Math 461 Homework 7

Paul Hacking

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Justify your answers carefully.

Recall that

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

denotes the sphere with center the origin and radius 1 in $\mathbb{R}^3$.

(1) Recall that a *spherical line* (or *great circle*) $L = \Pi \cap S^2$ is the intersection of a plane $\Pi \subset \mathbb{R}^3$ through the origin with the sphere S^2.

Recall that we say two points $P, Q \in S^2$ are *antipodal* if $\overrightarrow{OP} + \overrightarrow{OQ} = \mathbf{0}$.

Let $P, Q \in S^2$ be two points. Show that there is a unique spherical line through P and Q unless P and Q are antipodal. What happens in the antipodal case?

(2) Recall that, for two points $P, Q \in S^2$, the *spherical distance* $d(P, Q)$ from P to Q is the length of the shorter arc from P to Q of the spherical line passing through P and Q.

Let $P, Q \in S^2$ be two points. Show that $d(P, Q) \leq \pi$ with equality iff P and Q are antipodal.

(3) Let $P = \frac{1}{3}(1, 2, 2) \in S^2$ and $Q = \frac{1}{3}(2, 2, 1) \in S^2$.

(a) Compute the spherical distance from P to Q.

(b) Compute the equation of the spherical line through P and Q.

[Hint for (b): Review the cross product of two vectors from MATH 233.]

(4) Recall that the angle between two spherical lines L and M at a point $P \in L \cap M$ is by definition the angle between the tangent lines to L and M at P. Equivalently, let $\Pi_L, \Pi_M \subset \mathbb{R}^3$ be the planes through the origin such that $L = \Pi_L \cap S^2$ and $M = \Pi_M \cap S^2$. Then the angle between L and M is the dihedral angle between the planes Π_L and Π_M. (The *dihedral angle* between two planes $\Pi_1, \Pi_2 \subset \mathbb{R}^3$ is defined as follows: Let $\Pi \subset \mathbb{R}^3$ be a plane which is perpendicular to the line $\Pi_1 \cap \Pi_2$. Then the dihedral angle between Π_1 and Π_2 is the angle between the lines $l_1 = \Pi_1 \cap \Pi$ and $l_2 = \Pi_2 \cap \Pi$ in the plane Π.)

Let L and M be the spherical lines with equations $x + y + z = 0$ and $x + 2y + 3z = 0$.

(a) Find the intersection points of L and M.

(b) Compute the angle between L and M.

[Hint for (a): Solving two homogeneous linear equations in three variables as in MATH 235 gives solutions $\lambda(a, b, c)$ where $\lambda \in \mathbb{R}$ is arbitrary and $a, b, c \in \mathbb{R}$ are constants. This is a parametric description of the line through the origin in $\mathbb{R}^3$ that is the intersection of the two planes defined by the equations. Now determine the two intersection points of this line with the sphere S^2.]

(5) Let L be a spherical line on S^2 and P a point on S^2 not lying on L.

(a) Show that there is a spherical line M through P and perpendicular to L.

(b) Is the spherical line M uniquely determined by P and L?

(c) Determine the equation of M in the case that $P = \frac{1}{\sqrt{3}}(1, 1, 1)$ and L has equation $2x + 4y + z = 0$.

[Hint for (c): Review the cross product of two vectors from MATH 233.]

(6) Let $P \in S^2$ be a point and $r \in \mathbb{R}$, $0 < r < \pi$. We define the *spherical circle* $C(P, r)$ with center P and radius r by

$$C(P, r) = \{Q \in S^2 \mid d(P, Q) = r\}.$$

(a) Show that the spherical circle $C(P, r)$ is equal to the intersection $\Pi \cap S^2$ of a plane $\Pi \subset \mathbb{R}^3$ (not necessarily passing through the origin) with the sphere S^2. What is the normal vector of the plane Π?

(b) Show that the spherical circle $C(P, r)$ is a Euclidean circle in the plane Π and determine its Euclidean radius. Deduce a formula for the circumference of the spherical circle $C(P, r)$.

(c) Show that the circumference of a spherical circle of radius r is less than the circumference of a Euclidean circle of radius r.

(d) What happens to the circumference of a spherical circle of radius r as r approaches π? Interpret your answer geometrically.

(7) Let $P \in S^2$ be a point and $r \in \mathbb{R}$, $0 < r < \pi$. We define the *spherical disc* $D(P, r)$ with center P and radius r by

$$D(P, r) = \{Q \in S^2 \mid d(P, Q) \leq r\}.$$

(a) Show that the area of a spherical disc of radius r is equal to $2\pi(1 - \cos r)$.

[Hint: Use spherical polar coordinates and the integral formula

$$A = \int \left\| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right\| du dv$$

for the area of a parametrized surface $\mathbf{x} = \mathbf{x}(u, v)$ from MATH 233.]

(b) Show that the area of a spherical disc of radius r is less than the area of a Euclidean disc of radius r.

(c) What happens to the area of a spherical disc of radius r as r approaches π? Interpret your answer geometrically.

(8) (Optional) Let $f(r)$ be the circumference of a Euclidean circle of radius r minus the circumference of a spherical circle of radius r. Let $g(r)$ be the area of a Euclidean disc of radius r minus the area of a spherical disc of radius r. Determine approximations $f(r) \approx cr^k$ and $g(r) \approx dr^l$ for small r, where $c, d \in \mathbb{R}$, $c, d > 0$, and $k, l \in \mathbb{N}$.

[Hint: Use Q6b, Q7a, and the approximations to $\sin r$ and $\cos r$ for small r given by the first few terms in their Taylor expansions about $r = 0$.]