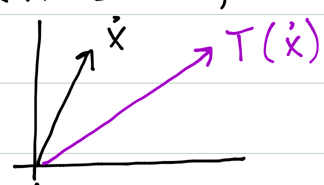
1) $\vec{v}$ is an eigenvec. of A w/ eigenvalue λ if
$A\vec{v} = \lambda\vec{v}$ (or $T(\vec{v}) = \lambda\vec{v}$)

2) a) If $\vec{v}$ $\parallel$ to the line, $T(\vec{v}) = \vec{v}$, so 1 is an eigenvalue w/ eigenspace $E_1 = \text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$

If $\vec{v} \perp$ to the line, $T(\vec{v}) = -\vec{v}$, so -1 is an eigenval w/ eigenspace $E_{-1} = sp\left\{\begin{bmatrix}2\\-1\end{bmatrix}\right\}$

b) If $\vec{v} \parallel$ to the line, $T(\vec{v}) = \vec{v}$, so 1 is eigenval with $E_1 = sp\left\{\begin{bmatrix}1\\3\end{bmatrix}\right\}$. If $\vec{v} \perp$ to the line, $T(\vec{v}) = \vec{0} = 0\cdot\vec{v}$, so 0 is an eigenvalue, $E_0 = \left\{\begin{bmatrix}3\\-1\end{bmatrix}\right\}$

c) $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x+y\\y\end{bmatrix}$; geometrically :



so any vector $\parallel$ to x-axis will get sent to itself ; Therefore, 1 is an eigenvalue, $E_1 = sp\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$
no other eigenvalues.

4) a) $\lambda = 2$, $E_2 = \ker\begin{pmatrix}0 & 1\\0 & 1\end{pmatrix} = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$

$\lambda = 3$, $E_3 = \ker\begin{bmatrix}-1 & 1\\0 & 0\end{bmatrix} = sp\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$

b) char. eq: $\lambda^2 - 5\lambda + 4 = 0 \to (\lambda - 4)(\lambda - 1) = 0$

$\lambda = 4$, $E_4 = \ker\begin{bmatrix}-2 & 1\\2 & -1\end{bmatrix} = \text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$

$\lambda = 1$, $E_1 = \ker\begin{bmatrix}1 & 1\\2 & 2\end{bmatrix} = \text{span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$

c) $\lambda^2 - 4\lambda + 4 = 0 \to (\lambda - 2)(\lambda - 2) = 0$

$\lambda = 2$, $E_2 = \ker\begin{bmatrix}-1 & 1\\-1 & 1\end{bmatrix} = \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$

d) $\det\begin{bmatrix}1-\lambda & 1 & 1\\ 0 & 1-\lambda & 0\\ 1 & 1 & 1-\lambda\end{bmatrix} = (1-\lambda)(1-\lambda)^2 - 1(0) + 1(-(1-\lambda))$

$= (1-\lambda)(1-2\lambda+\lambda^2-1)$

$= (1-\lambda)\cdot\lambda\cdot(\lambda-2) = 0$

$\lambda=1,\ E_1 = \mathrm{Ker}\begin{bmatrix}0 & 1 & 1\\ 0 & 0 & 0\\ 1 & 1 & 0\end{bmatrix} = \mathrm{span}\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$

$\lambda=0,\ E_0 = \mathrm{Ker}\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 1\end{bmatrix} = \mathrm{span}\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$

$\lambda=2,\ E_2 = \mathrm{Ker}\begin{bmatrix}-1 & 1 & 1\\ 0 & -1 & 0\\ 1 & 1 & -1\end{bmatrix} = \mathrm{span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$

e) $\det\begin{bmatrix}1-\lambda & 1 & 1\\ -1 & -\lambda & 0\\ 1 & 1 & 1-\lambda\end{bmatrix} = (1-\lambda)(-\lambda(1-\lambda)) - 1(-(1-\lambda)) + 1(-1+\lambda)$

$= (1-\lambda)(-\lambda+\lambda^2+1-1)$

$= \lambda(1-\lambda)^2 = 0$

$\lambda=0,\ E_0 = \mathrm{Ker}\begin{bmatrix}1 & 1 & 1\\ -1 & 0 & 0\\ 1 & 1 & 1\end{bmatrix} = \mathrm{sp}\left\{\begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$

$\lambda=1,\ E_1 = \mathrm{Ker}\begin{bmatrix}0 & 1 & 1\\ -1 & -1 & 0\\ 1 & 1 & 0\end{bmatrix} = \mathrm{sp}\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$

f) $\det\begin{bmatrix}3-\lambda & 1 & 0\\ 1 & 3-\lambda & 0\\ 0 & 0 & 4-\lambda\end{bmatrix} = (3-\lambda)(3-\lambda)(4-\lambda) - 1(4-\lambda) + 0$

$= (4-\lambda)(9-6\lambda+\lambda^2-1)$

$= (4-\lambda)(4-\lambda)(2-\lambda)$

$\lambda=4,\ E_4 = \mathrm{Ker}\begin{bmatrix}-1 & 1 & 0\\ 1 & -1 & 0\\ 0 & 0 & 0\end{bmatrix} = \mathrm{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

$\lambda=2,\ E_2 = \mathrm{Ker}\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 2\end{bmatrix} = \mathrm{sp}\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}\right\}$

5) If $\vec{v}$ is $\parallel$ to L, rotation about L will not change $\vec{v}$, so $T(\vec{v})=\vec{v}$. Therefore, 1 is an eigenvalue & the line will be E_1

$$E_1 = \ker\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{bmatrix} = \operatorname{sp}\left\{\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}\right\}$$

$\Rightarrow$ L is line $\parallel$ to $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

7) a) Yes, $\mathcal{B}=\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$, $B=\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

b) Yes, $\mathcal{B}=\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$, $B=\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ c) not diagonalizable

d) yes, $\mathcal{B}=\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$, $B=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

e) not diag'ble. f) yes, $\mathcal{B}=\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right\}$, $B=\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

8) Eigenvalues of A will be 2 and b. If $b \neq 2$, A will always be diag'ble, since it has distinct eigenvalues.

If $b=2$, need geometric mult ($\dim E_2$) to be 2. $E_2 = \ker\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$, will be 1-dim'l unless $a=0$.

so A will be diagonalizable if ① $b=2, a=0$ or ② $b\neq 2$, $a\in\mathbb{R}$

10) a) $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}^k = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}^k = \begin{bmatrix} 2^k & 3^k-2^k \\ 0 & 3^k \end{bmatrix}$$

b) $\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}^K = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 4^K & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} \cdot -\frac{1}{3}$

11) a) First, find orthogonal basis:

$$\vec{w}_1 = \vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \quad \vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1} \vec{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} - \frac{9}{9} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Now, normalize: $\vec{u}_1 = \frac{1}{\|\vec{w}_1\|} \vec{w}_1 = \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \quad \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$\mathcal{C} = \left\{ \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

b) $T(\vec{x}) = \left(\vec{x} \cdot \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix} \right) \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix} + \left(\vec{x} \cdot \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right) \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix} + 0 = \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

12) $\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 3 \\ 1 \\ 3 \\ 1 \end{bmatrix} - \frac{8}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

$$\vec{u}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 3 \\ 5 \\ 1 \\ 3 \end{bmatrix} \right) = \frac{1}{2}(3+5+1+3) \cdot \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2}(3-5+1-3)\frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 \\ 4 \\ 2 \\ 4 \end{bmatrix}$$

13) Check orthogonal: $\vec{u}_1 \cdot \vec{u}_2 = \frac{1}{81}(-28-4+32) = 0$

$$\vec{u}_1 \cdot \vec{u}_3 = \frac{1}{81}(16-8-8) = 0$$

$$\vec{u}_2 \cdot \vec{u}_3 = \frac{1}{81}(-28+32-4) = 0$$

check unit: $\vec{u}_1 \cdot \vec{u}_1 = \frac{1}{81}(16+1+64) = 1$

$\vec{u}_2 \cdot \vec{u}_2 = \frac{1}{81}(49+16+16) = 1$

$\vec{u}_3 \cdot \vec{u}_3 = \frac{1}{81}(16+64+1) = 1$

all orthonormal vectors are indep, so $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ form a basis of $\mathbb{R}^3$.

b) $\vec{v} = (\vec{v}\cdot\vec{u}_1)\vec{u}_1 + (\vec{v}\cdot\vec{u}_2)\vec{u}_2 + (\vec{v}\cdot\vec{u}_3)\vec{u}_3$, so

$$[v]_B = \begin{bmatrix} v\cdot u_1 \\ v\cdot u_2 \\ v\cdot u_3 \end{bmatrix} = \begin{bmatrix} -8/9 \\ -4/9 \\ 1/9 \end{bmatrix}$$

14) $$\left[\begin{array}{ccccc|c} 1 & -1 & 1 & 0 & 2 & 1 \\ 2 & -1 & 4 & 1 & 3 & 3 \\ -1 & 3 & 3 & 5 & -1 & 7 \end{array}\right] \begin{array}{l} \\ -2I \\ +I \end{array} \rightarrow \left[\begin{array}{ccccc|c} 1 & -1 & 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 1 & -1 & 1 \\ 0 & 2 & 4 & 5 & 1 & 8 \end{array}\right] \begin{array}{l} +II \\ \\ -2II \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 3 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 & -1 & 1 \\ 0 & 0 & 0 & 3 & 3 & 6 \end{array}\right] \begin{array}{l} -III \\ -III \\ \div 2 \end{array} \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & -2 & -1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{array}\right]$$

$x_1 = -3s$

$x_2 = -1 - 2s + 2t$

$x_3 = s$

$x_4 = 2 - t$

$x_5 = t$

$$\vec{x} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} t$$

$s, t \in \mathbb{R}$

15) T is linear if for any $\vec{v}_1, \vec{v}_2 \in V$ and $k \in \mathbb{R}$, $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$ and $T(k\vec{v}_1) = kT(\vec{v}_1)$.

$T(0) = 0$ for any lin. trans.

16) W is a subspace if $\vec{0} \in W$, and if for any $\vec{v}_1, \vec{v}_2 \in W$ and $k \in \mathbb{R}$, $(\vec{v}_1 + \vec{v}_2) \in W$ and $(k\vec{v}_1) \in W$

Let $W = E_\lambda$. $\vec{0} \in W$ since $T(\vec{0}) = \vec{0} = \lambda \cdot \vec{0}$

Let $\vec{v}_1$ and $\vec{v}_2$ be 2 elements of W. Then $T(v_1) = \lambda v_1$, $T(v_2) = \lambda v_2$.

Consider $T(v_1+v_2) = T(v_1) + T(v_2) = \lambda v_1 + \lambda v_2 = \lambda(v_1+v_2) \Rightarrow (v_1+v_2)$ is an eigenvector w/ eigenval $\lambda \Rightarrow (v_1+v_2) \in W$

Let $k \in \mathbb{R}$. $T(k\vec{v}) = kT(\vec{v}) = k\lambda\vec{v} = \lambda(k\vec{v})$ so $k\vec{v} \in W$.

Thus, W is a subspace.

17) R-N Theorem says if $T: \mathbb{R}^n \to \mathbb{R}^m$, $\text{rank}(T) + \text{nullity}(T) = n$

If $n > m$, since $\text{rank}(T) \leq m$, $\text{nullity}(T) \geq n - m$

18) $\mathcal{B}$-matrix of T is the matrix B such that $B[x]_\mathcal{B} = [T(x)]_\mathcal{B}$

$$B = \begin{bmatrix} | & | & & | \\ [T(b_1)]_\mathcal{B} & [T(b_2)]_\mathcal{B} & \cdots & [T(b_n)]_\mathcal{B} \\ | & | & & | \end{bmatrix}, \quad \mathcal{B} = \{b_1, \dots, b_n\}$$

a) $T: P_2 \to P_2$, $T(f) = f + f' + f''$ $\quad \mathcal{B} = \{x^2, x, 1\}$

$T(x^2) = x^2 + 2x + 2$

$T(x) = x + 1$

$T(1) = 1$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

T is an isomorphism b/c B is invertible

b) $T: \mathbb{R}^{2\times2} \to \mathbb{R}^{2\times2}$, $T(X) = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} X + X \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

If $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $T(X) = \begin{bmatrix} a & b \\ 2a+3c & 2b+3d \end{bmatrix} + \begin{bmatrix} a+b & b \\ c+d & d \end{bmatrix}$

$$= \begin{bmatrix} 2a+b & 2b \\ 2a+4c+d & 2b+4d \end{bmatrix}$$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix}$

$$B = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 4 & 1 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

Yes, isomorphism. Can see from formula for $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ that $\ker(T) = \{\vec{0}\}$.