# Math 461 Final review problems

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(1) Let $P = (1, 2)$ and $Q = (3, 4)$ be two points in the $xy$-plane.

(a) Find the equation of the perpendicular bisector of the line segment $PQ$.

(b) Let $C$ be the circle with center on the $x$-axis passing through $P$ and $Q$. Determine the center, radius, and equation of $C$.

(2) Find an isometry of $\mathbb{R}^2$ that sends the triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 3)$ to the triangle with vertices $(2, 3)$, $(2, 1)$, and $(5, 3)$. Describe the isometry geometrically and by an algebraic formula. Give a precise description of the isometry as a translation, rotation, reflection, or glide reflection.

(3) Let $T$ be an isometry of $\mathbb{R}^2$. What are the possible types of the isometry $T^2 = T \circ T$ ?

(4) Give a precise geometric description for each of the following isometries $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ as a translation, rotation, reflection, or glide reflection.

(a) $T(x, y) = (x + 5, y + 7)$.

(b) $T(x, y) = (1 - y, 3 - x)$.

(c) $T(x, y) = (y + 2, 8 - x)$.

(d) $T(x, y) = \frac{1}{13}(5x + 12y + 4, 12x - 5y - 6)$.

(5) Compute the following compositions. (Give a precise geometric description as a translation, rotation, reflection or glide reflection.)

(a) Reflection in $x = 2$ followed by reflection in $y = x + 3$.

(b) Reflection in $y = -x + 2$ followed by reflection in $y = -x + 4$.

(c) Rotation about the point $(1, 1)$ through angle $\pi/2$ counterclockwise followed by rotation about the point $(2, 2)$ through angle $\pi/2$ counterclockwise.

(d) Rotation about $(3, 2)$ through angle $\pi/2$ counter clockwise followed by translation by $(0, 4)$.

(6) Let $\Delta ABC$ be a triangle in the plane with vertices in counter-clockwise order and angles $a, b, c$. What is the composition of rotation about $B$ through angle $2b$ counterclockwise followed by rotation about $A$ through angle $2a$ counterclockwise?

(7) Let $P = \frac{1}{3}(1, 2, 2) \in S^2$ and $Q = \frac{1}{9}(4, 4, 7) \in S^2$. Find the length of the shortest path on the sphere $S^2$ from $P$ to $Q$.

(8) Let $P = \frac{1}{3}(2, 2, 1) \in S^2$ and $Q = \frac{1}{7}(2, 3, 6) \in S^2$. Find the equation of the spherical line through $P$ and $Q$.

(9) Let $L \subset S^2$ be the spherical line with equation $x + y + 2z = 0$ and $M \subset S^2$ the spherical line with equation $x + 2y + 3z = 0$.

(a) Find the two points of intersection of $L$ and $M$.

(b) Find the angle between $L$ and $M$.

(c) $L$ and $M$ divide the sphere into 4 regions. What are the areas of the regions?

(10) Let $T$ be the spherical triangle on $S^2$ with vertices $A = (1, 0, 0)$, $B = \frac{1}{\sqrt{2}}(0, 1, 1)$, and $C = (0, 0, 1)$.

(a) Compute the equations of the spherical lines given by the sides of $T$.

(b) Compute the angles of $T$. (You may assume that each angle of $T$ is $\leq \pi/2$.)

(c) Deduce the area of $T$.

(11) Let $P = \frac{1}{3}(2, 1, 2) \in S^2$ and let $L \subset S^2$ be the spherical line with equation $x + 2y + 3z = 0$. Find the equation of the spherical line $M$ such that $P \in M$ and $M$ is perpendicular to $L$.

(12) Find the spherical center, spherical radius, and circumference of the spherical circle $C = \Pi \cap S^2$, where $\Pi \subset \mathbb{R}^3$ is the plane with equation $3x + 4y + 5z = 6$.

(13) Let $\Delta ABC$ be a spherical triangle with angles $a$, $b$, and $c$ and opposite side lengths $\alpha$, $\beta$, and $\gamma$. Suppose $a = \pi/2$.

  (a) Show that if $\beta < \pi/2$ then $\alpha > \beta$.
  [Hint: What is the spherical cosine rule?]

  (b) Is it always true that the hypotenuse is the longest side of a right angled spherical triangle? Give a proof or a counterexample.

(14) Let $\Delta ABC$ be a spherical triangle with angles $a$, $b$, and $c$ and opposite side lengths $\alpha$, $\beta$, and $\gamma$. Show that if $\alpha = \beta$ then $a = b$.

[Hint: What is the spherical cosine rule?]

(15) (a) Suppose the sphere $S^2$ is tiled by congruent spherical triangles with angles $\pi/3$, $\pi/3$ and $2\pi/5$. How many triangles are there in the tiling?

  (b) Suppose the sphere $S^2$ is tiled by 20 congruent spherical equilateral triangles. What are the angles of the triangles?

(16) Let $C \subset S^2$ be the spherical circle defined by the plane $x = 1/2$.

  (a) Determine the image $F(C)$ of $C$ under stereographic projection (find the equation of $F(C)$ and describe it geometrically).

  (b) In class we showed that stereographic projection preserves angles. That is, given two curves on the sphere meeting at a point $P$, the angle between the curves at $P$ is equal to the angle between their images under stereographic projection at the point $F(P)$. Check this for property for the equator and the curve $C$ from part (a).

[Hint: Recall the stereographic projection $F\colon S^2 \setminus \{N\} \to \mathbb{R}^2$ is given by the algebraic formula

$$F(x, y, z) = \frac{1}{1-z}(x, y)$$

and its inverse $F^{-1}\colon \mathbb{R}^2 \to S^2 \setminus \{N\}$ is given by

$$F^{-1}(u,v) = \frac{1}{u^2+v^2+1}(2u, 2v, u^2+v^2-1).$$

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(17) Let $C \subset S^2$ be the spherical circle defined by the plane $x+y+z=1$. What is the image of $C$ under stereographic projection?

(18) In class we showed that spherical circles correspond to circles and lines in the plane under stereographic projection. Which circles and lines in the plane correspond to great circles on the sphere?

(19) Let $D = \{(u,v) \mid u^2+v^2 < 1\}$ be the region inside the circle with center the origin and radius 1 in $\mathbb{R}^2$. Let $T\colon S^2 \to S^2$ be the transformation given by rotation about the $x$-axis through angle $\pi/2$ counterclockwise (as viewed from $(1,0,0)$ looking towards the origin). Using the geometric description of the stereographic projection $F$ or otherwise, determine the image of $D$ under the composition $G = F \circ T \circ F^{-1}$.

(20) In class we showed that we can compute the length of a curve on the sphere in terms of its image under stereographic projection as follows. Suppose $\gamma$ is a curve on the sphere and $F(\gamma)$ is its image in the plane under stereographic projection, with parametrization $(u(t), v(t))$ for $a \le t \le b$. Then the length of $\gamma$ is given by

$$\text{length}(\gamma) = \int_a^b \frac{2}{u^2+v^2+1}\sqrt{(u')^2+(v')^2}dt$$

where we write $u' = \frac{du}{dt}$ and $v' = \frac{dv}{dt}$, while the (Euclidean) length of $F(\gamma)$ is given by

$$\text{length}(F(\gamma)) = \int_a^b \sqrt{(u')^2+(v')^2}dt$$

as in MATH 233. Here the factor $\frac{2}{u^2+v^2+1}$ in the first integrand encodes the distortion of distances that occurs under stereographic projection. Check the above formula for the length of $\gamma$ in the case that $\gamma$ is the portion of the spherical line connecting the south pole $(0,0,-1)$ and the point $(1,0,0)$.