Math 611 Homework 3

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Reading: Dummit and Foote, 5.5,4.5,3.4.
Justify your answers carefully.

(1) Let

$$G = O(2) = \{A \in \mathbb{R}^{2\times 2} \mid AA^T = I\}$$

be the group of 2×2 orthogonal matrices and

$$H = SO(2) = \{A \in O(2) \mid \det A = 1\}$$

the subgroup of matrices with determinant 1. Express G as a semidirect product $G = H \rtimes_\varphi K$ for some subgroup $K \leq G$ and homomorphism $\varphi \colon K \to \mathrm{Aut}(H)$.

(2) Determine the possible numbers of elements of order 11 in a group of order 44.

(3) Describe the Sylow 2-subgroups of the following groups.

(a) D_{10}, the dihedral group of order 20.

(b) T, the group of rotational symmetries of the tetrahedron.

(c) O, the group of rotational symmetries of the octahedron or the cube.

(d) I, the group of rotational symmetries of the icosahedron or the dodecahedron.

(4) Let $G = \mathrm{GL}_n(\mathbb{F}_p)$. Let $U \leq G$ be the subgroup consisting of upper triangular matrices with 1's on the diagonal.

(a) Show that U is a Sylow p-subgroup of G.
(b) Compute the number of Sylow p-subgroups of G.

[Hint for (b). A *complete flag* in $V = \mathbb{F}_p^n$ is a chain of subspaces

$$\{0\} \neq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = V$$

The subgroup U preserves the standard flag given by $V_i = \langle e_1, \ldots, e_i \rangle$. Show that the Sylow p-subgroups are in bijection with the set of complete flags.]

(5) In each of the following cases, show that one of the Sylow subgroups of G is normal.

(a) $|G| = 63$.
(b) $|G| = 56$.
(c) $|G| = pqr$, for distinct primes p,q,r.

(6) Classify groups G of the following orders: (a) 21, (b) 50, (c) 28.

[Hint: Use the Sylow theorems to express the groups as semi-direct products. You should also write the groups in terms of generators and relations and identify them with (direct products of) known groups where possible.]

(7) Let G be a finite group and let $\phi \colon G \to S_G$ be the homomorphism given by the action of G on itself by left multiplication. [Here S_G denotes the symmetric group of permutations of the set G.]

(a) Show that $\phi(g)$ is an odd permutation iff the order $\operatorname{ord}(g)$ of g is even and $|G|/\operatorname{ord}(g)$ is odd.
(b) Suppose $|G| = 2m$ where m is odd. Prove that G contains a normal subgroup of index 2.

(8) (a) Let G be a group. Let H and K be subgroups of G. Show that if $K \leq N(H)$ then

$$HK := \{hk \mid h \in H \text{ and } k \in K\} \subset G$$

is a subgroup of G, $H \subset HK$ is a normal subgroup, and $HK = KH$.

(b) Let G be a group, and H, K_1, K_2 subgroups such that

$$K_1 \lhd K_2 \leq N(H).$$

Show that $HK_1 \lhd HK_2$.

(9) Show that groups of the following orders are solvable: (a) 40, (b) 48, (c) 42.