Math 461 Homework 4

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Justify your answers carefully. Complete proofs are expected (as in MATH 300).

(1) Let ΔABC be a triangle. Let $\mathcal{C}$ be the unique circle passing through A, B, and C (see HW1Q3b). Let L be the tangent line to the circle at the point A. Prove that the angle between the line L and the line segment AC is equal to the angle $\angle ABC$.

[Hint: Review Section 2.7, Angles in a circle. What is the angle in a semicircle?]

(2) (a) Let ΔDEF be a triangle. Show that the perpendicular bisectors of the line segments DE, EF, and FD all meet at a point P.

(b) Let ΔABC be a triangle. Draw the line L through A parallel to BC, the line M through B parallel to CA, and the line N through C parallel to AB. Let D be the intersection point of L and M, let E be the intersection point of M and N, and let F be the intersection point of N and L. Show that $|AD| = |AF|$, $|BD| = |BE|$, and $|CE| = |CF|$.

(c) Let ΔABC be a triangle. Let H be the line through A perpendicular to BC, let I be the line through B perpendicular to AC, and let J be the line through C perpendicular to AB. Using parts (a) and (b) or otherwise, show that the lines H, I, and J all meet at a point.

[Hints: (a) Compare HW1Q3b. (b) Identify some congruent triangles in the diagram. (c) What does this have to do with part (a)?]

(3) Consider a triangle ΔABC with $|AB| = |AC| = 1$ and $\angle BAC = 2\theta$. Let L be the line through A perpendicular to BC and D the intersection point of L and BC. Let M be the line through B perpendicular to AC and E the intersection point of M and AC. Using basic trigonometry ("SOHCAHTOA") applied to this diagram, give a geometric proof of the formulas

$$\sin 2\theta = 2 \sin\theta \cos\theta$$

and

$$\cos 2\theta = 1 - 2(\sin\theta)^2.$$

[Remark: The second formula is equivalent to $\cos 2\theta = (\cos\theta)^2 - (\sin\theta)^2$ using $(\cos\theta)^2 + (\sin\theta)^2 = 1$.]

[Hint: First determine $\angle BAD$ and $\angle CBE$.]

(4) (a) Using the formulas

$$\cos(A+B) = \cos A \cos B - \sin A \sin B,$$

$$\cos 2\theta = (\cos\theta)^2 - (\sin\theta)^2,$$

$$\sin 2\theta = 2\sin\theta\cos\theta,$$

and

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

or otherwise, prove the formula

$$\cos 3\theta = 4(\cos\theta)^3 - 3\cos\theta.$$

[Hint: $\cos(3\theta) = \cos(2\theta + \theta)$.]
[Remark: An alternative approach uses the formula

$$\cos\theta = (e^{i\theta} + e^{-i\theta})/2,$$

which follows from *Euler's formula*

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

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(b) Using part (a) or otherwise, determine a cubic polynomial $p(X) = aX^3 + bX^2 + cX + d$ with integer coefficients a, b, c, d such that $p(\cos(2\pi/9)) = 0$.

(5) (a) Show that the regular 10-gon can be constructed using ruler and compass.

(b) Show that the regular 15-gon can be constructed using ruler and compass.

[Hints: The regular n-gon can be constructed if the angle $2\pi/n$ can be constructed (why?). You may assume (as proved in class) that the angle $2\pi/5$ can be constructed.]

(6) Using the cosine rule or otherwise, prove the *triangle inequality*: If a, b, c are the lengths of the sides of a triangle, then $a < b + c$.

(7) (a) Using Q8 below or otherwise, find a rational solution of the equation $X^3 + 3X - 14 = 0$.

(b) Check *Tartaglia's formula* in this case using your calculator: The solutions of $X^3 + aX + b = 0$ are given by

$$X = \sqrt[3]{-b/2 + \sqrt{D}} - \sqrt[3]{b/2 + \sqrt{D}}$$

where $D = (b/2)^2 + (a/3)^3$.

[Remark: If you prefer to avoid using your calculator, you can check by hand that $(1 + \sqrt{2})^3 = 7 + 5\sqrt{2}$ and $(-1 + \sqrt{2})^3 = -7 + 5\sqrt{2}$ using the binomial theorem $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$.]

(8) (Optional) Let $p(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_2X^2 + a_1X + a_0$ be a polynomial of degree n with integer coefficients $a_n, a_{n-1}, \ldots, a_0$. Suppose α is a rational number (a fraction) such that $p(\alpha) = 0$. Write α in its lowest terms: $\alpha = a/b$, where a and b are integers such that $b > 0$ and the greatest common divisor $\gcd(a, b) = 1$. Show that a divides a_0 and b divides a_n.

[Hint: Recall from MATH 300 that if a, b, c are integers, a divides bc, and $\gcd(a, b) = 1$, then a divides c. Clear denominators in the equation $p(\alpha) = 0$ and use this fact.]

(9) (Optional) For n a positive integer, the *Euler totient function* $\phi(n)$ is defined by

$$\phi(n) = |\{a \in \mathbb{N} \mid 1 \leq a \leq n \text{ and } \gcd(a,n) = 1\}|.$$

That is, $\phi(n)$ equals the number of positive integers a less than or equal to n such that $\gcd(a,n) = 1$.

(a) Show that if p is a prime then $\phi(p) = p - 1$.

(b) Show that if p is a prime and α is a positive integer then $\phi(p^\alpha) = p^{\alpha-1}(p-1)$.

(c) It follows from the *Chinese remainder theorem* that if $\gcd(m,n) = 1$ then $\phi(mn) = \phi(m)\phi(n)$. So, if

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$$

is the (unique) prime factorization of a positive integer n (here $p_1 < p_2 < \ldots < p_r$ are distinct primes and the exponents $\alpha_1, \ldots, \alpha_r$ are positive integers), then

$$\phi(n) = p_1^{\alpha_1 - 1}(p_1 - 1) p_2^{\alpha_2 - 1}(p_2 - 1) \cdots p_r^{\alpha_r - 1}(p_r - 1).$$

Gauss' theorem can be stated as follows: the regular n-gon can be constructed using ruler and compass if and only if $\phi(n)$ is a power of 2. Show that this condition holds if and only if

$$n = 2^\alpha p_1 p_2 \cdots p_s$$

where $p_1 < \ldots < p_s$ are distinct primes such that $p_i - 1$ is a power of 2 for each $i = 1, \ldots, s$.

(d) Suppose p is a prime and $p - 1 = 2^m$ is a power of 2. Show that m is necessarily a power of 2.

[Hint: Suppose for a contradiction that m is not a power of 2. Then we can write m as a product ab of positive integers a, b such that b is odd and $b > 1$. Now substitute $X = 2^a$ in the identity $X^b + 1 = (X+1)(X^{b-1} - X^{b-2} + \cdots + X^2 - X + 1)$ to show that $p = 2^m + 1$ is not prime, a contradiction.]

[Remark: The primes p of the form $p = 2^{(2^k)} + 1$ are called *Fermat primes*. The only known examples are $p = 3, 5, 17, 257, 65537$ corresponding to $k = 0, 1, 2, 3, 4$.]