Math 461 Midterm review problems

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October 10, 2019

(1) Let $\mathcal{C}$ be a circle and P be a point outside $\mathcal{C}$. Let L and M be the two tangent lines to the circle $\mathcal{C}$ that pass through the point P. Let A be the intersection point of L and $\mathcal{C}$ and let B be the intersection point of M and $\mathcal{C}$. Prove that $|PA| = |PB|$.

(2) Let A, B, C, D be points on a circle and suppose that the chords AB and CD meet at a point P inside the circle. Prove that $|AP||BP| = |CP||DP|$.

(3) (a) Let A, B, C be three points and let L be the bisector of the angle $\angle BAC$ (that is, the line through A that divides the angle $\angle BAC$ into two equal parts). Let P be a point on L. Let M be the line through P perpendicular to the line AB, and let Q be the intersection point of M with the line AB. Similarly, let N be the line through P perpendicular to the line AC, and let R be the intersection point of N with the line AC. Prove that $|PQ| = |PR|$.

(b) Let $ABCD$ be a convex quadrilateral and suppose that the bisectors of the angles $\angle ABC$, $\angle BCD$, $\angle CDA$, and $\angle DAB$ of the quadrilateral all meet at a point P. Using part (a) or otherwise, prove that there is a circle $\mathcal{C}$ such that the sides AB, BC, CD, and DA of the quadrilateral are all tangent to $\mathcal{C}$.

[Recall that a polygon is *convex* if all the interior angles are less than π.]

(c) Do the angle bisectors of a convex quadrilateral always meet at a point? Give a proof or a counterexample.

(4) Let ΔABC be a triangle such that $\angle ACB = \pi/2$. Let L be the line through the point C perpendicular to the the line AB. Let D be the

intersection point of L and the line AB. Prove that $|AB| \cdot |BD| = |BC|^2$.

(5) Suppose $\Delta ABC \sim \Delta A'B'C'$, so that $|A'B'|/|AB| = |B'C'|/|BC| = |C'A'|/|CA| = \lambda$ for some $\lambda > 0$. Let h be the perpendicular height from A to BC and h' the perpendicular height from A' to $B'C'$.

(a) Show that $h'/h = \lambda$.

(b) Deduce that $\mathrm{Area}(\Delta A'B'C') = \lambda^2 \, \mathrm{Area}(\Delta ABC)$.

(6) Let ΔABC be a triangle. Let $PQRS$ be a square with vertices P, Q lying on the line BC, vertex R lying on the line segment CA, and vertex S lying on the line segment AB. Given that $|BC| = a$ and the perpendicular height from A to BC equals h, determine the side length of the square $PQRS$.

(7) Let $ABCD$ be a convex quadrilateral. Let L be the line which bisects the angle $\angle BAD$ and M the line which bisects the angle $\angle BCD$. Suppose that L intersects the line segment BC at a point E, M intersects the line segment AD at a point F, and L is parallel to M. Prove that $\angle ABC = \angle ADC$.

(8) Let ΔABC be a triangle. Let $\mathcal{C}$ be the unique circle passing through A, B, and C. Let L be line through C which bisects the external angle of the triangle ΔABC at C. (Here, by the external angle of the triangle ΔABC at C, we mean the angle $\angle ACP$ where P is a point on the line BC on the opposite side of C to B.) Let D be the other intersection point of L with the circle $\mathcal{C}$ (besides C). Prove that $|AD| = |BD|$.

(9) Let ABC be a triangle. Let D be the midpoint of the line segment BC, and let E be a point on the line segment AD such that $|AE|/|ED| = 1/3$. Let L be the line through the point E parallel to the line AC, and let F be the intersection point of L and the line BC. Determine $|BF|/|FC|$.

(10) Let ΔABC and ΔCDE be two equilateral triangles with a common vertex C. Determine the angle between the lines AD and BE.

(11) Let $ABCD$ be a parallelogram of area 1. Let E be a point on the line segment $|BC|$ such that $|BE|/|EC| = 3/2$. Let F be the intersection

point of the lines AE and BD. Determine the area of the quadrilateral $CDFE$.

(12) Suppose given a circle $\mathcal{C}$ with center O and a point A on $\mathcal{C}$. Give a ruler and compass construction of a rectangle $ABCD$ such that the vertices A, B, C, D lie on the circle and the angle between the diagonals is $\pi/6$.

[Note: For this problem and the other ruler and compass problems below, you may use the ruler and compass constructions described in class and in the text book (perpendicular bisector of a line segment, angle bisector, etc.) as components of your construction.]

(13) Suppose given a circle $\mathcal{C}$ with center O and a point P on $\mathcal{C}$. Give a ruler and compass construction of an isosceles right-angled triangle ΔABC such that the sides are tangent to the circle.

(14) Describe a ruler and compass construction in each of the following cases.

(a) Suppose given a line segment AB. Construct a triangle ΔABC with vertices A, B and a third point C such that $\angle ABC = \pi/3$, $\angle BAC = \pi/6$, and $\angle ACB = \pi/2$.

(b) Suppose given a triangle ΔABC. Construct a triangle ΔABD such that $\angle ADB = \angle ACB$ and $\angle ABD = \angle BAD$.

(15) Show that the locus of points in the plane equidistant from a line L and a point P not on L is a parabola, given in appropriate coordinates by the equation $y = ax^2 + b$ for some positive real numbers a, b. (Here the distance from a point Q to a line L is defined as follows: let M be the line through Q perpendicular to L, and R the intersection point of L and M. Then the distance from Q to L equals $|QR|$.)

(16) (Optional) Let A, B be two points and write $c = |AB|$. Let d be a positive real number such that $d > c$. Show that the locus of points P such that $|PA|+|PB| = d$ is an ellipse, given in appropriate coordinates by the equation $x^2/a^2 + y^2/b^2 = 1$ for some positive real numbers a and b.

Hints:

1 What is the angle between a tangent and the radius at a point of $\mathcal{C}$? What is Pythagoras' theorem?

2 What are similar triangles? Review angles in a circle.

3 (a) What is the angle sum of a triangle?

4 What are similar triangles?

6 Use Q5a. Obtain an equation for the side length x of the square using similar triangles, and solve for x.

7 Review angles determined by a transversal line to parallel lines. What is the angle sum of a triangle?

8 What is the isosceles triangle theorem? Review angles in a circle.

9 What is Thales' theorem?

10 Identify some congruent triangles.

11 Use Q5b.

14 Hints:(b) Review angles in a circle.

15 Without loss of generality, we can choose coordinates such that the line L is the x-axis and the point P has coordinates $(0, c)$ for some $b > 0$.

16 Choose coordinates such that $A = (-c/2, 0)$ and $B = (c/2, 0)$, write down the equation $|PA| + |PB| = d$ in coordinates, and simplify (you will need to square both sides, then rearrange and square again to get rid of the remaining square root, and finally simplify to obtain the equation of an ellipse).