrightarrow{\delta} \operatorname{coker}(\theta_L) \xrightarrow{\bar{\alpha}'} \operatorname{coker}(\theta_M) \xrightarrow{\bar{\beta}'} \operatorname{coker}(\theta_N)$$

is exact. Here δ is the so called *boundary* or *connecting* homomorphism.

In class we showed that δ is a well-defined R-module homomorphism and checked exactness of the sequence at $\ker(\theta_M)$ and $\operatorname{coker}(\theta_L)$ by "diagram chasing". Complete the proof of the snake lemma by showing exactness at $\ker(\theta_N)$ and $\operatorname{coker}(\theta_M)$.

(6) Consider a commutative diagram of R-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \theta_L} & & \downarrow{\scriptstyle \theta_M} & & \downarrow{\scriptstyle \theta_N} & & \\
0 & \longrightarrow & L' & \xrightarrow{\alpha'} & M' & \xrightarrow{\beta'} & N' & \longrightarrow & 0
\end{array}$$

Show that if any two of θ_L, θ_M, and θ_N are isomorphisms then so is the third.

(7) The following assertions are called the 5-*lemma* and are exercises in diagram chasing. Consider a commutative diagram of R-modules

$$\begin{array}{ccccccccc}
M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 & \longrightarrow & N_5
\end{array}$$

with exact rows.

(a) If f_1 is surjective and f_2 and f_4 are injective then f_3 is injective.

(b) If f_5 is injective and f_2 and f_4 are surjective then f_3 is surjective.

(c) If f_1, f_2, f_4 and f_5 are isomorphisms, then f_3 is an isomorphism.

(8) Show that $e_1 \otimes e_2 + e_2 \otimes e_1 \in \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$ is *not* a pure tensor.

(9) (a) Let $m, n \in \mathbb{N}$. Show that $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \simeq \mathbb{Z}/d\mathbb{Z}$ where $d = \gcd(m, n)$.

(b) More generally, let R be a ring and I and J ideals of R. Show that $R/I \otimes_R R/J \simeq R/(I + J)$.

(10) Let F be a field, $R = F[x, y]$, and $m = (x, y) \subset R$. Compute the tensor product $m \otimes_R (R/m)$.

(11) Let $R = \mathbb{Z}[x]$ and $I = (2, x) \subset R$. Show that $2 \otimes 2 + x \otimes x \in I \otimes_R I$ is *not* a pure tensor.

[Hint: Consider the R-bilinear map $I \times I \to R$, $(a, b) \mapsto ab$.]

(12) Let F be a field, $R = F[x, y]$, and $m = (x, y) \subset R$.

(a) Compute a presentation of the R-module (i) m (ii) $m \otimes_R m$.

(b) Let $t = x \otimes y - y \otimes x \in m \otimes_R m$. Show that $t \neq 0$ and $xt = yt = 0$.