235 Final exam review questions

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(1) Let A be an $n \times n$ matrix and $T\colon \mathbb{R}^n \to \mathbb{R}^n$, $T(\mathbf{x}) = A\mathbf{x}$ the linear transformation with matrix A. What does it mean to say that a vector $\mathbf{v} \in \mathbb{R}^n$ is an eigenvector of A (or T) with eigenvalue λ?

(2) Arguing geometrically, describe the eigenvalues and eigenvectors of the following linear transformations.

(a) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by reflection in the line $y = 2x$.

(b) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by orthogonal projection onto the line $y = 3x$.

(c) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ the horizontal shear given by $T(\mathbf{x}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{x}$.

(3) Let A be an $n \times n$ matrix. Here is the strategy to find the eigenvalues and eigenvectors of A:

(a) Solve the characteristic equation $\det(A - \lambda I) = 0$ to find the eigenvalues.

(b) For each eigenvalue λ solve the equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$ to find the eigenvectors $\mathbf{v}$ with eigenvalue λ.

[Why does this work? The equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$ is obtained from the equation $A\mathbf{v} = \lambda\mathbf{v}$ by rearranging the terms. This equation has a nonzero solution $\mathbf{v} \in \mathbb{R}^n$ exactly when $(A - \lambda I)$ is *not* invertible, equivalently $\det(A - \lambda I) = 0$.]

The function $\det(A-\lambda I)$ is a polynomial of degree n in the variable λ. In particular if $n = 2$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$\det(A - \lambda I) = \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$= (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad - bc)$$

and we can solve the characteristic equation using the quadratic formula. If $n = 3$ we can determine the polynomial $\det(A - \lambda I)$ by computing the determinant using either Sarrus' rule or expansion along a row or column.

(4) For each of the following matrices, find all the eigenvalues and eigenvectors.

(a) $\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

(5) Let

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

The linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$, $T(\mathbf{x}) = A\mathbf{x}$ is given by rotation about a line L through some angle θ. Find the line L.

[Hint: A vector $\mathbf{v}$ in the direction of L is an eigenvector of A (why?). What is the corresponding eigenvalue?]

(6) Let A be an $n \times n$ matrix. We say A is *diagonalizable* if there is a basis $\mathcal{B}$ of $\mathbb{R}^n$ consisting of eigenvectors of A. In this case, let $\mathcal{B} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be the basis of eigenvectors, with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then the $\mathcal{B}$-matrix of the transformation $T(\mathbf{x}) = A\mathbf{x}$ is the diagonal matrix D with diagonal entries the eigenvalues $\lambda_1, \ldots, \lambda_n$ (why?). Equivalently, writing S for the matrix with columns the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, we have

$$A = SDS^{-1}.$$

We can determine whether A is diagonalizable as follows: for each eigenvalue λ, find a basis of the *eigenspace* $E_\lambda = \ker(A - \lambda I)$ (the subspace of $\mathbb{R}^n$ consisting of all the eigenvectors with eigenvalue λ together with the zero vector). Now combine the bases of all the eigenspaces. These vectors are linearly independent. If there are n vectors, then they form a basis $\mathcal{B}$ of $\mathbb{R}^n$ and A is diagonalizable, otherwise A is not diagonalizable.

(7) For each of the matrices A of Q4, determine whether A is diagonalizable. If A is diagonalizable identify a basis $\mathcal{B}$ of $\mathbb{R}^n$ consisting of eigenvectors of A and write down the $\mathcal{B}$-matrix of the linear transformation $T(\mathbf{x}) = A\mathbf{x}$.

(8) For which values of a and b is the matrix $A = \begin{pmatrix} 2 & a \\ 0 & b \end{pmatrix}$ diagonalizable?

(9) If A is diagonalizable we can compute an explicit formula for powers of A as follows: Write $A = SDS^{-1}$ as above where D is the diagonal matrix with diagonal entries the eigenvalues $\lambda_1, \ldots, \lambda_n$. Then for any positive integer k we have

$$A^k = SD^kS^{-1}$$

(why?) and D^k is the diagonal matrix with diagonal entries $\lambda_1^k, \ldots, \lambda_n^k$.

(10) For the matrices A of Q4(a) and (b) compute a formula for A^k.

(11) Let $W \subset \mathbb{R}^3$ be the subspace with basis $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2)$ where

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}.$$

(a) Using the Gram-Schmidt process, find an orthonormal basis $\mathcal{C} = (\mathbf{u}_1, \mathbf{u}_2)$ for W.

(b) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by orthogonal projection onto W. Write down a formula for $T(\mathbf{x})$ in terms of $\mathbf{u}_1$ and $\mathbf{u}_2$, and use it to compute $T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(12) Let $W \subset \mathbb{R}^4$ be the subspace with basis $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2)$ where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 3 \\ 1 \end{pmatrix}.$$

(a) Using the Gram-Schmidt process, find an orthonormal basis $\mathcal{C} = (\mathbf{u}_1, \mathbf{u}_2)$ of W.

(b) Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be orthogonal projection onto W. Compute $T\begin{pmatrix} 3 \\ 5 \\ 1 \\ 3 \end{pmatrix}$.

(13) Let

$$\mathbf{u}_1 = \frac{1}{9}\begin{pmatrix} 4 \\ -1 \\ -8 \end{pmatrix}, \mathbf{u}_2 = \frac{1}{9}\begin{pmatrix} -7 \\ 4 \\ -4 \end{pmatrix}, \mathbf{u}_3 = \frac{1}{9}\begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix}.$$

(a) Show that $\mathcal{B} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ is an orthonormal basis of $\mathbb{R}^3$.

(b) Let $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Using part (a) or otherwise, compute the $\mathcal{B}$-coordinate vector $[\mathbf{v}]_{\mathcal{B}}$ of $\mathbf{v}$.

(14) Find all solutions of the following system of linear equations. Write your answer as a linear combination of vectors in $\mathbb{R}^5$.

$$\begin{array}{rcrcrcrcrcl} x_1 & - & x_2 & + & x_3 & & & + & 2x_5 & = & 1 \\ 2x_1 & - & x_2 & + & 4x_3 & + & x_4 & + & 3x_5 & = & 3 \\ -x_1 & + & 3x_2 & + & 3x_3 & + & 5x_4 & - & x_5 & = & 7 \end{array}$$

(15) Let V be a linear space and $T\colon V \to V$ a function (or transformation) from V to V. What does it mean to say that T is linear? (There are two conditions that must be satisfied.) If T is linear what is $T(0)$?

(16) What does it mean to say that a subset $W \subset \mathbb{R}^n$ is a subspace? (There are 3 conditions that must be satisfied.) If $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and $\lambda \in \mathbb{R}$, let W be the subset of $\mathbb{R}^n$ consisting of all the vectors $\mathbf{v}$ such that $T(\mathbf{v}) = \lambda\mathbf{v}$. Show that W is a subspace of $\mathbb{R}^n$. [Remark: The subspace W is the *eigenspace* E_λ consisting of all the eigenvectors of T with eigenvalue λ together with the zero vector.]

(17) What is the rank-nullity theorem? If $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, what can you say about the kernel of T if $n > m$?

(18) Let V be a linear space and $\mathcal{B}$ a basis of V. Let $T\colon V \to V$ be a linear transformation. What is the $\mathcal{B}$-matrix of T and how can it be computed? In each of the following examples, write down a basis $\mathcal{B}$ of V, compute the $\mathcal{B}$-matrix of T, and determine whether T is an isomorphism.

(a) $V = \mathcal{P}_2$, the linear space of polynomials $f(x)$ of degree ≤ 2, and $T\colon V \to V$, $T(f(x)) = f(x) + f'(x) + f''(x)$.

(b) $V = \mathbb{R}^{2\times 2}$, the linear space of 2×2 matrices, and $T\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$, $T(X) = AX + XB$ where $A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.