Math 461 Homework 6

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Reading: Stillwell, Sections 3.7, 3.8, and 4.7.

Justify your answers carefully. Complete proofs are expected (as in MATH 300).

(1) Let L be a line in $\mathbb{R}^2$. Recall that Refl_L denotes reflection in the line L, an isometry of $\mathbb{R}^2$.

Let P, Q be two points in $\mathbb{R}^2$ on the same side as L. Suppose we place a mirror on the line L (standing vertically, perpendicular to the plane), with the silvered side pointing towards P and Q. Show that the perceived position of P for an observer at Q looking in the mirror is $\mathrm{Refl}_L(P)$. (In other words, light from the object at P reflected in the mirror travels the same distance to Q and arrives in the same direction as light from an object at $\mathrm{Refl}_L(P)$ when the mirror is removed.)

[Hint: For reflection of light rays in a mirror, the incoming ray makes the same angle with the mirror as the outgoing ray ("angle of incidence equals angle of reflection").]

(2) Give a precise geometric description of each of the following compositions of isometries as a translation, rotation, reflection, or glide reflection.

(a) Reflection in the line L_1 with equation $y = 3$ followed by reflection in the line L_2 with equation $y = x + 1$.

(b) Rotation about the point $(1, 4)$ through angle π counterclockwise followed by rotation about the point $(3, 4)$ through angle $\pi/2$ counterclockwise.

(c) Translation by the vector $(0, 6)$ followed by rotation about the point $(1, 2)$ through angle $\pi/3$.

(3) Let L and M be two distinct lines in the plane. Show that $\mathrm{Refl}_M \circ \mathrm{Refl}_L = \mathrm{Refl}_L \circ \mathrm{Refl}_M$ if and only if L and M are perpendicular.

(4) Recall that $\mathrm{Rot}(P, \theta)$ denotes rotation about a point P through angle θ counterclockwise. Let $ABCD$ be a square such that the vertices A, B, C, D are in counterclockwise order. Determine the composition

$$\mathrm{Rot}(A, \pi/2) \circ \mathrm{Rot}(B, \pi/2) \circ \mathrm{Rot}(C, \pi/2) \circ \mathrm{Rot}(D, \pi/2).$$

Justify your answer carefully.

(5) Let ΔABC be an equilateral triangle with vertices A, B, C in counterclockwise order. Determine the composition

$$\mathrm{Rot}(A, 2\pi/3) \circ \mathrm{Rot}(B, 2\pi/3) \circ \mathrm{Rot}(C, 2\pi/3).$$

(6) Express the isometry $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + 6, 4 - y)$ as a composite of at most 3 reflections.

(7) Let $A = (0, 0), B = (1, 0), C = (0, 2)$, and $A' = (3, 5), B' = (3, 4), C' = (1, 5)$. Find an isometry T sending ΔABC to $\Delta A'B'C'$. (Give a geometric description and an explicit algebraic formula.)

(8) Give an algebraic proof that the composition $\mathrm{Rot}(Q, \phi) \circ \mathrm{Rot}(P, \theta)$ of two rotations is a translation (or the identity) if $\theta + \phi$ is a multiple of 2π and a rotation about another point R through angle $\theta + \phi$ otherwise.

[Hint: Any isometry $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ has an algebraic formula $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ where A is a 2×2 orthogonal matrix and $\mathbf{b} \in \mathbb{R}^2$ is a vector. What is the matrix A if $T = \mathrm{Rot}(P, \theta)$ is rotation about a point P through angle θ counterclockwise? If $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ and $U(\mathbf{x}) = C\mathbf{x} + \mathbf{d}$, what is the algebraic formula for the composition $U \circ T$?]