Geometry vs Algebra. An excerpt from Mathematics in the 20th century

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Let me try to explain my own view of the difference between geometry and algebra. Geometry is, of course, about space, of that there is no question. If I look out at the audience in this room I can see a lot; in one single second or microsecond. I can take in a vast amount of information, and that is of course not an accident. Our brains have been constructed in such a way that they are extremely concerned with vision. Vision, I understand from friends who work in neurophysiology, uses up something like 80 or 90 percent of the cortex of the brain. There are about 17 different centres in the brain, each of which is specialised in a different part of the process of vision: some parts are concerned with vertical, some parts with horizontal, some parts with colour, or perspective, and finally some parts are concerned with meaning and interpretation. Understanding, and making sense of, the world that we see is a very important part of our evolution. Therefore, spatial intuition or spatial perception is an enormously powerful tool, and that is why geometry is actually such a powerful part of mathematics — not only for things that are obviously geometrical, but even for things that are not. We try to put them into geometrical form because that enables us to use our intuition. Our intuition is our most powerful tool. That is quite clear if you try to explain a piece of mathematics to a student or a colleague. You have a long difficult argument, and finally the student understands. What does the student say? The student says, I see! Seeing is synonymous with understanding, and we use the word perception to mean both things as well. At least this is true of the English language. It would be interesting to compare this with other languages. I think it is very fundamental that the human mind has evolved with this enormous capacity to absorb a vast amount of information, by instantaneous visual action, and mathematics takes that and perfects it.

Algebra, on the other hand (and you may not have thought about it like this), is concerned essentially with time. Whatever kind of algebra you are doing, a sequence of operations is performed one after the other and one after the other means you have got to have time. In a static universe you cannot imagine algebra, but geometry is essentially static. I can just sit here and see, and nothing may change, but I can still see. Algebra, however, is concerned with time, because you have operations which are performed sequentially and, when I say algebra, I do not just mean modern algebra. Any algorithm, any process for calculation, is a sequence of steps performed one after the other; the modern computer makes that quite clear. The modern computer takes its information in a stream of zeros and ones, and it gives the answer.

Algebra is concerned with manipulation in time and geometry is concerned with space. These are two orthogonal aspects of the world, and they represent two different points of view in mathematics. Thus the argument or dialogue between mathematicians in the past about the relative importance of geometry and algebra represents something very, very fundamental.

Of course it does not pay to think of this as an argument in which one side loses and the other side wins. I like to think of this in the form of an analogy: Should you just be an algebraist or a geometer? is like saying Would you rather be deaf or blind? If you are blind, you do not see space: if you are deaf, you do not hear, and hearing takes place in time. On the whole, we prefer to have both faculties.

In physics, there is an analogous, roughly parallel, division between the concepts of physics and the experiments. Physics has two parts to it: theory — concepts, ideas, words, laws — and experimental apparatus. I think that concepts are in some broad sense geometrical, since they are concerned with things taking place in the real world. An experiment, on the other hand, is more like an algebraic computation. You do something in time; you measure some numbers; you insert them into formulae, but the basic concepts behind the experiments are a part of the geometrical tradition.

One way to put the dichotomy in a more philosophical or literary framework is to say that algebra is to the geometer what you might call the Faustian offer. As you know, Faust in Goethe's story was offered whatever he wanted (in his case the love of a beautiful woman), by the devil, in return for selling his soul. Algebra is the offer made by the devil to the mathematician. The devil says: I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine. (Nowadays you can think of it as a computer!) Of course we like to have things both ways; we would probably cheat on the devil, pretend we are selling our soul, and not give it away. Nevertheless, the danger to our soul is there, because when you pass over into algebraic calculation, essentially you stop thinking; you stop thinking geometrically, you stop thinking about the meaning.

I am a bit hard on the algebraists here, but fundamentally the purpose of algebra always was to produce a formula which one could put into a machine, turn a handle and get the answer. You took something that had a meaning; you converted it into a formula, and you got out the answer. In that process you do not need to think any more about what the different stages in the algebra correspond to in the geometry. You lose the insights, and this can be important at different stages. You must not give up the insight altogether! You might want to come back to it later on. That is what I mean by the Faustian offer. I am sure it is provocative.

This choice between geometry and algebra has led to hybrids which confuse the two, and the division between algebra and geometry is not as straightforward and nave as I just said. For example, algebraists frequently will use diagrams. What is a diagram except a concession to geometrical intuition?