Factorization algebras in quantum field theory
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Contents

CHAPTER 1

Overview

In this chapter, we give an overview of the contents of the book.

1.1. Classical field theory and factorization algebras

The main aim of this book is to present a deformation-quantization approach to quantum field theory. In this section we will outline how a classical field theory gives rise to the classical algebraic structure we consider.

We use the Lagrangian formulation throughout. Thus, classical field theory means the study of the critical locus of an action functional. In fact, we use the language of derived geometry, in which it becomes clear that functions on a derived critical locus (section 5.1) should form a P_0 algebra (section 2.3), that is, a commutative algebra with a Poisson bracket of cohomological degree 1. (For an overview of these ideas, see the section 1.4.)

Applying these ideas to infinite-dimensional spaces, such as the space of smooth functions on a manifold, one runs into analytic problems. Although there is no difficulty in constructing a commutative algebra Obs^{cl} of classical observabes, we find that the Poisson bracket on Obs^{cl} is not always well-defined. However, we show the following.

1.1.0.1 Theorem. *For a classical field theory (section 5.4) on a manifold M, there is a sub-commutative factorization algebra $\widetilde{\mathrm{Obs}}^{c}l$ of the commutative factorization algebra Obs^{cl} on which the Poisson bracket is defined, so that $\widetilde{\mathrm{Obs}}^{cl}$ forms a P_0 factorization algebra. Further, the inclusion $\widetilde{\mathrm{Obs}}^{cl} \to \mathrm{Obs}^{cl}$ is a quasi-isomorphism of factorization algebras.*

Remark: Our approach to field theory involves both cochain complexes of infinite-dimensional vector spaces and families over manifolds (and dg manifolds). The infinite-dimensional vector spaces that appear are of the type studied in functional analysis: for example, spaces of smooth functions and of distributions. One approach to working with such vector spaces is to treat them as topological vector spaces. In this book, we will instead treat them as *differentiable* vector spaces. In particular, Obs^{cl} will be a factorization algebra valued in differentiable vector spaces. For a careful discussion of differential vector spaces, see Appendix **??**. The basic idea is as follows: a differentiable vector space is a vector space V with a smooth structure, meaning that we have a well-defined set of smooth maps from any manifold X into V; and further, we have enough structure to be able to differentiate any smooth map into V. These notions make it possible to efficiently study cochain complexes of vector spaces in families over manifolds.

1.1.1. A gloss of the main ideas. In the rest of this section, we will outline why one would expect that classical observables should form a P_0 algebra. More details are available in section 3.

The idea of the construction is very simple: if $U \subset M$ is an open subset, we will let $\mathcal{EL}(U)$ be the derived space of solutions to the Euler-Lagrange equation on U. Since we are dealing with perturbative field theory, we are interested in those solutions to the equations of motion which are infinitely close to a given solution.

The differential graded algebra $\mathrm{Obs}^{cl}(U)$ is defined to be the space of functions on $\mathcal{EL}(U)$. (Since $\mathcal{EL}(U)$ is an infinite dimensional space, it takes some work to define $\mathrm{Obs}^{cl}(U)$. Details will be presented later (Chapter **??**).

On a compact manifold M, the solutions to the Euler-Lagrange equations are the critical points of the action functional. If we work on an open subset $U \subset M$, this is no longer strictly true, because the integral of the action functional over U is not defined. However, fields on U have a natural foliation, where tangent vectors lying in the leaves of the foliation correspond to variations $\phi \to \phi + \delta\phi$, where $\delta\phi$ has compact support. In this case, the Euler-Lagrange equations are the critical points of a closed one-form $\mathrm{d}S$ defined along the leaves of this foliation.

Any derived scheme which arises as the derived critical locus (section 5.1) of a function acquires an extra structure: it's ring of functions is equipped with the structure of a P_0 algebra. The same holds for a derived scheme arising as the derived critical locus of a closed one-form define along some foliation. Thus, we would expect that $\mathrm{Obs}^{cl}(U)$ is equipped with a natural structure of P_0 algebra; and that, more generally, the commutative factorization algebra Obs^{cl} should be equipped with the structure of P_0 factorization algebra.

1.2. Quantum field theory and factorization algebras

Another aim of the book is to relate perturbative quantum field theory, as developed in [**Cos11b**], to factorization algebras. We give a natural definition of an *observable* of a quantum field theory, which leads to the following theorem.

1.2.0.1 Theorem. *For a classical field theory (section 5.4) and a choice of BV quantization (section 8.2), the quantum observables* Obs^q *form a factorization algebra over the ring* $\mathbb{R}[[\hbar]]$*. Moreover, the factorization algebra of classical observables* Obs^{cl} *is homotopy equivalent to* $\mathrm{Obs}^q \bmod \hbar$ *as a factorization algebra.*

Thus, the quantum observables form a factorization algebra and, in a very weak sense, are related to the classical observables. The quantization theorems will sharpen the relationship between classical and quantum observables.

The main result of [**Cos11b**] allows one to construct perturbative quantum field theories, term by term in $\hbar$, using cohomological methods. This theorem therefore gives a general method to quantize the factorization algebra associated to classical field theory.

1.3. The quantization theorem

We have explained how a classical field theory gives rise to P_0 factorization algebra Obs^{cl}, and how a quantum field theory (in the sense of [**Cos11b**]) gives rise to a factorization algebra Obs^q over $\mathbb{R}[[\hbar]]$, which specializes at $\hbar = 0$ to the factorization algebra Obs^{cl} of classical observables. In this section we will state our *quantization theorem*, which says that the Poisson bracket on Obs^{cl} is compatible, in a certain sense, with the quantization given by Obs^q.

This statement is the analog, in our setting, of a familiar statement in quantum-mechanical deformation quantization. Recall (section **??**) that in that setting, we require that the associative product on the algebra A^q of quantum observables is related to the Poisson bracket on the Poisson algebra A^{cl} of classical observables by the formula

$$\{a, b\} = \lim_{\hbar \to 0} \hbar^{-1} [\widetilde{a}, \widetilde{b}]$$

where $\widetilde{a}, \widetilde{b}$ are any lifts of the elements $a, b \in A^{cl}$ to A^q.

One can make a similar definition in the world of P_0 algebras. If A^{cl} is any commutative differential graded algebra, and A^q is a cochain complex flat over $\mathbb{R}[[\hbar]]$ which reduces to A^{cl} modulo $\hbar$, then we can define a cochain map

$$\{-,-\}_{A^q} : A^{cl} \otimes A^{cl} \to A^{cl}$$

which measures the failure of the commutative product on A^{cl} to lift to a product on A^q, to first order in $\hbar$. (A precise definition is given in section 2.3).

Now, suppose that A^{cl} is a P_0 algebra. Let A^q be a cochain complex flat over $\mathbb{R}[[\hbar]]$ which reduces to A^{cl} modulo $\hbar$. We say that A^q is a *quantization* of A^{cl} if the bracket $\{-,-\}_{A^q}$ on A^{cl}, induced by A^q, is homotopic to the given Poisson bracket on A^{cl}.

This is a very loose notion, because the bracket $\{-,-\}_{A^q}$ on A^{cl} need not be a Poisson bracket; it is simply a bilinear map. When we discuss the notion of rigid quantization (section 1.4), we will explain how to put a certain operadic structure on A^q which guarantees that this induced bracket is a Poisson bracket.

1.3.1. The quantization theorem. Now that we have the definition of quantization at hand, we can state our quantization theorem.

For every open subset $U \subset M$, $\mathrm{Obs}^{cl}(U)$ is a lax P_0 algebra. Given a BV quantization of our classical field theory, $\mathrm{Obs}^q(U)$ is a cochain complex flat over $\mathbb{R}[[\hbar]]$ which coincides, modulo $\hbar$, with $\mathrm{Obs}^{cl}(U)$. Our definition of quantization makes sense with minor modifications for lax P_0 algebras as well as for ordinary P_0 algebras.

1.3.1.1 Theorem (The quantization theorem). *For every $U \subset M$, the cochain complex* $\mathrm{Obs}^q(U)$ *of classical observables on U is a quantization of the lax P_0 algebra* $\mathrm{Obs}^{cl}(U)$.

1.4. The rigid quantization conjecture

We have seen (section 1.3) how the observables of a quantum field theory are a quantization, in a weak sense, of the lax P_0 algebra of observables of a quantum field theory. The definition of quantization appearing in this theorem is somewhat unsatisfactory, however, because the bracket on the classical observables arising from the quantum observables is not a Poisson bracket.

In this section we will explain a stricter notion of quantization. We would like to show that the quantization of the classical observables of a field theory we construct lifts to a rigid quantization. However, this is unfortunately is still a conjecture (except for the case of free fields).

1.4.0.1 Definition. *A* BD algebra *is a cochain complex A, flat over $\mathbb{C}[[\hbar]]$, equipped with a commutative product and a Poisson bracket of cohomological degree* 1*, satisfying the identity*

$$\mathrm{d}(a \cdot b) = a \cdot (\mathrm{d}b) \pm (\mathrm{d}a) \cdot b + \hbar\{a, b\}. \tag{1.4.0.1}$$

The BD operad is investigated in detail in section 2.4. Note that, modulo $\hbar$, a BD algebra is a P_0 algebra.

1.4.0.2 Definition. *A* quantization *of a P_0 algebra A^{cl} is a BD algebra A^q, flat over $\mathbb{C}[[\hbar]]$, together with an equivalence of P_0 algebras between $A^q/\hbar$ and A^{cl}.*

More generally, one can (using standard operadic techniques) define a concept of *homotopy BD algebra*. This leads to a definition of a homotopy quantization of a P_0 algebra.

Recall that the classical observables Obs^{cl} of a classical field theory have the structure of a P_0 factorization algebra on our space-time manifold M.

1.4.0.3 Definition. *Let $\mathcal{F}^{cl}$ be a P_0 factorization algebra on M. Then, a* rigid quantization *of $\mathcal{F}^{cl}$ is a lift of $\mathcal{F}^{cl}$ to a homotopy BD factorization algebra $\mathcal{F}^q$, such that $\mathcal{F}^q(U)$ is a quantization (in the sense described above) of $\mathcal{F}^{cl}$.*

We conjecture that our construction of the factorization algebra of quantum observables associated to a quantum field theory has this structure. More precisely,

Conjecture. *Suppose we have a classical field theory on M, and a BV quantization of the theory. Then, Obs^q has the structure of a homotopy BD factorization algebra quantizing the P_0 factorization algebra Obs^{cl}.*

CHAPTER 2

Structured factorization algebras and quantization

In this chapter we will define what it means to have a factorization algebra endowed with the structure of an algebra over an operad. Not all operads work for this construction: only operads endowed with an extra structure – that of a *Hopf operad* – can be used. The issue is that we need to mix the structure maps of the factorization algebra with those of an algebra over an operad P, so we need to know how to tensor together P-algebras. (See the definition **??** in appendix A.)

After explaining the relevant machinery, we focus on the cases of interest for us: the P_0 and BD operads that appear in the classical and quantum BV formalisms, respectively. These operads play a central role in our quantization theorem, the main result of this book, and thus we will have formulated the goal toward which the next two parts of the book are devoted.

Since, in this book, we are principally concerned with factorization algebras taking values in the category of differentiable cochain complexes we will restrict attention to this case in the present section.

2.1. Structured factorization algebras

2.1.0.1 Definition. *A* Hopf operad *is an operad in the category of differential graded cocommutative coalgebras.*

Any Hopf operad P is, in particular, a differential graded operad. In addition, the cochain complexes $P(n)$ are endowed with the structure of differential graded commutative coalgebra. The operadic composition maps

$$\circ_i : P(n) \otimes P(m) \to P(n+m-1)$$

are maps of coalgebras, as are the maps arising from the symmetric group action on $P(n)$.

If P is a Hopf operad, then the category of dg P-algebras becomes a symmetric monoidal category. If A, B are P-algebras, the tensor product $A \otimes_{\mathbb{C}} B$ is also a P-algebra. The structure map

$$P_{A\otimes B} : P(n) \otimes (A \otimes B)^{\otimes n} \to A \otimes B$$

is defined to be the composition

$$P(n) \otimes (A \otimes B)^{\otimes n} \xrightarrow{c(n)} P(n) \otimes P(n) \otimes A^{\otimes n} \otimes B^{\otimes n} \xrightarrow{P_A \otimes P_B} A \otimes B.$$

In this diagram, $c(n) : P(n) \to P(n)^{\otimes 2}$ is the comultiplication on $c(n)$.

Any dg operad that is the homology operad of an operad in topological spaces is a Hopf operad (because topological spaces are automatically cocommutative coalgebras, with comultiplication defined by the diagonal map). For example, the commutative operad Com is a Hopf operad, with coproduct defined on the generator $\star \in \mathrm{Com}(2)$ by

$$c(\star) = \star \otimes \star.$$

With the comultiplication defined in this way, the tensor product of commutative algebras is the usual one. If A and B are commutative algebras, the product on $A \otimes B$ is defined by

$$(a \otimes b) \star (a' \otimes b') = (-1)^{|a'||b|}(a \star a') \otimes (b \star b').$$

The Poisson operad is also a Hopf operad, with coproduct defined (on the generators $\star, \{-,-\}$ by

$$\begin{aligned} c(\star) &= \star \otimes \star \\ c(\{-,-\}) &= \{-,-\} \otimes \star + \star \otimes \{-,-\}. \end{aligned}$$

If A, B are Poisson algebras, then the tensor product $A \otimes B$ is a Poisson algebra with product and bracket defined by

$$\begin{aligned} (a \otimes b) \star (a' \otimes b') &= (-1)^{|a'||b|}(a \star a') \otimes (b \star b') \\ \{a \otimes b, a' \otimes b'\} &= (-1)^{|a'||b|} \left(\{a, a'\} \otimes (b \star b') + (a \star a') \otimes \{b, b'\} \right). \end{aligned}$$

2.1.0.2 Definition. *Let P be a differential graded Hopf operad. A* prefactorization P-algebra *is a prefactorization algebra with values in the multicategory of P-algebras. A* factorization P-algebra *is a prefactorization P-algebra, such that the underlying prefactorization algebra with values in cochain complexes is a factorization algebra.*

We can unpack this definition as follows. Suppose that $\mathcal{F}$ is a factorization P-algebra. Then $\mathcal{F}$ is a factorization algebra; and, in addition, for all $U \subset M$, $\mathcal{F}(U)$ is a P-algebra. The structure maps

$$\mathcal{F}(U_1) \times \cdots \times \mathcal{F}(U_n) \to \mathcal{F}(V)$$

(defined when $U_1, \ldots, U_n$ are disjoint open subsets of V) are required to be P-algebra maps in the sense defined above.

2.2. Commutative factorization algebras

One of the most important examples is when P is the operad Com of commutative algebras. Then, we find that $\mathcal{F}(U)$ is a commutative algebra for each U. Further, if $U_1, \dots, U_k \subset V$ are as above, the product map

$$m : \mathcal{F}(U_1) \times \dots \times \mathcal{F}(U_k) \to \mathcal{F}(V)$$

is compatible with the commutative algebra structures, in the following sense.

(1) If $1 \in \mathcal{F}(U_i)$ is the unit for the commutative product on each $F(U_i)$, then

$$m(1, \dots, 1) = 1.$$

(2) If $\alpha_i, \beta_i \in \mathcal{F}(U_i)$, then

$$m(\alpha_1\beta_1, \dots, \alpha_k\beta_k) = \pm m(\alpha_1, \dots, \alpha_k) m(\beta_1, \dots, \beta_k)$$

where $\pm$ indicates the usual Koszul rule of signs.

Note that the axioms of a factorization algebra imply that $\mathcal{F}(\emptyset)$ is the ground ring k (which we normally take to be $\mathbb{R}$ or $\mathbb{C}$ for classical theories and $\mathbb{R}[[\hbar]]$ or $\mathbb{C}[[\hbar]]$ for quantum field theories). The axioms above, in the case that $k = 1$ and $U_1 = \emptyset$, imply that the map

$$\mathcal{F}(\emptyset) \to \mathcal{F}(U)$$

is a map of unital commutative algebras.

If $\mathcal{F}$ is a commutative prefactorization algebra, then we can recover $\mathcal{F}$ uniquely from the underlying cosheaf of commutative algebras. Indeed, the maps

$$\mathcal{F}(U_1) \times \dots \times \mathcal{F}(U_k) \to \mathcal{F}(V)$$

can be described in terms of the commutative product on $\mathcal{F}(V)$ and the maps $\mathcal{F}(U_i) \to \mathcal{F}(V)$.

2.3. The P_0 operad

Recall that the collection of observables in quantum mechanics form an associative algebra. The observables of a classical mechanical system form a Poisson algebra. In the deformation quantization approach to quantum mechanics, one starts with a Poisson algebra A^{cl}, and attempts to construct an associative algebra A^q, which is an algebra flat over the ring $\mathbb{C}[[\hbar]]$, together with an isomorphism of associative algebras $A^q/\hbar \cong A^{cl}$. In addition, if $a, b \in A^{cl}$, and $\widetilde{a}, \widetilde{b}$ are any lifts of a, b to A^q, then

$$\lim_{\hbar \to 0} \frac{1}{\hbar} [\widetilde{a}, \widetilde{b}] = \{a, b\} \in A^{cl}.$$

This book concerns the analog, in quantum field theory, of the deformation quantization picture in quantum mechanics. We have seen that the sheaf of solutions to the Euler-Lagrange equation of a classical field theory can be encoded by a commutative factorization algebra. A commutative factorization algebra is the analog, in our setting, of the commutative algebra appearing in deformation quantization. We have argued (section 1.2) that the observables of a quantum field theory should form a factorization algebra. This factorization algebra is the analog of the associative algebra appearing in deformation quantization.

In deformation quantization, the commutative algebra of classical observables has an extra structure – a Poisson bracket – which makes it "want" to deform into an associative algebra. In this section we will explain the analogous structure on a commutative factorization algebra which makes it want to deform into a factorization algebra. Later (section 6.2) we will see that the commutative factorization algebra associated to a classical field theory has this extra structure.

2.3.1. The E_0 operad.

2.3.1.1 Definition. *Let E_0 be the operad defined by*

$$E_0(n) = \begin{cases} 0 & \textit{if } n > 0 \\ \mathbb{R} & \textit{if } n = 0 \end{cases}$$

Thus, an E_0 algebra in the category of real vector spaces is a real vector space with a distinguished element in it. More generally, an E_0 algebra in a symmetric monoidal category $\mathcal{C}$ is the same thing as an object A of $\mathcal{C}$ together with a map $1_{\mathcal{C}} \to A$

The reason for the terminology E_0 is that this operad can be interpreted as the operad of little 0-discs.

The inclusion of the empty set into every open set implies that, for any factorization algebra $\mathcal{F}$, there is a unique map from the unit factorization algebra $\mathbb{R} \to \mathcal{F}$.

2.3.2. The P_0 operad. The Poisson operad is an object interpolating between the commutative operad and the associative (or E_1) operad. We would like to find an analog of the Poisson operad which interpolates between the commutative operad and the E_0 operad.

Let us define the P_k operad to be the operad whose algebras are commutative algebras equipped with a Poisson bracket of degree $1 - k$. With this notation, the usual Poisson operad is the P_1 operad.

Recall that the homology of the E_n operad is the P_n operad, for $n > 1$. Thus, just as the semi-classical version of an algebra over the E_1 operad is a Poisson algebra in the usual sense (that is, a P_1 algebra), the semi-classical version of an E_n algebra is a P_n algebra.

Thus, we have the following table:

Classical	Quantum
?	E_0 operad
P_1 operad	E_1 operad
P_2 operad	E_2 operad
$\vdots$	$\vdots$

This immediately suggests that the P_0 operad is the semi-classical version of the E_0 operad.

Note that the P_0 operad is a Hopf operad: the coproduct is defined by

$$\begin{aligned} c(\star) &= \star \otimes \star \\ c(\{-,-\}) &= \{-,-\} \otimes \star + \star \otimes \{-,-\}. \end{aligned}$$

In concrete terms, this means that if A and B are P_0 algebras, their tensor product $A \otimes B$ is again a P_0 algebra, with product and bracket defined by

$$\begin{aligned} (a \otimes b) \star (a' \otimes b') &= (-1)^{|a'||b|} (a \star a') \otimes (b \star b') \\ \{a \otimes b, a' \otimes b'\} &= (-1)^{|a'||b|} \left(\{a, a'\} \otimes (b \star b') + (a \star a') \otimes \{b, b'\} \right). \end{aligned}$$

2.3.3. P_0 **factorization algebras.** Since the P_0 operad is a Hopf operad, it makes sense to talk about P_0 factorization algebras. We can give an explicit description of this structure. A P_0 factorization algebra is a commutative factorization algebra $\mathcal{F}$, together with a Poisson bracket of cohomological degree 1 on each commutative algebra $\mathcal{F}(U)$, with the following additional properties. Firstly, if $U \subset V$, the map

$$\mathcal{F}(U) \to \mathcal{F}(V)$$

must be a homomorphism of P_0 algebras.

The second condition is that observables coming from disjoint sets must Poisson commute. More precisely, let U_1, U_2 be disjoint subsets f V. . Let $j_i : \mathcal{F}(U_i) \to \mathcal{F}(V)$ be the natural maps. Let $\alpha_i \in \mathcal{F}(U_i)$, and $j_i(\alpha_i) \in \mathcal{F}(V)$. Then, we require that

$$\{j_1(\alpha_1), j_2(\alpha_2)\} = 0 \in \mathcal{F}(V)$$

where $\{-,-\}$ is the Poisson bracket on $\mathcal{F}(V)$.

2.3.4. Quantization of P_0 algebras. We know what it means to quantize an Poisson algebra in the ordinary sense (that is, a P_1 algebra) into an E_1 algebra.

There is a similar notion of quantization for P_0 algebras. A quantization is simply an E_0 algebra over $\mathbb{R}[[\hbar]]$ which, modulo $\hbar$, is the original P_0 algebra, and for which there is a certain compatibility between the Poisson bracket on the P_0 algebra and the quantized E_0 algebra.

Let A be a commutative algebra in the category of cochain complexes. Let A_1 be an E_0 algebra flat over $\mathbb{R}[[\hbar]]/\hbar^2$, and suppose that we have an isomorphism of chain complexes

$$A_1 \otimes_{\mathbb{R}[[\hbar]]/\hbar^2} \mathbb{R} \cong A.$$

In this situation, we can define a bracket on A of degree 1, as follows.

We have an exact sequence

$$0 \to \hbar A \to A_1 \to A \to 0.$$

The boundary map of this exact sequence is a cochain map

$$D : A \to A$$

(well-defined up to homotopy).

Let us define a bracket on A by the formula

$$\{a, b\} = D(ab) - (-1)^{|a|} a D b - (Da) b.$$

Because D is well-defined up to homotopy, so is this bracket. However, unless D is an order two differential operator, this bracket is simply a cochain map $A \otimes A \to A$, and not a Poisson bracket of degree 1.

In particular, this bracket induces one on the cohomology $H^*(A)$ of A. The cohomological bracket is independent of any choices.

2.3.4.1 Definition. *Let A be a P_0 algebra in the category of cochain complexes. Then a* quantization *of A is an E_0 algebra $\widetilde{A}$ over $\mathbb{R}[[\hbar]]$, together with a quasi-isomorphism of E_0 algebras*

$$\widetilde{A} \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R} \cong A,$$

which satisfies the following correspondence principle*: the bracket on $H^*(A)$ induced by $\widetilde{A}$ must coincide with that given by the P_0 structure on A.*

In the next section we will consider a more sophisticated, operadic notion of quantization, which is strictly stronger than this one. To distinguish between the two notions, one could call the definition of quantization presented here a *quantization*, while the definition introduced later will be called a *rigid quantization*.

2.4. The Beilinson-Drinfeld operad

Beilinson and Drinfeld [**BD04**] constructed an operad over the formal disc which generically is equivalent to the E_0 operad, but which at 0 is equivalent to the P_0 operad. We call this operad the Beilinson-Drinfeld operad.

The operad P_0 is generated by a commutative associative product $- \star -$, of degree 0; and a Poisson bracket $\{-,-\}$ of degree $+1$.

2.4.0.1 Definition. *The* Beilinson-Drinfeld (or *BD*) operad *is the differential graded operad over the ring* $\mathbb{R}[[\hbar]]$ *which, as a graded operad, is simply*

$$BD = P_0 \otimes \mathbb{R}[[\hbar]];$$

but with differential defined by

$$\mathrm{d}(- \star -) = \hbar\{-,-\}.$$

If M is a flat differential graded $\mathbb{R}[[\hbar]]$ module, then giving M the structure of a BD algebra amounts to giving M a commutative associative product, of degree 0, and a Poisson bracket of degree 1, such that the differential on M is a derivation of the Poisson bracket, and the following identity is satisfied:

$$\mathrm{d}(m \star n) = (\mathrm{d}m) \star n + (-1)^{|m|} m \star (\mathrm{d}n) + (-1)^{|m|}\hbar\{m,n\}$$

2.4.0.2 Lemma. *There is an isomorphism of operads,*

$$BD \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R} \cong P_0,$$

and a quasi-isomorphism of operads over $\mathbb{R}((\hbar))$*,*

$$BD \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R}((\hbar)) \simeq E_0 \otimes \mathbb{R}((\hbar)).$$

Thus, the operad BD interpolates between the P_0 operad and the E_0 operad.

BD is an operad in the category of differential graded $\mathbb{R}[[\hbar]]$ modules. Thus, we can talk about BD algebras in this category, or in any symmetric monoidal category enriched over the category of differential graded $\mathbb{R}[[\hbar]]$ modules.

The BD algebra is, in addition, a Hopf operad, with coproduct defined in the same way as in the P_0 operad. Thus, one can talk about BD factorization algebras.

2.4.1. BD quantization of P_0 algebras.

2.4.1.1 Definition. *Let A be a P_0 algebra (in the category of cochain complexes). A* BD quantization of *A is a flat* $\mathbb{R}[[\hbar]]$ *module A^q, flat over* $\mathbb{R}[[\hbar]]$*, which is equipped with the structure of a*

BD algebra, and with an isomorphism of P_0 algebras

$$A^q \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R} \cong A.$$

Similarly, an order k BD quantization of A is a differential graded $\mathbb{R}[[\hbar]]/\hbar^{k+1}$ module A^q, flat over $\mathbb{R}[[\hbar]]/\hbar^{k+1}$, which is equipped with the structure of an algebra over the operad

$$BD \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R}[[\hbar]]/\hbar^{k+1},$$

and with an isomorphism of P_0 algebras

$$A^q \otimes_{\mathbb{R}[[\hbar]]/\hbar^{k+1}} \mathbb{R} \cong A.$$

This definition applies without any change in the world of factorization algebras.

2.4.1.2 Definition. *Let $\mathcal{F}$ be a P_0 factorization algebra on M. Then a* BD quantization of $\mathcal{F}$ *is a BD factorization algebra $\widetilde{\mathcal{F}}$ equipped with a quasi-isomorphism*

$$\widetilde{\mathcal{F}} \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R} \simeq \mathcal{F}$$

of P_0 factorization algebras on M.

2.4.2. Operadic description of ordinary deformation quantization. We will finish this section by explaining how the ordinary deformation quantization picture can be phrased in similar operadic terms.

Consider the following operad BD_1 over $\mathbb{R}[[\hbar]]$. BD_1 is generated by two binary operations, a product $*$ and a bracket $[-,-]$. The relations are that the product is associative; the bracket is antisymmetric and satisfies the Jacobi identity; the bracket and the product satisfy a certain Leibniz relation, expressed in the identity

$$[ab, c] = a[b, c] \pm [b, c]a$$

(where $\pm$ indicates the Koszul sign rule); and finally the relation

$$a * b \mp b * a = \hbar[a, b]$$

holds. This operad was introduced by Ed Segal [Seg10].

Note that, modulo $\hbar$, BD_1 is the ordinary Poisson operad P_1. If we set $\hbar = 1$, we find that BD_1 is the operad E_1 of associative algebras. Thus, BD_1 interpolates between P_1 and E_1 in the same way that BD_0 interpolates between P_0 and E_0.

Let A be a P_1 algebra. Let us consider possible lifts of A to a BD_1 algebra.

2.4.2.1 Lemma. *A lift of A to a BD_1 algebra, flat over $\mathbb{R}[[\hbar]]$, is the same as a deformation quantization of A in the usual sense.*

PROOF. We need to describe BD_1 structures on $A[[\hbar]]$ compatible with the given Poisson structure. To give such a BD_1 structure is the same as to give an associative product on $A[[\hbar]]$, linear over $\mathbb{R}[[\hbar]]$, and which modulo $\hbar$ is the given commutative product on A. Further, the relations in the BD_1 operad imply that the Poisson bracket on A is related to the associative product on $A[[\hbar]]$ by the formula

$$\hbar^{-1}(a * b \mp b * a) = \{a, b\} \mod \hbar.$$

□

Part 1

Classical field theory

CHAPTER 3

Introduction to classical field theory

Our goal here is to describe how the observables of a classical field theory naturally form a factorization algebra (section **??**). More accurately, we are interested in what might be called classical perturbative field theory. "Classical" means that the main object of interest is the sheaf of solutions to the Euler-Lagrange equations for some local action functional. "Perturbative" means that we will only consider those solutions which are infinitesimally close to a given solution. Much of this part of the book is devoted to providing a precise mathematical definition of these ideas, with inspiration taken from deformation theory and derived geometry. In this chapter, then, we will simply sketch the essential ideas.

3.1. The Euler-Lagrange equations

The fundamental objects of a physical theory are the observables of a theory, that is, the measurements one can make in that theory. In a classical field theory, the fields that appear "in nature" are constrained to be solutions to the Euler-Lagrange equations (also called the equations of motion). Thus, the measurements one can make are the functions on the space of solutions to the Euler-Lagrange equations.

However, it is essential that we do not take the naive moduli space of solutions. Instead, we consider the *derived* moduli space of solutions. Since we are working perturbatively — that is, infinitesimally close to a given solution — this derived moduli space will be a "formal moduli problem" [**?, Lur11**]. In the physics literature, the procedure of taking the derived critical locus of the action functional is implemented by the BV formalism. Thus, the first step (chapter 4.1.3) in our treatment of classical field theory is to develop a language to treat formal moduli problems cut out by systems of partial differential equations on a manifold M. Since it is essential that the differential equations we consider are elliptic, we call such an object a *formal elliptic moduli problem*.

Since one can consider the solutions to a differential equation on any open subset $U \subset M$, a formal elliptic moduli problem $\mathcal{F}$ yields, in particular, a sheaf of formal moduli problems on M. This sheaf sends U to the formal moduli space $\mathcal{F}(U)$ of solutions on U.

We will use the notation $\mathcal{EL}$ to denote the formal elliptic moduli problem of solutions to the Euler-Lagrange equation on M; thus, $\mathcal{EL}(U)$ will denote the space of solutions on an open subset $U \subset M$.

3.2. Observables

In a field theory, we tend to focus on measurements that are localized in spacetime. Hence, we want a method that associates a set of observables to each region in M. If $U \subset M$ is an open subset, the observables on U are

$$\mathrm{Obs}^{cl}(U) = \mathscr{O}(\mathcal{EL}(U)),$$

our notation for the algebra of functions on the formal moduli space $\mathcal{EL}(U)$ of solutions to the Euler-Lagrange equations on U. (We will be more precise about which class of functions we are using later.) As we are working in the derived world, $\mathrm{Obs}^{cl}(U)$ is a differential-graded commutative algebra. Using these functions, we can answer any question we might ask about the behavior of our system in the region U.

The factorization algebra structure arises naturally on the observables in a classical field theory. Let U be an open set in M, and $V_1, \ldots, V_k$ a disjoint collection of open subsets of U. Then restriction of solutions from U to each V_i induces a natural map

$$\mathcal{EL}(U) \to \mathcal{EL}(V_1) \times \cdots \times \mathcal{EL}(V_k).$$

Since functions pullback under maps of spaces, we get a natural map

$$\mathrm{Obs}^{cl}(V_1) \otimes \cdots \otimes \mathrm{Obs}^{cl}(V_k) \to \mathrm{Obs}^{cl}(U)$$

so that Obs^{cl} forms a *pre*factorization algebra. To see that Obs^{cl} is indeed a factorization algebra, it suffices to observe that the functor $\mathcal{EL}$ is a sheaf.

Since the space $\mathrm{Obs}^{cl}(U)$ of observables on a subset $U \subset M$ is a commutative algebra, and not just a vector space, we see that the observables of a classical field theory form a commutative factorization algebra (section 2).

3.3. The symplectic structure

Above, we outlined a way to construct, from the elliptic moduli problem associated to the Euler-Lagrange equations, a commutative factorization algebra. This construction, however, would apply equally well to any system of differential equations. The Euler-Lagrange equations, of course, have the special property that they arise as the critical points of a functional.

In finite dimensions, a formal moduli problem which arises as the derived critical locus (section 5.1) of a function is equipped with an extra structure: a symplectic form of cohomological degree -1. For us, this symplectic form is an intrinsic way of indicating that a formal moduli problem arises as the critical locus of a functional. Indeed, any formal moduli problem with such a symplectic form can be expressed (non-uniquely) in this way.

We give (section 5.2) a definition of symplectic form on an elliptic moduli problem. We then simply *define* a classical field theory to be a formal elliptic moduli problem equipped with a symplectic form of cohomological degree -1.

Given a local action functional satisfying certain non-degeneracy properties, we construct (section 5.3.1) an elliptic moduli problem describing the corresponding Euler-Lagrange equations, and show that this elliptic moduli problem has a symplectic form of degree -1.

In ordinary symplectic geometry, the simplest construction of a symplectic manifold is as a cotangent bundle. In our setting, there is a similar construction: given any elliptic moduli problem $\mathcal{F}$, we construct (section 5.6) a new elliptic moduli problem $T^*[-1]\mathcal{F}$ which has a symplectic form of degree -1. It turns out that many examples of field theories of interest in mathematics and physics arise in this way.

3.4. The P_0 structure

In finite dimensions, if X is a formal moduli problem with a symplectic form of degree -1, then the dg algebra $\mathscr{O}(X)$ of functions on X is equipped with a Poisson bracket of degree 1. In other words, $\mathscr{O}(X)$ is a P_0 algebra (section 2.3).

In infinite dimensions, we show that something similar happens. If $\mathcal{F}$ is a classical field theory, then we show that on every open U, the commutative algebra $\mathscr{O}(\mathcal{F}(U)) = \mathrm{Obs}^{cl}(U)$ has a P_0 structure. We then show that the commutative factorization algebra Obs^{cl} forms a P_0 factorization algebra. This is not quite trivial; it is at this point that we need the assumption that our Euler-Lagrange equations are elliptic.

CHAPTER 4

Elliptic moduli problems

The essential data of a classical field theory is the moduli space of solutions to the equations of motion of the field theory. For us, it is essential that we take not the naive moduli space of solutions, but rather the *derived* moduli space of solutions. In the physics literature, the procedure of taking the derived moduli of solutions to the Euler-Lagrange equations is known as the classical Batalin-Vilkovisky formalism.

The derived moduli space of solutions to the equations of motion of a field theory on X is a sheaf on X. In this chapter we will introduce a general language for discussing sheaves of "derived spaces" on X that are cut out by differential equations.

Our focus in this book is on perturbative field theory, so we sketch the heuristic picture from physics before we introduce a mathematical language that formalizes the picture. Suppose we have a field theory and we have found a solution to the Euler-Lagrange equations ϕ_0. We want to find the nearby solutions, and a time-honored approach is to consider a formal series expansion around ϕ_0,

$$\phi_t = \phi_0 + t\phi_1 + t^2\phi_2 + \cdots ,$$

and to solve iteratively the Euler-Lagrange equations for the higher terms ϕ_n. Of course, such an expansion is often not convergent in any reasonable sense, but this perturbative method has provided insights into many physical problems. In mathematics, particularly the deformation theory of algebraic geometry, this method has also flourished and acquired a systematic geometric interpretation. Here, though, we work in place of t with a parameter ε that is nilpotent, so that there is some integer n such that $\varepsilon^{n+1} = 0$. Let

$$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots + \varepsilon^n\phi_n.$$

Again, the Euler-Lagrange equation applied to ϕ becomes a system of simpler differential equations organized by each power of ε. As we let the order of ε go to infinity and find the nearby solutions, we describe the *formal neighborhood* of ϕ_0 in the space of all solutions to the Euler-Lagrange equations. (Although this procedure may seem narrow in scope, its range expands considerably by considering families of solutions, rather a single fixed solution. Our formalism is built to work in families.)

In this chapter we will introduce a mathematical formalism for this procedure, which includes derived perturbations (i.e., ε has nonzero cohomological degree). In mathematics, this formalism is part of derived deformation theory or formal derived geometry. Thus, before we discuss the concepts specific to classical field theory, we will explain some general techniques from deformation theory. A key role is played by a deep relationship between Lie algebras and formal moduli spaces.

4.1. Formal moduli problems and Lie algebras

In ordinary algebraic geometry, the fundamental objects are commutative algebras. In derived algebraic geometry, commutative algebras are replaced by commutative differential graded algebras concentrated in non-positive degrees (or, if one prefers, simplicial commutative algebras; over $\mathbb{Q}$, there is no difference).

We are interested in formal derived geometry, which is described by nilpotent commutative dg algebras.

4.1.0.1 Definition. *An* Artinian dg algebra *over a field K of characteristic zero is a differential graded commutative K-algebra R, concentrated in degrees ≤ 0, such that*

(1) each graded component R^i is finite dimensional, and $R^i = 0$ for $i \ll 0$;
(2) R has a unique maximal differential ideal m such that $R/m = K$, and such that $m^N = 0$ for $N \gg 0$.

Given the first condition, the second condition is equivalent to the statement that $H^0(R)$ is Artinian in the classical sense.

The category of Artinian dg algebras is simplicially enriched in a natural way. A map $R \to S$ is simply a map of dg algebras taking the maximal ideal m_R to that of m_S. Equivalently, such a map is a map of non-unital dg algebras $m_R \to m_S$. An n-simplex in the space $\mathrm{Maps}(R, S)$ of maps from R to S is defined to be a map of non-unital dg algebras

$$m_R \to m_S \otimes \Omega^*(\triangle^n)$$

where $\Omega^*(\triangle^n)$ is some commutative algebra model for the cochains on the n-simplex. (Normally, we will work over $\mathbb{R}$, and $\Omega^*(\triangle^n)$ will be the usual de Rham complex.)

We will (temporarily) let Art_k denote the simplicially enriched category of Artinian dg algebras over k.

4.1.0.2 Definition. *A formal moduli problem over a field k is a functor (of simplicially enriched categories)*

$$F : \mathrm{Art}_k \to \mathrm{sSets}$$

from Art_k to the category sSets of simplicial sets, with the following additional properties.

(1) $F(k)$ is contractible.
(2) F takes surjective maps of dg Artinian algebras to fibrations of simplicial sets.
(3) Suppose that A, B, C are dg Artinian algebras, and that $B \to A$, $C \to A$ are surjective maps. Then we can form the fiber product $B \times_A C$. We require that the natural map

$$F(B \times_A C) \to F(B) \times_{F(A)} F(C)$$

is a weak homotopy equivalence.

We remark that such a moduli problem F is *pointed*: F assigns to k a point, up to homotopy, since $F(k)$ is contractible. Since we work mostly with pointed moduli problems in this book, we will not emphasize this issue. Whenever we work with more general moduli problems, we will indicate it explicitly.

Note that, in light of the second property, the fiber product $F(B) \times_{F(A)} F(C)$ coincides with the homotopy fiber product.

The category of formal moduli problems is itself simplicially enriched, in an evident way. If F, G are formal moduli problems, and $\phi : F \to G$ is a map, we say that ϕ is a weak equivalence if for all dg Artinian algebras R, the map

$$\phi(R) : F(R) \to G(R)$$

is a weak homotopy equivalence of simplicial sets.

4.1.1. Formal moduli problems and L_∞ algebras. One very important way in which formal moduli problems arise is as the solutions to the Maurer-Cartan equation in an L_∞ algebra. As we will see later, all formal moduli problems are equivalent to formal moduli problems of this form.

If $\mathfrak{g}$ is an L_∞ algebra, and (R, m) is a dg Artinian algebra, we will let

$$\mathrm{MC}(\mathfrak{g} \otimes m)$$

denote the simplicial set of solutions to the Maurer-Cartan equation in $\mathfrak{g} \otimes m$. Thus, an n-simplex in this simplicial set is an element

$$\alpha \in \mathfrak{g} \otimes m \otimes \Omega^*(\triangle^n)$$

of cohomological degree 1, which satisfies the Maurer-Cartan equation

$$\mathrm{d}\alpha + \sum_{n \geq 2} \tfrac{1}{n!} l_n(\alpha, \dots, \alpha) = 0.$$

It is a well-known result in derived deformation theory that sending R to $\mathrm{MC}(\mathfrak{g} \otimes m)$ defines a formal moduli problem (see [Get09], [Hin01]). We will often use the notation $B\mathfrak{g}$ to denote this formal moduli problem.

If $\mathfrak{g}$ is finite dimensional, then a Maurer-Cartan element of $\mathfrak{g} \otimes m$ is the same thing as a map of commutative dg algebras

$$C^*(\mathfrak{g}) \to R$$

which takes the maximal ideal of $C^*(\mathfrak{g})$ to that of R.

Thus, we can think of the Chevalley-Eilenberg cochain complex $C^*(\mathfrak{g})$ as the algebra of functions on $B\mathfrak{g}$.

Under the dictionary between formal moduli problems and L_∞ algebras, a dg vector bundle on $B\mathfrak{g}$ is the same thing as a dg module over $\mathfrak{g}$. The cotangent complex to $B\mathfrak{g}$ corresponds to the $\mathfrak{g}$-module $\mathfrak{g}^\vee[-1]$, with the shifted coadjoint action. The tangent complex corresponds to the $\mathfrak{g}$-module $\mathfrak{g}[1]$, with the shifted adjoint action.

If M is a $\mathfrak{g}$-module, then sections of the corresponding vector bundle on $B\mathfrak{g}$ is the Chevalley-Eilenberg cochains with coefficients in M. Thus, we can define $\Omega^1(B\mathfrak{g})$ to be

$$\Omega^1(B\mathfrak{g}) = C^*(\mathfrak{g}, \mathfrak{g}^\vee[-1]).$$

Similarly, the complex of vector fields on $B\mathfrak{g}$ is

$$\mathrm{Vect}(B\mathfrak{g}) = C^*(\mathfrak{g}, \mathfrak{g}[1]).$$

Note that, if $\mathfrak{g}$ is finite dimensional, this is the same as the cochain complex of derivations of $C^*(\mathfrak{g})$. Even if $\mathfrak{g}$ is not finite dimensional, the complex $\mathrm{Vect}(B\mathfrak{g})$ is, up to a shift of one, the Lie algebra controlling deformations of the L_∞ structure on $\mathfrak{g}$.

4.1.2. The fundamental theorem of deformation theory. The following statement is at the heart of the philosophy of deformation theory:

> There is an equivalence of $(\infty, 1)$ categories between the category of differential graded Lie algebras and the category of formal pointed moduli problems.

In a different guise, this statement goes back to Quillen's work [**Qui69**] on rational homotopy theory. A precise formulation of this theorem has been proved by Hinich [**Hin01**]; more general theorems of this nature are considered in [**Lur11**], [**?**] and in [**?**], which is also an excellent survey of these ideas.

It would take us too far afield to describe the language in which this statement can be made precise. We will simply use this statement as motivation: we will only consider formal moduli problems described by L_∞ algebras, and this statement asserts that we lose no information in doing so.

4.1.3. Elliptic moduli problems. We are interested in formal moduli problems which describe solutions to differential equations on a manifold M. Since we can discuss solutions to a differential equation on any open subset of M, such an object will give a sheaf of derived moduli problems on M, described by a sheaf of homotopy Lie algebras. Let us give a formal definition of such a sheaf.

4.1.3.1 Definition. *Let M be a manifold. A* local L_∞ algebra *on M consists of the following data.*

(1) A graded vector bundle L on M, whose space of smooth sections will be denoted $\mathcal{L}$.
(2) A differential operator $\mathrm{d} : \mathcal{L} \to \mathcal{L}$, of cohomological degree 1 and square 0.
(3) A collection of poly-differential operators

$$l_n : \mathcal{L}^{\otimes n} \to \mathcal{L}$$

for $n \geq 2$, which are alternating, are of cohomological degree $2-n$, and endow $\mathcal{L}$ with the structure of L_∞ algebra.

4.1.3.2 Definition. *An* elliptic L_∞ algebra *is a local L_∞ algebra $\mathcal{L}$ as above with the property that $(\mathcal{L}, \mathrm{d})$ is an elliptic complex.*

Remark: The reader who is not comfortable with the language of L_∞ algebras will lose little by only considering elliptic dg Lie algebras. Most of our examples of classical field theories will be described using dg Lie algebra rather than L_∞ algebras.

If $\mathcal{L}$ is a local L_∞ algebra on a manifold M, then it yields a presheaf $B\mathcal{L}$ of formal moduli problems on M. This presheaf sends a dg Artinian algebra (R, m) and an open subset $U \subset M$ to the simplicial set

$$B\mathcal{L}(U)(R) = \mathrm{MC}(\mathcal{L}(U) \otimes m)$$

of Maurer-Cartan elements of the L_∞ algebra $\mathcal{L}(U) \otimes m$ (where $\mathcal{L}(U)$ refers to the sections of L on U). We will think of this as the R-points of the formal pointed moduli problem associated to $\mathcal{L}(U)$. One can show, using the fact that $\mathcal{L}$ is a fine sheaf, that this sheaf of formal moduli problems is actually a homotopy sheaf, i.e. it satisfies Čech descent. Since this point plays no role in our work, we will not elaborate further.

4.1.3.3 Definition. *A* formal pointed elliptic moduli problem *(or simply* elliptic moduli problem*) is a sheaf of formal moduli problems on M that is represented by an elliptic L_∞ algebra.*

The basepoint of the moduli problem corresponds, in the setting of field theory, to the distinguished solution we are expanding around.

4.2. Examples of elliptic moduli problems related to scalar field theories

4.2.1. The free scalar field theory. Let us start with the most basic example of an elliptic moduli problem, that of harmonic functions. Let M be a Riemannian manifold. We want to consider the formal moduli problem describing functions ϕ on M that are harmonic, namely, functions that satisfy $\mathrm{D}\phi = 0$ where D is the Laplacian. The base point of this formal moduli problem is the zero function.

The elliptic L_∞ algebra describing this formal moduli problem is defined by

$$\mathcal{L} = C^\infty(M)[-1] \xrightarrow{\mathrm{D}} C^\infty(M)[-2].$$

This complex is thus situated in degrees 1 and 2. The products l_n in this L_∞ algebra are all zero for $n \geq 2$.

In order to justify this definition, let us analyze the Maurer-Cartan functor of this L_∞ algebra. Let R be an ordinary (not dg) Artinian algebra, and let m be the maximal ideal of R. The set of 0-simplices of the simplicial set $\mathrm{MC}_{\mathcal{L}}(R)$ is the set

$$\{\phi \in C^\infty(M) \otimes m \mid \mathrm{D}\phi = 0.\}$$

Indeed, because the L_∞ algebra $\mathcal{L}$ is Abelian, the set of solutions to the Maurer-Cartan equation is simply the set of closed degree 1 elements of the cochain complex $\mathcal{L} \otimes m$. All higher simplices in the simplicial set $\mathrm{MC}_{\mathcal{L}}(R)$ are constant. To see this, note that if $\phi \in \mathcal{L} \otimes m \otimes \Omega^*(\triangle^n)$ is a closed element in degree 1, then ϕ must be in $C^\infty(M) \otimes m \otimes \Omega^0(\triangle^n)$. The fact that ϕ is closed amounts to the statement that $\mathrm{D}\phi = 0$ and that $\mathrm{d}_{dR}\phi = 0$, where d_{dR} is the de Rham differential on $\Omega^*(\triangle^n)$.

Let us now consider the Maurer-Cartan simplicial set associated to a differential graded Artinian algebra (R, m) with differential d_R. The the set of 0-simplices of $\mathrm{MC}_{\mathcal{L}}(R)$ is the set

$$\{\phi \in C^\infty(M) \otimes m^0, \;\; \psi \in C^\infty(M) \otimes m^{-1} \mid \mathrm{D}\phi = \mathrm{d}_R \psi.\}$$

(The superscripts on m indicate the cohomological degree.) Thus, the 0-simplices of our simplicial set can be identified with the set R-valued smooth functions ϕ on M that are harmonic up to a homotopy given by ψ and also vanish modulo the maximal ideal m.

Next, let us identify the set of 1-simplices of the Maurer-Cartan simplicial set $\mathrm{MC}_{\mathcal{L}}(R)$. This is the set of closed degree 1 elements of $\mathcal{L} \otimes m \otimes \Omega^*([0,1])$. Such a closed degree 1 element has four terms:

$$\begin{aligned}
\phi_0(t) &\in C^\infty(M) \otimes m^0 \otimes \Omega^0([0,1]) \\
\phi_1(t)\mathrm{d}t &\in C^\infty(M) \otimes m^{-1} \otimes \Omega^1([0,1]) \\
\psi_0(t) &\in C^\infty(M) \otimes m^{-1} \otimes \Omega^0([0,1]) \\
\psi_1(t)\mathrm{d}t &\in C^\infty(M) \otimes m^{-2} \otimes \Omega^1([0,1]).
\end{aligned}$$

Being closed amounts to satisfying the three equations

$$\begin{aligned} \mathrm{D}\,\phi_0(t) &= \mathrm{d}_R \psi_0(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\phi_0(t) &= \mathrm{d}_R \phi_1(t) \\ \mathrm{D}\,\phi_1(t) + \frac{\mathrm{d}}{\mathrm{d}t}\psi_0(t) &= \mathrm{d}_R \psi_1(t). \end{aligned}$$

These equations can be interpreted as follows. We think of $\phi_0(t)$ as providing a family of R-valued smooth functions on M, which are harmonic up to a homotopy specified by $\psi_0(t)$. Further, $\phi_0(t)$ is independent of t, up to a homotopy specified by $\phi_1(t)$. Finally, we have a coherence condition among our two homotopies.

The higher simplices of the simplicial set have a similar interpretation.

4.2.2. Interacting scalar field theories. Next, we will consider an elliptic moduli problem that arises as the Euler-Lagrange equation for an interacting scalar field theory. Let ϕ denote a smooth function on the Riemannian manifold M with metric g. The action functional is

$$S(\phi) = \int_M \tfrac{1}{2}\phi\,\mathrm{D}\,\phi + \tfrac{1}{4!}\phi^4 \,\mathrm{dvol}_g\,.$$

The Euler-Lagrange equation for the action functional S is

$$\mathrm{D}\,\phi + \tfrac{1}{3!}\phi^3 = 0,$$

a nonlinear PDE, whose space of solutions is hard to describe.

Instead of trying to describe the actual space of solutions to this nonlinear PDE, we will describe the formal moduli problem of solutions to this equation where ϕ is infinitesimally close to zero.

The formal moduli problem of solutions to this equation can be described as the solutions to the Maurer-Cartan equation in a certain elliptic L_∞ algebra which continue we call $\mathcal{L}$. As a cochain complex, $\mathcal{L}$ is

$$\mathcal{L} = C^\infty(M)[-1] \xrightarrow{\mathrm{D}} C^\infty(M)[-2].$$

Thus, $C^\infty(M)$ is situated in degrees 1 and 2, and the differential is the Laplacian.

The L_∞ brackets l_n are all zero except for l_3. The cubic bracket l_3 is the map

$$\begin{aligned} l_3 : C^\infty(M)^{\otimes 3} &\to C^\infty(M) \\ \phi_1 \otimes \phi_2 \otimes \phi_3 &\mapsto \phi_1\phi_2\phi_3. \end{aligned}$$

Here, the copy of $C^\infty(M)$ appearing in the source of l_3 is the one situated in degree 1, whereas that appearing in the target is the one situated in degree 2.

If R is an ordinary (not dg) Artinian algebra, then the Maurer-Cartan simplicial set $\mathrm{MC}_{\mathcal{L}}(R)$ associated to R has for 0-simplices the set $\phi \in C^\infty(M) \otimes m$ such that $\mathrm{D}\,\phi + \frac{1}{3!}\phi^3 = 0$. This equation may look as complicated as the full nonlinear PDE, but it is substantially simpler than the original problem. For example, consider $R = \mathbb{R}[\varepsilon]/(\varepsilon^2)$, the "dual numbers." Then $\phi = \varepsilon\phi_1$ and the Maurer-Cartan equation becomes $\mathrm{D}\,\phi_1 = 0$. For $R = \mathbb{R}[\varepsilon]/(\varepsilon^4)$, we have $\phi = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3$ and the Maurer-Cartan equation becomes a triple of simpler *linear* PDE:

$$\mathrm{D}\,\phi_1 = 0,\ \mathrm{D}\,\phi_2 = 0,\ \text{and}\ D\phi_3 + \tfrac{1}{2}\phi_1^3 = 0.$$

We are simply reading off the ε^k components of the Maurer-Cartan equation. The higher simplices of this simplicial set are constant.

If R is a dg Artinian algebra, then the simplicial set $\mathrm{MC}_{\mathcal{L}}(R)$ has for 0-simplices the set of pairs $\phi \in C^\infty(M) \otimes m^0$ and $\psi \in C^\infty(M) \otimes m^{-1}$ such that

$$\mathrm{D}\,\phi + \tfrac{1}{3!}\phi^3 = \mathrm{d}_R\psi.$$

We should interpret this as saying that ϕ satisfies the Euler-Lagrange equations up to a homotopy given by ψ.

The higher simplices of this simplicial set have an interpretation similar to that described for the free theory.

4.3. Examples of elliptic moduli problems related to gauge theories

4.3.1. Flat bundles. Next, let us discuss a more geometric example of an elliptic moduli problem: the moduli problem describing flat bundles on a manifold M. In this case, because flat bundles have automorphisms, it is more difficult to give a direct definition of the formal moduli problem.

Thus, let G be a Lie group, and let $P \to M$ be a principal G-bundle equipped with a flat connection ∇_0. Let $\mathfrak{g}_P$ be the adjoint bundle (associated to P by the adjoint action of G on its Lie algebra $\mathfrak{g}$). Then $\mathfrak{g}_P$ is a bundle of Lie algebras on M, equipped with a flat connection that we will also denote ∇_0.

For each Artinian dg algebra R, we want to define the simplicial set $\mathrm{Def}_P(R)$ of R-families of flat G-bundles on M that deform P. The question is "what local L_∞ algebra yields this elliptic moduli problem?"

The *answer* is $\mathcal{L} = \Omega^*(M, \mathfrak{g}_P)$, where the differential is d_{∇_0}, the de Rham differential coupled to our connection ∇_0. But we need to explain how to find this answer so we will provide the reasoning behind our answer. This reasoning is a model for finding the local L_∞ algebras associated to field theories.

Let us start by being more precise about the formal moduli problem that we are studying. We will begin by considering only on the deformations before we examine the issue of gauge equivalence. In other words, we start by just discussing the 0-simplices of our formal moduli problem.

As the underlying topological bundle of P is rigid, we can only deform the flat connection on P. Let's consider deformations over a dg Artinian ring R with maximal ideal m. A deformation of the connection ∇_0 on P is given by an element

$$A \in \Omega^1(M, \mathfrak{g}_P) \otimes m^0,$$

since the difference $\nabla - \nabla_0$ between any connection and our initial connection is a $\mathfrak{g}_P$-valued 1-form. The curvature of the deformed connection $\nabla_0 + A$ is

$$F(A) = \mathrm{d}_{\nabla_0} A + \tfrac{1}{2}[A, A] \in \Omega^2(M, \mathfrak{g}_P) \otimes m.$$

Note that, by the Bianchi identity, $\mathrm{d}_{\nabla_0} F(A) + [A, F(A)] = 0$.

Our first attempt to define the formal moduli functor Def_P might be that our moduli problem only returns deformations A such that $F(A) = 0$. From a homotopical perspective, it is more natural to loosen up this strict condition by requiring instead that $F(A)$ be *exact* in the cochain complex $\Omega^2(M, \mathfrak{g}_P) \otimes m$ of m-valued 2-forms on M. In other words, we ask for A to be flat up to homotopy. However, we should also ask that $F(A)$ is exact in a way compatible with the Bianchi identity, because a curvature always satisfies this condition.

Thus, as a tentative version of the formal moduli functor Def_P, we will define the 0-simplices of the deformation functor by

$$\begin{aligned}&\mathrm{Def}_P^{prelim}(R)[0] = \\ &\quad \{A \in \Omega^1(M, \mathfrak{g}_P) \otimes m^0, B \in \Omega^2(M, \mathfrak{g}_P) \otimes m^{-1} \mid F(A) = \mathrm{d}_R B,\ \mathrm{d}_{\nabla_0} B + [A, B] = 0\}.\end{aligned}$$

These equations say precisely that there exists a term B making $F(A)$ exact and that B satisfies a condition that enforces the Bianchi identity on $F(A)$.

This functor $\mathrm{Def}_P^{prelim}[0]$ does not behave the way that we want, though. Consider fixing our Artinian algebra to be $R = \mathbb{R}[\varepsilon_n]/(\varepsilon_n^2)$, where $|\varepsilon_n| = -n$, which is a shifted version of the "dual numbers." The functor $\mathrm{Def}_P^{prelim}[0](R)$ is then a presheaf of sets on M, which assigns to each open U the set

$$\{a \in \Omega^1(U, \mathfrak{g}_P), b \in \Omega^2(U, \mathfrak{g}_P) \mid d_{\nabla_0} a = 0,\ \mathrm{d}_{\nabla_0} b = 0\}.$$

In other words, we obtain the sheaf of sets $\Omega^1_{cl}(-, \mathfrak{g}_P) \times \Omega^2_{cl}(-, \mathfrak{g}_P)$, which returns closed 1-forms and closed 2-forms. This sheaf is *not*, however, a homotopy sheaf, because these sheaves are not fine and hence have higher cohomology groups.

How do we ensure that we obtain a homotopy sheaf of formal moduli problems? We will ask that B satisfy the Bianchi constraint up a sequence of higher homotopies, rather than satisfy the constraint strictly. Thus, the 0-simplices of our simplicial set of deformations are defined by

$$\mathrm{Def}_P(R)[0] = \{A \in \Omega^1(M, \mathfrak{g}_P) \otimes m^0, B \in \bigoplus_{k \geq 2} \Omega^k(M, \mathfrak{g}_P) \otimes m^{1-k} \\ \mid F(A) + \mathrm{d}B + [A, B] + \tfrac{1}{2}[B, B] = 0.\}.$$

Here, d refers to the total differential $\mathrm{d}_{\nabla_0} + \mathrm{d}_R$ on the tensor product cochain complex $\Omega^{\geq 2}(M, \mathfrak{g}_P) \otimes m$.

If we let $B_i \in \Omega^i(M, \mathfrak{g}_P) \otimes m^{1-i}$, then the first few constraints on the B_i can be written as

$$\begin{aligned} \mathrm{d}_{\nabla_0} B_2 + [A, B_2] + \mathrm{d}_R B_3 &= 0 \\ \mathrm{d}_{\nabla_0} B_3 + [A, B_3] + \tfrac{1}{2}[B_2, B_2] + \mathrm{d}_R B_4 &= 0. \end{aligned}$$

Thus, B_2 satisfies the Bianchi constraint up to a homotopy defined by B_3, and so on.

The higher simplices of this simplicial set must relate gauge-equivalent solutions. If we restricted our attention to ordinary Artinian algebras — i.e., to dg algebras R concentrated in degree 0 (and so with zero differential) — then we could define the simplicial set $\mathrm{Def}_P(R)$ to be the homotopy quotient of $\mathrm{Def}_P(R)[0]$ by the nilpotent group associated to the nilpotent Lie algebra $\Omega^0(M, \mathfrak{g}_P) \otimes m$, which acts on $\mathrm{Def}_P(R)[0]$ in the standard way (see, for instance, [**KS**] or [**Man09**]).

This approach, however, does not extend well to the dg Artinian algebras. When the algebra R is not concentrated in degree 0, the higher simplices of $\mathrm{Def}_P(R)$ must also involve elements of R of negative cohomological degree. Indeed, degree 0 elements of R should be thought of as homotopies between degree 1 elements of R, and so should contribute 1-simplices to our simplicial set.

A slick way to define a simplicial set with both desiderata is to set

$$\mathrm{Def}_P(R)[n] = \{A \in \Omega^*(M, \mathfrak{g}_P) \otimes m \otimes \Omega^*(\triangle^n) \mid \mathrm{d}_{\nabla_0} A + \mathrm{d}_R A + \mathrm{d}_{\triangle^n} A + \tfrac{1}{2}[A, A] = 0\},$$

where $\mathrm{d}_{\triangle^n}$ denotes the exterior derivative on $\Omega^*(\triangle^n)$.

Suppose that R is concentrated in degree 0 (so that the differential on R is zero). Then, the higher forms on M don't play any role, and

$$\mathrm{Def}_P(R)[0] = \{A \in \Omega^1(M, \mathfrak{g}_P) \otimes m \mid \mathrm{d}_{\nabla_0} A + \tfrac{1}{2}[A, A] = 0\}.$$

One can show (see [**Get09**]) that in this case, the simplicial set $\mathrm{Def}_P(R)$ is weakly homotopy equivalent to the homotopy quotient of $\mathrm{Def}_P(R)[0]$ by the nilpotent group associated to the nilpotent Lie algebra $\Omega^0(M, \mathfrak{g}_P) \otimes m$. Indeed, a 1-simplex in the simplicial set

$\mathrm{Def}_P(R)$ is given by a family of the form $A_0(t) + A_1(t)\mathrm{d}t$, where $A_0(t)$ is a smooth family of elements of $\Omega^1(M, \mathfrak{g}_P) \otimes m$ depending on $t \in [0,1]$, and $A_1(t)$ is a smooth family of elements of $\Omega^0(M, \mathfrak{g}_P) \otimes m$. The Maurer-Cartan equation in this context says that

$$\begin{aligned} \mathrm{d}_{\nabla_0} A_0(t) + \tfrac{1}{2}[A_0(t), A_0(t)] &= 0 \\ \tfrac{\mathrm{d}}{\mathrm{d}t} A_0(t) + [A_1(t), A_0(t)] &= 0. \end{aligned}$$

The first equation says that $A_0(t)$ defines a family of flat connections. The second equation says that the gauge equivalence class of $A_0(t)$ is independent of t. In this way, gauge equivalences are represented by 1-simplices in $\mathrm{Def}_P(R)$.

It is immediate that the formal moduli problem $\mathrm{Def}_P(R)$ is represented by the elliptic dg Lie algebra

$$\mathcal{L} = \Omega^*(M, \mathfrak{g}).$$

The differential on $\mathcal{L}$ is the de Rham differential d_{∇_0} on M coupled to the flat connection on $\mathfrak{g}$. The only nontrivial bracket is l_2, which just arises by extending the bracket of $\mathfrak{g}$ over the commutative dg algebra $\Omega^*(M)$ in the appropriate way.

4.3.2. Self-dual bundles. Next, we will discuss the formal moduli problem associated to the self-duality equations on a 4-manifold. We won't go into as much detail as we did for flat connections; instead, we will simply write down the elliptic L_∞ algebra representing this formal moduli problem. (For a careful explanation, see the original article [AHS78].)

Let M be an oriented 4-manifold. Let G be a Lie group, and let $P \to M$ be a principal G-bundle, and let $\mathfrak{g}_P$ be the adjoint bundle of Lie algebras. Suppose we have a connection A on P with anti-self-dual curvature:

$$F(A)_+ = 0 \in \Omega^2_+(M, \mathfrak{g}_P)$$

(here $\Omega^2_+(M)$ denotes the space of self-dual two-forms).

Then, the elliptic Lie algebra controlling deformations of (P, A) is described by the diagram

$$\Omega^0(M, \mathfrak{g}_P) \xrightarrow{\mathrm{d}} \Omega^1(M, \mathfrak{g}_P) \xrightarrow{\mathrm{d}_+} \Omega^2_+(M, \mathfrak{g}_P).$$

Here d_+ is the composition of the de Rham differential (coupled to the connection on $\mathfrak{g}_P$) with the projection onto $\Omega^2_+(M, \mathfrak{g}_P)$.

Note that this elliptic Lie algebra is a quotient of that describing the moduli of flat G-bundles on M.

4.3.3. Holomorphic bundles. In a similar way, if M is a complex manifold and if $P \to M$ is a holomorphic principal G-bundle, then the elliptic dg Lie algebra $\Omega^{0,*}(M, \mathfrak{g}_P)$, with differential $\overline{\partial}$, describes the formal moduli space of holomorphic G-bundles on M.

4.4. Cochains of a local L_∞ algebra

Let L be a local L_∞ algebra on M. If $U \subset M$ is an open subset, then $\mathcal{L}(U)$ denotes the L_∞ algebra of smooth sections of L on U. Let $\mathcal{L}_c(U) \subset \mathcal{L}(U)$ denote the sub-L_∞ algebra of compactly supported sections.

In the appendix (section B.1) we defined the algebra of functions on the space of sections on a vector bundle on a manifold. We are interested in the algebra

$$\mathscr{O}(\mathcal{L}(U)[1]) = \prod_{n \geq 0} \operatorname{Hom}\left((\mathcal{L}(U)[1])^{\otimes n}, \mathbb{R}\right)_{S_n}$$

where the tensor product is the completed projective tensor product, and Hom denotes the space of continuous linear maps.

This space is naturally a graded differentiable vector space (that is, we can view it as a sheaf of graded vector spaces on the site of smooth manifolds). However, it is important that we treat this object as a differentiable pro-vector space. Basic facts about differentiable pro-vector spaces are developed in the Appendix **??**. The pro-structure comes from the filtration

$$F^i \mathscr{O}(\mathcal{L}(U)[1]) = \prod_{n \geq i} \operatorname{Hom}\left((\mathcal{L}(U)[1])^{\otimes n}, \mathbb{R}\right)_{S_n},$$

which is the usual filtration on "power series."

The L_∞ algebra structure on $\mathcal{L}(U)$ gives, as usual, a differential on $\mathscr{O}(\mathcal{L}(U)[1])$, making $\mathscr{O}(\mathcal{L}(U)[1])$ into a differentiable pro-cochain complex.

4.4.0.1 Definition. *Define the* Lie algebra cochain complex $C^*(\mathcal{L}(U))$ *to be*

$$C^*(\mathcal{L}(U)) = \mathscr{O}(\mathcal{L}(U)[1])$$

equipped with the usual Chevalley-Eilenberg differential. Similarly, define

$$C^*_{red}(\mathcal{L}(U)) \subset C^*(\mathcal{L}(U))$$

to be the reduced Chevalley-Eilenberg complex*, that is, the kernel of the natural augmentation map* $C^*(\mathcal{L}(U)) \to \mathbb{R}$*. These are both differentiable pro-cochain complexes.*

One defines $C^*(\mathcal{L}_c(U))$ in the same way, everywhere substituting $\mathcal{L}_c$ for $\mathcal{L}$.

We will think of $C^*(\mathcal{L}(U))$ as the algebra of functions on the formal moduli problem $B\mathcal{L}(U)$ associated to the L_∞ algebra $\mathcal{L}(U)$.

4.4.1. Cochains with coefficients in a module.

Let L be a local L_∞ algebra on M, and let $\mathcal{L}$ denote the smooth sections. Let E be a graded vector bundle on M and equip the global smooth sections $\mathscr{E}$ with a differential that is a differential operator.

4.4.1.1 Definition. *A* local action of $\mathcal{L}$ on $\mathscr{E}$ *is an action of $\mathcal{L}$ on $\mathscr{E}$ with the property that the structure maps*

$$\mathcal{L}^{\otimes n} \otimes \mathscr{E} \to \mathscr{E}$$

(defined for $n \geq 1$) are all polydifferential operators.

Note that $\mathcal{L}$ has an action on itself, called the adjoint action, where the differential on $\mathcal{L}$ is the one coming from the L_∞ structure, and the action map

$$\mu_n : \mathcal{L}^{\otimes n} \otimes \mathcal{L} \to \mathcal{L}$$

is the L_∞ structure map l_{n+1}.

Let $L^! = L^\vee \otimes_{C^\infty_M} \mathrm{Dens}_M$. Then, $\mathcal{L}^!$ has a natural local $\mathcal{L}$-action, which we should think of as the coadjoint action. This action is defined by saying that if $\alpha_1, \dots, \alpha_n \in \mathcal{L}$, the differential operator

$$\mu_n(\alpha_1, \dots, \alpha_n, -) : \mathcal{L}^! \to \mathcal{L}^!$$

is the formal adjoint to the corresponding differential operator arising from the action of $\mathcal{L}$ on itself.

has the structure of a local module over $\mathcal{L}$.

If E is a local module over L, then, for each $U \subset M$, we can define the Chevalley-Eilenberg cochains

$$C^*(\mathcal{L}(U), \mathscr{E}(U))$$

of $\mathcal{L}(U)$ with coefficients in $\mathscr{E}(U)$. As above, one needs to take account of the topologies on the vector spaces $\mathcal{L}(U)$ and $\mathscr{E}(U)$ when defining this Chevalley-Eilenberg cochain complex. Thus, as a graded vector space,

$$C^*(\mathcal{L}(U), \mathscr{E}(U)) = \prod_{n \geq 0} \mathrm{Hom}((\mathcal{L}(U)[1])^{\otimes n}, \mathscr{E}(U))_{S_n}$$

where the tensor product is the completed projective tensor product, and Hom denotes the space of continuous linear maps. Again, we treat this object as a differentiable pro-cochain complex.

As explained in the section on formal moduli problems (section 4.1), we should think of a local module E over L as providing, on each open subset $U \subset M$, a vector bundle on the formal moduli problem $B\mathcal{L}(U)$ associated to $\mathcal{L}(U)$. Then the Chevalley-Eilenberg cochain complex $C^*(\mathcal{L}(U), \mathscr{E}(U))$ should be thought of as the space of sections of this vector bundle.

4.5. D-modules and local L_∞ algebras

Our definition of a local L_∞ algebra is designed to encode the derived moduli space of solutions to a system of non-linear differential equations. An alternative language for describing differential equations is the theory of D-modules. In this section we will show how our local L_∞ algebras can also be viewed as L_∞ algebras in the symmetric monoidal category of D-modules.

The main motivation for this extra layer of formalism is that local action functionals — which play a central role in classical field theory — are elegantly described using the language of D-modules.

Let C^∞_M denote the sheaf of smooth functions on the manifold M, let Dens_M denote the sheaf of smooth densities, and let D_M the sheaf of differential operators with smooth coefficients. The ∞-jet bundle $Jet(E)$ of a vector bundle E is the vector bundle whose fiber at a point $x \in M$ is the space of jets (or formal germs) at x of sections of E. The sheaf of sections of $Jet(E)$, denoted $J(E)$, is equipped with a canonical D_M-module structure, i.e., the natural flat connection sometimes known as the Cartan distribution. This flat connection is characterized by the property that flat sections of $J(E)$ are those sections which arise by taking the jet at every point of a section of the vector bundle E. (For motivation, observe that a field ϕ (a section of E) gives a section of $Jet(E)$ that encodes all the *local* information about ϕ.)

The category of D_M modules has a symmetric monoidal structure, given by tensoring over C^∞_M. The following lemma allows us to translate our definition of local L_∞ algebra into the world of D-modules.

4.5.0.1 Lemma. *Let $E_1, \dots, E_n, F$ be vector bundles on M, and let $\mathscr{E}_i, \mathscr{F}$ denote their spaces of global sections. Then, there is a natural bijection*

$$\mathrm{PolyDiff}(\mathscr{E}_1 \times \cdots \times \mathscr{E}_n, \mathscr{F}) \cong \mathrm{Hom}_{D_M}(J(E_1) \otimes \cdots \otimes J(E_n), J(F))$$

where PolyDiff *refers to the space of polydifferential operators. On the right hand side, we need to consider maps which are continuous with respect to the natural adic topology on the bundle of jets.*

Further, this bijection is compatible with composition.

A more formal statement of this lemma is that the multi-category of vector bundles on M, with morphisms given by polydifferential operators, is a full subcategory of the symmetric monoidal category of D_M modules. The embedding is given by taking jets. The proof of this lemma (which is straightforward) is presented in [Cos11b], Chapter 5.

This lemma immediately tells us how to interpret a local L_∞ algebra in the language of D-modules.

4.5.0.2 Corollary. *Let L be a local L_∞ algebra on M. Then $J(L)$ has the structure of L_∞ algebra in the category of D_M modules.*

Indeed, the lemma implies that to give a local L_∞ algebra on M is the same as to give a graded vector bundle L on M together with an L_∞ structure on the D_M module $J(L)$.

We are interested in the Chevalley-Eilenberg cochains of $J(L)$, but taken now in the category of D_M modules. Because $J(L)$ is an inverse limit of the sheaves of finite-order jets, some care needs to be taken when defining this Chevalley-Eilenberg cochain complex.

In general, if E is a vector bundle, let $J(E)^\vee$ denote the sheaf $\operatorname{Hom}_{C^\infty_M}(J(E), C^\infty_M)$, where $\operatorname{Hom}_{C^\infty_M}$ denotes continuous linear maps of C^∞_M-modules. This sheaf is naturally a D_M-module. We can form the completed symmetric algebra

$$\begin{aligned}\mathscr{O}_{red}(J(E)) &= \prod_{n>0} \operatorname{Sym}^n_{C^\infty_M}\left(J(E)^\vee\right) \\ &= \prod_{n>0} \operatorname{Hom}_{C^\infty_M}(J(E)^{\otimes n}, C^\infty_M)_{S_n}.\end{aligned}$$

Note that $\mathscr{O}_{red}(J(E)$ is a D_M-algebra, as it is defined by taking the completed symmetric algebra of $J(E)^\vee$ in the symmetric monoidal category of D_M-modules where the tensor product is taken over C^∞_M.

We can equivalently view $J(E)^\vee$ as an infinite-rank vector bundle with a flat connection. The symmetric power sheaf $\operatorname{Sym}^n_{C^\infty_M}(J(E)^\vee)$ is the sheaf of sections of the infinite-rank bundle whose fibre at x is the symmetric power of the fibre of $J(E)^\vee$ at x.

In the case that E is the trivial bundle $\underline{\mathbb{R}}$, the sheaf $J(\underline{\mathbb{R}})^\vee$ is naturally isomorphic to D_M as a left D_M-module. In this case, sections of the sheaf $\operatorname{Sym}^n_{C^\infty_M}(D_M)$ are objects which in local coordinates are finite sums of expressions like

$$f(x_i)\partial_{I_1}\dots\partial_{I_n}.$$

where ∂_{I_j} is the partial differentiation operator corresponding to a multi-index.

We should think of an element of $\mathscr{O}_{red}(J(E))$ as a Lagrangian on the space $\mathscr{E}$ of sections of E (a Lagrangian in the sense that an action functional is given by a Lagrangian density). Indeed, every element of $\mathscr{O}_{red}(J(E))$ has a Taylor expansion $F = \sum F_n$ where each F_n is a section

$$F_n \in \operatorname{Hom}_{C^\infty_M}(J(E)^{\otimes n}, C^\infty_M)^{S_n}.$$

Each such F_n is a multilinear map which takes sections $\phi_1, \dots, \phi_n \in \mathscr{E}$ and yields a smooth function $F_n(\phi_1, \dots, \phi_n) \in C^\infty(M)$, with the property that $F_n(\phi_1, \dots, \phi_n)(x)$ only depends on the ∞-jet of ϕ_i at x.

In the same way, we can interpret an element $F \in \mathscr{O}_{red}(J(E))$ as something that takes a section $\phi \in \mathscr{E}$ and yields a smooth function

$$\sum F_n(\phi, \ldots, \phi) \in C^\infty(M),$$

with the property that $F(\phi)(x)$ only depends on the jet of ϕ at x.

Of course, the functional F is a formal power series in the variable ϕ. One cannot evaluate most formal power series, since the putative infinite sum makes no sense. Instead, it only makes sense to evaluate a formal power series on infinitesimal elements. In particular, one can always evaluate a formal power series on nilpotent elements of a ring.

Indeed, a formal way to characterize a formal power series is to use the functor of points perspective on Artinian algebras: if R is an auxiliary graded Artinian algebra with maximal ideal m and if $\phi \in \mathscr{E} \otimes m$, then $F(\phi)$ is an element of $C^\infty(M) \otimes m$. This assignment is functorial with respect to maps of graded Artin algebras.

4.5.1. Local functionals. We have seen that we can interpret $\mathscr{O}_{red}(J(E))$ as the sheaf of Lagrangians on a graded vector bundle E on M. Thus, the sheaf

$$\mathrm{Dens}_M \otimes_{C^\infty_M} \mathscr{O}_{red}(J(E))$$

is the sheaf of Lagrangian densities on M. A section F of this sheaf is something which takes as input a section $\phi \in \mathscr{E}$ of $\mathscr{E}$ and produces a density $F(\phi)$ on M, in such a way that $F(\phi)(x)$ only depends on the jet of ϕ at x. (As before, F is a formal power series in the variable ϕ.)

The sheaf of local action functionals is the sheaf of Lagrangian densities modulo total derivatives. Two Lagrangian densities that differ by a total derivative define the same local functional on (compactly supported) sections because the integral of total derivative vanishes. Thus, we do not want to distinguish them, as they lead to the same physics. The formal definition is as follows.

4.5.1.1 Definition. *Let E be a graded vector bundle on M, whose space of global sections is $\mathscr{E}$. Then the space of* local action functionals *on $\mathscr{E}$ is*

$$\mathscr{O}_{loc}(\mathscr{E}) = \mathrm{Dens}_M \otimes_{D_M} \mathscr{O}_{red}(J(E)).$$

Here, Dens_M is the right D_M-module of densities on M.

Let $\mathscr{O}_{red}(\mathscr{E}_c)$ denote the algebra of functionals modulo constants on the space $\mathscr{E}_c$ of compactly supported sections of E. Integration induces a natural inclusion

$$\iota : \mathscr{O}_{loc}(\mathscr{E}) \to \mathscr{O}_{red}(\mathscr{E}_c),$$

where the Lagrangian density $S \in \mathscr{O}_{loc}(\mathscr{E})$ becomes the functional $\iota(S) : \phi \mapsto \int_M S(\phi)$. (Again, ϕ must be nilpotent and compactly supported.) From here on, we will use this inclusion without explicitly mentioning it.

4.5.2. Local Chevalley-Eilenberg complex of a local L_∞ algebra. Let L be a local L_∞ algebra. Then we can form, as above, the reduced Chevalley-Eilenberg cochain complex $C^*_{red}(J(L))$ of L. This is the D_M-algebra $\mathscr{O}_{red}(J(L)[1])$ equipped with a differential encoding the L_∞ structure on L.

4.5.2.1 Definition. *If $\mathcal{L}$ is a local L_∞-algebra, define the local Chevalley-Eilenberg complex to be*

$$C^*_{red,loc}(\mathcal{L}) = \mathrm{Dens}_M \otimes_{D_M} C^*_{red}(J(L)).$$

This is the space of local action functionals on $\mathcal{L}[1]$, equipped with the Chevalley-Eilenberg differential. In general, if $\mathfrak{g}$ is an L_∞ algebra, we think of the Lie algebra cochain complex $C^*(\mathfrak{g})$ as being the algebra of functions on $B\mathfrak{g}$. In this spirit, we sometimes use the notation $\mathscr{O}_{loc}(B\mathcal{L})$ for the complex $C^*_{red,loc}(\mathcal{L})$.

Note that $C^*_{red,loc}(\mathcal{L})$ is *not* a commutative algebra. Although the D_M-module $C^*_{red}(J(L))$ is a commutative D_M-module, the functor $\mathrm{Dens}_M \otimes_{D_M} -$ is not a symmetric monoidal functor from D_M-modules to cochain complexes, so it does not take commutative algebras to commutative algebras.

Note that there's a natural inclusion of cochain complexes

$$C^*_{red,loc}(\mathcal{L}) \to C^*_{red}(\mathcal{L}_c(M)),$$

where $\mathcal{L}_c(M)$ denotes the L_∞ algebra of compactly supported sections of L. The complex on the right hand side was defined earlier (see definition 4.4.0.1) and includes *non*local functionals.

4.5.3. Central extensions and local cochains. In this section we will explain how local cochains are in bijection with certain central extensions of a local L_∞ algebra. To avoid some minor analytical difficulties, we will only consider central extensions that are split as precosheaves of graded vector spaces.

4.5.3.1 Definition. *Let $\mathcal{L}$ be a local L_∞ algebra on M. A k-*shifted local central extension of *$\mathcal{L}$ is an L_∞ structure on the precosheaf $\mathcal{L}_c \oplus \underline{\mathbb{C}}[k]$, where $\underline{\mathbb{C}}$ is the constant precosheaf which takes value $\mathbb{C}$ on any open subset. We use the notation $\widetilde{\mathcal{L}}_c$ for the precosheaf $\mathcal{L}_c \oplus \underline{\mathbb{C}}[k]$. We require that this L_∞ structure has the following properties.*

(1) The sequence

$$0 \to \underline{\mathbb{C}}[k] \to \widetilde{\mathcal{L}}_c \to \mathcal{L}_c \to 0$$

is an exact sequence of precosheaves of L_∞ algebras, where $\underline{\mathbb{C}}[k]$ is given the abelian structure and $\mathcal{L}_c$ is given its original structure.

(2) *This implies that the L_∞ structure on $\widetilde{\mathcal{L}}_c$ is determined from that on $\mathcal{L}_c$ by L_∞ structure maps*
$$\widetilde{l}_n : \mathcal{L}_c \to \underline{\mathbb{C}}[k]$$
for $n \geq 1$. We require that these structure maps are given by local *action functionals.*

Two such central extensions, say $\widetilde{\mathcal{L}}_c$ and $\widetilde{\mathcal{L}}'_c$, are isomorphic *if there is an L_∞-isomorphism*
$$\widetilde{\mathcal{L}}_c \to \widetilde{\mathcal{L}}'_c$$
that is the identity on $\underline{\mathbb{C}[k]}$ and on the quotient $\mathcal{L}_c$. This L_∞ isomorphism must satisfy an additional property: the terms in this L_∞ -isomorphism, which are given (using the decomposition of $\widetilde{\mathcal{L}}_c$ and $\widetilde{\mathcal{L}}'_c$ as $\mathcal{L}_c \oplus \underline{\mathbb{C}}[k]$) by functionals
$$\mathcal{L}_c^{\otimes n} \to \underline{\mathbb{C}}[k],$$
must be local.

This definition refines the definition of central extension given in section **??** to include an extra locality property.

Example: Let Σ be a Riemann surface, and let $\mathfrak{g}$ be a Lie algebra with an invariant pairing. Let $\mathcal{L} = \Omega^{0,*}_\Sigma \otimes \mathfrak{g}$. Consider the Kac-Moody central extension, as defined in section **??** of **??** We let
$$\widetilde{\mathcal{L}}_c = \underline{\mathbb{C}} \cdot c \oplus \mathcal{L}_c,$$
where the central parameter c is of degree 1 and the Lie bracket is defined by
$$[\alpha, \beta]_{\widetilde{\mathcal{L}}_c} = [\alpha, \beta]_{\mathcal{L}_c} + c \int \alpha \partial \beta.$$
This is a local central extension. As shown in section **??** of chapter **??**, the factorization envelope of this extension recovers the vertex algebra of an associated affine Kac-Moody algebra. ◊

4.5.3.2 Lemma. *Let $\mathcal{L}$ be a local L_∞ algebra on a manifold M. There is a bijection between isomorphism classes of k-shifted local central extensions of $\mathcal{L}$ and classes in $H^{k+2}(\mathscr{O}_{loc}(B\mathcal{L}))$.*

Proof. This result is almost immediate. Indeed, any closed degree $k+2$ element of $\mathscr{O}_{loc}(B\mathcal{L})$ give a local L_∞ structure on $\underline{\mathbb{C}}[k] \oplus \mathcal{L}_c$, where the L_∞ structure maps
$$\widetilde{l}_n : \mathcal{L}_c(U) \to \mathbb{C}[k]$$
arise from the natural cochain map $\mathscr{O}_{loc}(B\mathcal{L}) \to C^*_{red}(\mathcal{L}_c(U))$. The fact that we start with a closed element of $\mathscr{O}_{loc}(B\mathcal{L})$ corresponds to the fact that the L_∞ axioms hold. Isomorphisms of local central extensions correspond to adding an exact cocycle to a closed degree $k+2$ element in $\mathscr{O}_{loc}(B\mathcal{L})$. ☐

Particularly important is the case when we have a -1-shifted central extension. As explained in subsection **??** in Chapter **??**, in this situation we can form the twisted factorization envelope, which is a factorization algebra over $\mathbb{C}[t]$ (where t is of degree 0) defined by sending an open subset U to the Chevalley-Eilenberg chain complex

$$U \mapsto C_*(\widetilde{\mathcal{L}}_c(U)).$$

We think of $\mathbb{C}[t]$ as the Chevalley-Eilenberg chains of the Abelian Lie algebra $\mathbb{C}[-1]$. In this situation, we can set t to be a particular value, leading to a *twisted* factorization envelope of $\mathcal{L}$. Twisted factorization envelopes will play a central role in our formulation of Noether's theorem at the quantum level in chapter 12.

4.5.4. Calculations of local L_∞ algebra cohomology play an important role in quantum field theory. Indeed, the obstruction-deformation complex describing quantizations of a classical field theory are local L_∞ algebra cohomology groups. Thus, it will be helpful to be able to compute some examples.

Before we start, let us describe a general result which will facilitate computation.

4.5.4.1 Lemma. *Let M be an oriented manifold and let $\mathcal{L}$ be a local L_∞-algebra on M. Then, there is a natural quasi-isomorphism*

$$\Omega^*(M, C^*_{red}(J(L)))[\mathrm{dimM}] \cong C^*_{red,loc}(\mathcal{L}).$$

PROOF. By definition,

$$\mathscr{O}(B\mathcal{L}) = \mathrm{Dens}_M \otimes_{D_M} C^*_{red} J(\mathcal{L})$$

where D_M is the sheaf of C^∞ differential operators. The D_M-module $C^*_{red}(J(\mathcal{L}))$ is flat (this was checked in [**Cos11b**]), so we can replace the tensor product over D_M with the left-derived tensor product.

Since M is oriented, we can replace Dens_M by Ω^d_M where $d = \dim M$. The right D_M-module Ω^d_M has a free resolution of the form

$$\cdots \to \Omega^{d-1}_M \otimes_{C^\infty_M} D_M \to \Omega^d M \otimes_{C^\infty_M} D_M$$

where $\Omega^i_M \otimes_{C^\infty_M} D_M$ is in cohomological degree $-i$, and the differential in this complex is the de Rham differential coupled to the left D_M-module structure on D_M. (This is sometimes called the Spenser resolution).

It follows that we the derived tensor product can be represented as

$$\Omega^d_M \otimes^{mbbL}_{D_M} C^*_{red}(J(\mathcal{L})) = \Omega^*(M, C^*_{red}(J(L)))[d]$$

as desired.

$\square$

4.5.4.2 Lemma. *Let Σ be a Riemann surface. Let $\mathcal{L}$ be the local L_∞ algebra on Σ defined by $\mathcal{L}(U) = \Omega^{0,*}(U, TU)$. In other words, $\mathcal{L}$ is the Dolbeault resolution of the sheaf of holomorphic vector fields on Σ.*

Then,

$$H^i(\mathscr{O}(B\mathcal{L})) = H^*(\Sigma)[-1].$$

Remark: The class in $H^1(\mathscr{O}(B\mathcal{L}))$ corresponding to the class $1 \in H^0(\Sigma)$ leads to a local central extension of $\mathcal{L}$. One can check that the corresponding twisted factorization envelope corresponds to the Virasoro vertex algebra, in the same way that we showed in section **??** that the Kac-Moody extension above leads to the Kac-Moody vertex algebra. $\Diamond$

PROOF. The previous lemma tells us that we need to compute the de Rham cohomology with coefficients in the D_Σ-module $C^*_{red}(J(L))[2]$. Suppose we want to compute the de Rham cohomology with coefficients in any complex M of D_Σ-modules. There is a spectral sequence converging to this cohomology, associated to the filtration on $\Omega^*(\Sigma, M)$ by form degree. The E_2 page of this spectral sequence is the de Rham complex $\Omega^*(\Sigma, \mathcal{H}^*(M))$ with coefficients in the cohomology D_Σ-module $\mathcal{H}^*(M)$.

We will use this spectral sequence in our example. The first step is to compute the cohomology of the D_Σ-module $C^*_{red}(J(\mathcal{L}))$. We will compute the cohomology of the fibres of this sheaf at an arbitrary point $x \in \Sigma$. Let us choose a holomorphic coordinate z at x. The fibre $J_x(\mathcal{L})$ at x is the dg Lie algebra $\mathbb{C}[[z, \overline{z}, \mathrm{d}\overline{z}]]\partial_z$ with differential $\overline{\partial}$. This dg Lie algebra is quasi-isomorphic to the Lie algebra of formal vector fields $\mathbb{C}[[z]]\partial_z$.

A calculation performed by Gelfand-Fuchs [] shows that the reduced Lie algebra cohomology of $\mathbb{C}[[z]]\partial_z$ is concentrated in degree 3, where it is one-dimensional. A cochain representative for the unique non-zero cohomology class is $\partial_z^\vee (z\partial_z)^\vee (z^2\partial_z)^\vee$ where $(z^k\partial_z)^\vee$ refers to the element in $(\mathbb{C}[[z]]\partial_z)^\vee$ in the dual basis.

Thus, we find that the cohomology of $C^*_{red}(J(L))$ is a rank one local system situated in cohomological degree 3. Choosing a formal coordinate at a point in a Riemann surface trivializes the fibre of this line bundle. The trivialization is independent of the coordinate choice, and compatible with the flat connection. From this we deduce that

$$\mathcal{H}^*(C^*_{red}(J(\mathcal{L}))) = C^\infty_\Sigma[-3]$$

is the trivial rank one local system, situated in cohomological degree 3.

Therefore, the cohomology of $\mathscr{O}_{loc}(B\mathcal{L})$ is a shift by -1 of the de Rham cohomology of this trivial flat line bundle, completing the result. $\square$

4.5.5. Cochains with coefficients in a local module for a local L_∞ algebras. Let L be a local L_∞ algebra on M, and let E be a local module for L. Then $J(E)$ has an action of the L_∞ algebra $J(L)$, in a way compatible with the D_M-module on both $J(E)$ and $J(L)$.

4.5.5.1 Definition. *Suppose that E has a local action of L. Then the* local cochains $C^*_{loc}(\mathcal{L}, \mathscr{E})$ *of $\mathcal{L}$ with coefficients in $\mathscr{E}$ is defined to be the flat sections of the D_M-module of cochains of $J(L)$ with coefficients in $J(E)$.*

More explicitly, the D_M-module $C^*(J(L), J(E)$ is

$$\prod_{n\geq 0} \mathrm{Hom}_{C^\infty_M}\left((J(L)[1])^{\otimes n}, J(E)\right)_{S_n},$$

equipped with the usual Chevalley-Eilenberg differential. The sheaf of flat sections of this D_M module is the subsheaf

$$\prod_{n\geq 0} \mathrm{Hom}_{D_M}\left((J(L)[1])^{\otimes n}, J(E)\right)_{S_n},$$

where the maps must be D_M-linear. In light of the fact that

$$\mathrm{Hom}_{D_M}\left(J(L)^{\otimes n}, J(E)\right) = \mathrm{PolyDiff}(\mathcal{L}^{\otimes n}, \mathscr{E}),$$

we see that $C^*_{loc}(\mathcal{L}, \mathscr{E})$ is precisely the subcomplex of the Chevalley-Eilenberg cochain complex

$$C^*(\mathcal{L}, \mathscr{E}) = \prod_{n\geq 0} \mathrm{Hom}_{\mathbb{R}}((\mathcal{L}[1])^{\otimes n}, \mathscr{E})_{S_n}$$

consisting of those cochains built up from polydifferential operators.

CHAPTER 5

The classical Batalin-Vilkovisky formalism

In the preceding chapter we explained how to encode the formal neighborhood of a solution to the Euler-Lagrange equations — a formal elliptic moduli problem — by an elliptic L_∞ algebra. As we explain in this chapter, the elliptic moduli problems arising from action functionals possess even more structure: a shifted symplectic form, so that the formal moduli problem is a derived symplectic space.

Our starting point is the finite-dimensional model that motivates the Batalin-Vilkovisky formalism for classical field theory. With this model in mind, we then develop the relevant definitions in the language of elliptic L_∞ algebras. The end of the chapter is devoted to several examples of classical BV theories, notably *cotangent* field theories, which are the analogs of cotangent bundles in ordinary symplectic geometry.

5.1. The classical BV formalism in finite dimensions

Before we discuss the Batalin-Vilkovisky formalism for classical field theory, we will discuss a finite-dimensional toy model (which we can think of as a 0-dimensional classical field theory). Our model for the space of fields is a finite-dimensional smooth manifold manifold M. The "action functional" is given by a smooth function $S \in C^\infty(M)$. Classical field theory is concerned with solutions to the equations of motion. In our setting, the equations of motion are given by the subspace $\mathrm{Crit}(S) \subset M$. Our toy model will not change if M is a smooth algebraic variety or a complex manifold, or indeed a smooth formal scheme. Thus we will write $\mathscr{O}(M)$ to indicate whatever class of functions (smooth, polynomial, holomorphic, power series) we are considering on M.

If S is not a nice function, then this critical set can by highly singular. The classical Batalin-Vilkovisky formalism tells us to take, instead the *derived* critical locus of S. (Of course, this is exactly what a derived algebraic geometer — see [**Lur09**], [**Toë06**] — would tell us to do as well.) We will explain the essential idea without formulating it precisely inside any particular formalism for derived geometry. For such a treatment, see [**Vez11**].

The critical locus of S is the intersection of the graph

$$\Gamma(\mathrm{d}S) \subset T^*M$$

with the zero-section of the cotangent bundle of M. Algebraically, this means that we can write the algebra $\mathscr{O}(\mathrm{Crit}(S))$ of functions on $\mathrm{Crit}(S)$ as a tensor product

$$\mathscr{O}(\mathrm{Crit}(S)) = \mathscr{O}(\Gamma(\mathrm{d}S)) \otimes_{\mathscr{O}(T^*M)} \mathscr{O}(M).$$

Derived algebraic geometry tells us that the derived critical locus is obtained by replacing this tensor product with a derived tensor product. Thus, the derived critical locus of S, which we denote $\mathrm{Crit}^h(S)$, is an object whose ring of functions is the commutative dg algebra

$$\mathscr{O}(\mathrm{Crit}^h(S)) = \mathscr{O}(\Gamma(\mathrm{d}S)) \otimes^{\mathbb{L}}_{\mathscr{O}(T^*M)} \mathscr{O}(M).$$

In derived algebraic geometry, as in ordinary algebraic geometry, spaces are determined by their algebras of functions. In derived geometry, however, one allows differential-graded algebras as algebras of functions (normally one restricts attention to differential-graded algebras concentrated in non-positive cohomological degrees).

We will take this derived tensor product as a definition of $\mathscr{O}(\mathrm{Crit}^h(S))$.

5.1.1. An explicit model. It is convenient to consider an explicit model for the derived tensor product. By taking a standard Koszul resolution of $\mathscr{O}(M)$ as a module over $\mathscr{O}(T^*M)$, one sees that $\mathscr{O}(\mathrm{Crit}^h(S))$ can be realized as the complex

$$\mathscr{O}(\mathrm{Crit}^h(S)) \simeq \dots \xrightarrow{\vee \mathrm{d}S} \Gamma(M, \wedge^2 TM) \xrightarrow{\vee \mathrm{d}S} \Gamma(M, TM) \xrightarrow{\vee \mathrm{d}S} \mathscr{O}(M).$$

In other words, we can identify $\mathscr{O}(\mathrm{Crit}^h(S))$ with functions on the "graded manifold" $T^*[-1]M$, equipped with the differential given by contracting with the 1-form $\mathrm{d}S$. This notation $T^*[-1]M$ denotes the ordinary smooth manifold M equipped with the graded-commutative algebra $\mathrm{Sym}_{C^\infty_M}(\Gamma(M, TM)[1])$ as its ring of functions.

Note that

$$\mathscr{O}(T^*[-1]M) = \Gamma(M, \wedge^* TM)$$

has a Poisson bracket of cohomological degree 1, called the Schouten-Nijenhuis bracket. This Poisson bracket is characterized by the fact that if $f, g \in \mathscr{O}(M)$ and $X, Y \in \Gamma(M, TM)$, then

$$\begin{aligned} \{X, Y\} &= [X, Y] \\ \{X, f\} &= Xf \\ \{f, g\} &= 0 \end{aligned}$$

and the Poisson bracket between other elements of $\mathscr{O}(T^*[-1]M)$ is inferred from the Leibniz rule.

The differential on $\mathscr{O}(T^*[-1]M)$ corresponding to that on $\mathscr{O}(\mathrm{Crit}^h(S))$ is given by

$$\mathrm{d}\phi = \{S, \phi\}$$

for $\phi \in \mathscr{O}(T^*[-1]M)$.

The derived critical locus of any function thus has a symplectic form of cohomological degree -1. It is manifest in this model and hence can be found in others. In the Batalin-Vilkovisky formalism, the space of fields always has such a symplectic structure. However, one does not require that the space of fields arises as the derived critical locus of a function.

5.2. The classical BV formalism in infinite dimensions

We would like to consider classical field theories in the BV formalism. We have already explained how the language of elliptic moduli problems captures the formal geometry of solutions to a system of PDE. Now we need to discuss the shifted symplectic structures possessed by a derived critical locus. For us, a classical field theory will be specified by an elliptic moduli problem equipped with a symplectic form of cohomological degree -1.

We defined the notion of formal elliptic moduli problem on a manifold M using the language of L_∞ algebras. Thus, in order to give the definition of a classical field theory, we need to understand the following question: what extra structure on an L_∞ algebra $\mathfrak{g}$ endows the corresponding formal moduli problem with a symplectic form?

In order to answer this question, we first need to understand a little about what it means to put a shifted symplectic form on a (formal) derived stack.

In the seminal work of Schwarz [**Sch93**, **AKSZ97**], a definition of a shifted symplectic form on a dg manifold is given. Dg manifolds where an early attempt to develop a theory of derived geometry. It turns out that dg manifolds are sufficient to capture some aspects of the modern theory of derived geometry, including formal derived geometry.

In the world of dg manifolds, as in any model of derived geometry, all spaces of tensors are cochain complexes. In particular, the space of i-forms $\Omega^i(\mathcal{M})$ on a dg manifold is a cochain complex. The differential on this cochain complex is called the internal differential on i-forms. In addition to the internal differential, there is also a de Rham differential $\mathrm{d}_{dR} : \Omega^i(\mathcal{M}) \to \Omega^{i+1}(\mathcal{M})$ which is a cochain map. Schwarz defined a symplectic form on a dg manifold $\mathcal{M}$ to be a two-form ω which is both closed in the differential on the complex of two-forms, and which is also closed under the de Rham differential mapping two-forms to three-forms. A symplectic form is also required to be non-degenerate. The symplectic two-form ω will have some cohomological degree, which for the case relevant to the BV formalsim is -1.

Following these ideas, Pantev et al. [**PTVV11**] give a definition of (shifted) symplectic structure in the more modern language of derived stacks. In this approach, instead of asking that the two-form defining the symplectic structure be closed both in the internal differential on two-forms and closed under the de Rham differential, one constructs a double complex

$$\Omega^{\geq 2} = \Omega^2 \to \Omega^3[-1] \to \dots$$

as the subcomplex of the de Rham complex consisting of 2 and higher forms. One then looks for an element of this double complex which is closed under the total differential (the sum of the de Rham differential and the internal differential on each space of k-forms) and whose 2-form component is non-degenerate in a suitable sense.

However, it turns out that, in the case of formal derived stacks, the definition given by Schwarz and that given by Pantev et al. coincides. One can also show that in this situation there is a Darboux lemma, showing that we can take the symplectic form to have constant coefficients. In order to explain what we mean by this, let us explain how to understand forms on a formal derived stack in terms of the associated L_∞-algebra.

Given a pointed formal moduli problem $\mathcal{M}$, the associated L_∞ algebra $\mathfrak{g}_{\mathcal{M}}$ has the property that

$$\mathfrak{g}_{\mathcal{M}} = T_p\mathcal{M}[-1].$$

Further, we can identify geometric objects on $\mathcal{M}$ in terms of $\mathfrak{g}_{\mathcal{M}}$ as follows.

$C^*(\mathfrak{g}_{\mathcal{M}})$	the algebra $\mathscr{O}(\mathcal{M})$ of functions on $\mathcal{M}$
$\mathfrak{g}_{\mathcal{M}}$-modules	$\mathscr{O}_{\mathcal{M}}$-modules
$C^*(\mathfrak{g}_{\mathcal{M}}, V)$	the $\mathscr{O}_{\mathcal{M}}$-module corresponding to the $\mathfrak{g}_{\mathcal{M}}$-module V
the $\mathfrak{g}_{\mathcal{M}}$-module $g_{\mathcal{M}}[1]$	$T\mathcal{M}$

Following this logic, we see that the complex of 2-forms on $\mathcal{M}$ is identified with $C^*(\mathfrak{g}_{\mathcal{M}}, \wedge^2(\mathfrak{g}_{\mathcal{M}}^\vee[-1]))$.

As we have seen, according to Schwarz, a symplectic form on $\mathcal{M}$ is a two-form on $\mathcal{M}$ which is closed for both the internal and de Rham differentials. Any constant-coefficient two-form is automatically closed under the de Rham differential. A constant-coefficient two-form of degree k is an element of $\mathrm{Sym}^2(\mathfrak{g}_{\mathcal{M}})^\vee$ of cohomological degree $k-2$, i.e. a symmetric pairing on $\mathfrak{g}_{\mathcal{M}}$ of this degree. Such a two-form is closed for the internal differential if and only if it is invariant.

To give a formal pointed moduli problem with a symplectic form of cohomological degree k is the same as to give an L_∞ algebra with an invariant and non-degenerate pairing of cohomological degree $k-2$.

Thus, we find that constant coefficient symplectic two-forms of degree k on $\mathcal{M}$ are precisely the same as non-degenerate symmetric invariant pairings on $\mathfrak{g}_{\mathcal{M}}$. The relation between derived symplectic geometry and invariant pairings on Lie algebras was first developed by Kontsevich [**Kon93**].

The following formal Darboux lemma makes this relationship into an equivalence.

5.2.0.1 Lemma. *Let $\mathfrak{g}$ be a finite-dimensional L_∞ algebra. Then, k-shifted symplectic structures on the formal derived stack $B\mathfrak{g}$ (in the sense of Pantev et al.) are the same as symmetric invariant non-degenerate pairings on $\mathfrak{g}$ of cohomological degree $k-2$.*

The proof is a little technical, and appears in an appendix **??**. The proof of a closely related statement in a non-commutative setting was given by Kontsevich and Soibelman [KS06]. In the statement of the lemma, "the same" means that simplicial sets parametrizing the two objects are canonically equivalent.

Following this idea, we will define a classical field theory to be an elliptic L_∞ algebra equipped with a non-degenerate invariant pairing of cohomological degree -3. Let us first define what it means to have an invariant pairing on an elliptic L_∞ algebra.

5.2.0.2 Definition. *Let M be a manifold, and let E be an elliptic L_∞ algebra on M. An* invariant pairing on E of cohomological degree k *is a symmetric vector bundle map*

$$\langle -,-\rangle_E : E \otimes E \to \mathrm{Dens}(M)[k]$$

satisfying some additional conditions:

1. Non-degeneracy*: we require that this pairing induces a vector bundle isomorphism*
$$E \to E^\vee \otimes \mathrm{Dens}(M)[-3].$$
2. Invariance*: let $\mathscr{E}_c$ denotes the space of compactly supported sections of E. The pairing on E induces an inner product on $\mathscr{E}_c$, defined by*
$$\langle -,-\rangle : \mathscr{E}_c \otimes \mathscr{E}_c \to \mathbb{R}$$
$$\alpha \otimes \beta \to \int_M \langle \alpha, \beta \rangle .$$
We require it to be an invariant pairing on the L_∞ algebra $\mathscr{E}_c$.

Recall that a symmetric pairing on an L_∞ algebra $\mathfrak{g}$ is called invariant if, for all n, the linear map

$$\mathfrak{g}^{\otimes n+1} \to \mathbb{R}$$
$$\alpha_1 \otimes \cdots \otimes \alpha_{n+1} \mapsto \langle l_n(\alpha_1, \ldots, \alpha_n), \alpha_{n+1} \rangle$$

is graded anti-symmetric in the α_i.

5.2.0.3 Definition. *A* formal pointed elliptic moduli problem with a symplectic form of cohomological degree k *on a manifold M is an elliptic L_∞ algebra on M with an invariant pairing of cohomological degree $k-2$.*

5.2.0.4 Definition. *In the BV formalism, a* (perturbative) classical field theory *on M is a formal pointed elliptic moduli problem on M with a symplectic form of cohomological degree -1.*

5.3. The derived critical locus of an action functional

The critical locus of a function f is, of course, the zero locus of the 1-form $\mathrm{d}f$. We are interested in constructing the derived critical locus of a local functional $S \in \mathscr{O}_{loc}(B\mathcal{L})$ on the formal moduli problem associated to a local L_∞ algebra $\mathcal{L}$ on a manifold M. Thus, we need to understand what kind of object the exterior derivative $\mathrm{d}S$ of such an action functional S is.

If $\mathfrak{g}$ is an L_∞ algebra, then we should think of $C^*_{red}(\mathfrak{g})$ as the algebra of functions on the formal moduli problem $B\mathfrak{g}$ that vanish at the base point. Similarly, $C^*(\mathfrak{g}, \mathfrak{g}^\vee[-1])$ should be the thought of as the space of 1-forms on $B\mathfrak{g}$. The exterior derivative is thus a map

$$\mathrm{d} : C^*_{red}(\mathfrak{g}) \to C^*(\mathfrak{g}, \mathfrak{g}^\vee[-1]),$$

namely the universal derivation.

We will define a similar exterior derivative for a local Lie algebra $\mathcal{L}$ on M. The analog of $\mathfrak{g}^\vee$ is the $\mathcal{L}$-module $\mathcal{L}^!$, whose sections are (up to completion) the Verdier dual of the sheaf $\mathcal{L}$. Thus, our exterior derivative will be a map

$$\mathrm{d} : \mathscr{O}_{loc}(B\mathcal{L}) \to C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]).$$

Recall that $\mathscr{O}_{loc}(B\mathcal{L})$ denotes the subcomplex of $C^*_{red}(\mathcal{L}_c(M))$ consisting of local functionals. The exterior derivative for the L_∞ algebra $\mathcal{L}_c(M)$ is a map

$$\mathrm{d} : C^*_{red}(\mathcal{L}_c(M)) \to C^*(\mathcal{L}_c(M), \mathcal{L}_c(M)^\vee[-1]).$$

Note that the dual $\mathcal{L}_c(M)^\vee$ of $\mathcal{L}_c(M)$ is the space $\overline{\mathcal{L}}^!(M)$ of distributional sections of the bundle $L^!$ on M. Thus, the exterior derivative is a map

$$\mathrm{d} : C^*_{red}(\mathcal{L}_c(M)) \to C^*(\mathcal{L}_c(M), \overline{\mathcal{L}}^!(M)[-1]).$$

Note that

$$C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]) \subset C^*(\mathcal{L}_c(M), \mathcal{L}^!(M)) \subset C^*(\mathcal{L}_c(M), \overline{\mathcal{L}}^!(M)).$$

We will now show that d preserves locality and more.

5.3.0.1 Lemma. *The exterior derivative takes the subcomplex $\mathscr{O}_{loc}(B\mathcal{L})$ of $C^*_{red}(\mathcal{L}_c(M))$ to the subcomplex $C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1])$ of $C^*(\mathcal{L}_c(M), \overline{\mathcal{L}}^!(M))$.*

PROOF. The content of this lemma is the familiar statement that the Euler-Lagrange equations associated to a local action functional are differential equations. We will give a formal proof, but the reader will see that we only use integration by parts.

Any functional

$$F \in \mathscr{O}_{loc}(B\mathcal{L})$$

can be written as a sum $F = \sum F_n$ where

$$F_n \in \mathrm{Dens}_M \otimes_{D_M} \mathrm{Hom}_{C^\infty_M} \left(J(L)^{\otimes n}, C^\infty_M \right)_{S_n}.$$

Any such F_n can be written as a finite sum

$$F_n = \sum_i \omega D_1^i \dots D_n^i$$

where ω is a section of Dens_M and D_j^i are differential operators from $\mathcal{L}$ to C^∞_M. (The notation $\omega D_1^i \dots D_n^i$ means simply to multiply the density ω by the outputs of the differential operators, which are smooth functions.)

If we view $F \in \mathscr{O}(\mathcal{L}_c(M))$, then the nth Taylor component of F is the linear map

$$\mathcal{L}_c(M)^{\otimes n} \to \mathbb{R}$$

defined by

$$\phi_1 \otimes \dots \otimes \phi_n \to \sum_i \int_M \omega (D_1^i \phi_1) \dots (D_n^i \phi_n).$$

Thus, the $(n-1)$th Taylor component of $\mathrm{d}F$ is given by the linear map

$$\begin{aligned} \mathrm{d}F_n : \mathcal{L}_c(M)^{\otimes n-1} &\to \overline{L}^!(M) = \mathcal{L}_c(M)^\vee \\ \phi_1 \otimes \dots \otimes \phi_{n-1} \sum_i &\mapsto \omega (D_1^i \phi_1) \dots (D_{n-1}^i \phi_{n-1}) D_n^i(-) + \text{symmetric terms} \end{aligned}$$

where the right hand side is viewed as a linear map from $\mathcal{L}_c(M)$ to $\mathbb{R}$. Now, by integration by parts, we see that

$$(\mathrm{d}F_n)(\phi_1, \dots, \phi_{n-1})$$

is in the subspace $\mathcal{L}^!(M) \subset \overline{L}^!(M)$ of smooth sections of the bundle $L^!(M)$, inside the space of distributional sections.

It is clear from the explicit expressions that the map

$$\mathrm{d}F_n : \mathcal{L}_c(M)^{\otimes n-1} \to \mathcal{L}^!(M)$$

is a polydifferential operator, and so defines an element of $C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1])$ as desired. $\square$

5.3.1. Field theories from action functionals. Physicists normally think of a classical field theory as being associated to an action functional. In this section we will show how to construct a classical field theory in our sense from an action functional.

We will work in a very general setting. Recall (section 4.1.3) that we defined a local L_∞ algebra on a manifold M to be a sheaf of L_∞ algebras where the structure maps are given by differential operators. We will think of a local L_∞ algebra $\mathcal{L}$ on M as defining a formal moduli problem cut out by some differential equations. We will use the notation $B\mathcal{L}$ to denote this formal moduli problem.

We want to take the derived critical locus of a local action functional

$$S \in \mathscr{O}_{loc}(B\mathcal{L})$$

of cohomological degree 0. (We also need to assume that S is at least quadratic: this condition insures that the base-point of our formal moduli problem $B\mathcal{L}$ is a critical point of S). We have seen (section 5.3) how to apply the exterior derivative to a local action functional S yields an element

$$\mathrm{d}S \in C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]),$$

which we think of as being a local 1-form on $B\mathcal{L}$.

The critical locus of S is the zero locus of $\mathrm{d}S$. We thus need to explain how to construct a new local L_∞ algebra that we interpret as being the derived zero locus of $\mathrm{d}S$.

5.3.2. Finite dimensional model. We will first describe the analogous construction in finite dimensions. Let $\mathfrak{g}$ be an L_∞ algebra, M be a $\mathfrak{g}$-module of finite total dimension, and α be a closed, degree zero element of $C^*_{red}(\mathfrak{g}, M)$. The subscript *red* indicates that we are taking the reduced cochain complex, so that α is in the kernel of the augmentation map $C^*(\mathfrak{g}, M) \to M$.

We think of M as a dg vector bundle on the formal moduli problem $B\mathfrak{g}$, and so α is a section of this vector bundle. The condition that α is in the reduced cochain complex translates into the statement that α vanishes at the basepoint of $B\mathfrak{g}$. We are interested in constructing the L_∞ algebra representing the zero locus of α.

We start by writing down the usual Koszul complex associated to a section of a vector bundle. In our context, the commutative dg algebra representing this zero locus of α is given by the total complex of the double complex

$$\cdots \to C^*(\mathfrak{g}, \wedge^2 M^\vee) \xrightarrow{\vee\alpha} C^*(\mathfrak{g}, M^\vee) \xrightarrow{\vee\alpha} C^*(\mathfrak{g}).$$

In words, we have written down the symmetric algebra on the dual of $\mathfrak{g}[1] \oplus M[-1]$. It follows that this commutative dg algebra is the Chevalley-Eilenberg cochain complex of $\mathfrak{g} \oplus M[-2]$, equipped with an L_∞ structure arising from the differential on this complex.

Note that the direct sum $\mathfrak{g} \oplus M[-2]$ (without a differential depending on α) has a natural semi-direct product L_∞ structure, arising from the L_∞ structure on $\mathfrak{g}$ and the action of $\mathfrak{g}$ on $M[-2]$. This L_∞ structure corresponds to the case $\alpha = 0$.

5.3.2.1 Lemma. *The L_∞ structure on $\mathfrak{g} \oplus M[-2]$ describing the zero locus of α is a deformation of the semidirect product L_∞ structure, obtained by adding to the structure maps l_n the maps*

$$\begin{aligned} D_n\alpha : \mathfrak{g}^{\otimes n} &\to M \\ X_1 \otimes \cdots \otimes X_n &\mapsto \frac{\partial}{\partial X_1} \cdots \frac{\partial}{\partial X_n} \alpha. \end{aligned}$$

This is a curved L_∞ algebra unless the section α vanishes at $0 \in \mathfrak{g}$.

PROOF. The proof is a straightforward computation. □

Note that the maps $D_n\alpha$ in the statement of the lemma are simply the homogeneous components of the cochain α.

We will let $Z(\alpha)$ denote $\mathfrak{g} \oplus M[-2]$, equipped with this L_∞ structure arising from α.

Recall that the formal moduli problem $B\mathfrak{g}$ is the functor from dg Artin rings (R, m) to simplicial sets, sending (R, m) to the simplicial set of Maurer-Cartan elements of $\mathfrak{g} \otimes m$. In order to check that we have constructed the correct derived zero locus for α, we should describe the formal moduli problem associated $Z(\alpha)$.

Thus, let (R, m) be a dg Artin ring, and $x \in \mathfrak{g} \otimes m$ be an element of degree 1, and $y \in M \otimes m$ be an element of degree -1. Then (x, y) satisfies the Maurer-Cartan equation in $Z(\alpha)$ if and only if

(1) x satisfies the Maurer-Cartan equation in $\mathfrak{g} \otimes m$ and
(2) $\alpha(x) = \mathrm{d}_x y \in M$, where
$$\mathrm{d}_x = \mathrm{d}y + \mu_1(x, y) + \tfrac{1}{2!}\mu_2(x, x, y) + \cdots : M \to M$$
is the differential obtained by deforming the original differential by that arising from the Maurer-Cartan element x. (Here $\mu_n : \mathfrak{g}^{\otimes n} \otimes M \to M$ are the action maps.)

In other words, we see that an R-point of $BZ(\alpha)$ is both an R-point x of $B\mathfrak{g}$ and a homotopy between $\alpha(x)$ and 0 in the fiber M_x of the bundle M at $x \in B\mathfrak{g}$. The fibre M_x is the cochain complex M with differential d_x arising from the solution x to the Maurer-Cartan equation. Thus, we are described the homotopy fiber product between the section α and the zero section in the bundle M, as desired.

Let us make thigs

5.3.3. The derived critical locus of a local functional. Let us now return to the situation where $\mathcal{L}$ is a local L_∞ algebra on a manifold M and $S \in \mathscr{O}(B\mathcal{L})$ is a local functional that is at least quadratic. Let
$$\mathrm{d}S \in C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1])$$
denote the exterior derivative of S. Note that $\mathrm{d}S$ is in the reduced cochain complex, i.e. the kernel of the augmentation map $C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]) \to \mathcal{L}^![-1]$.

Let
$$\mathrm{d}_n S : \mathcal{L}^{\otimes n} \to \mathcal{L}^!$$

be the nth Taylor component of $\mathrm{d}S$. The fact that $\mathrm{d}S$ is a local cochain means that $\mathrm{d}_n S$ is a polydifferential operator.

5.3.3.1 Definition. *The* derived critical locus of S *is the local L_∞ algebra obtained by adding the maps*

$$\mathrm{d}_n S : \mathcal{L}^{\otimes n} \to \mathcal{L}^!$$

to the structure maps l_n of the semi-direct product L_∞ algebra $\mathcal{L} \oplus \mathcal{L}^![-3]$. We denote this local L_∞ algebra by $\mathrm{Crit}(S)$*.*

If (R, m) is an auxiliary Artinian dg ring, then a solution to the Maurer-Cartan equation in $\mathrm{Crit}(S) \otimes m$ consists of the following data:

(1) a Maurer-Cartan element $x \in \mathcal{L} \otimes m$ and
(2) an element $y \in \mathcal{L}^! \otimes m$ such that

$$(\mathrm{d}S)(x) = \mathrm{d}_x y.$$

Here $\mathrm{d}_x y$ is the differential on $\mathcal{L}^! \otimes m$ induced by the Maurer-Cartan element x. These two equations say that x is an R-point of $B\mathcal{L}$ that satisfies the Euler-Lagrange equations up to a homotopy specified by y.

5.3.4. Symplectic structure on the derived critical locus. Recall that a classical field theory is given by a local L_∞ algebra that is elliptic and has an invariant pairing of degree -3. The pairing on the local L_∞ algebra $\mathrm{Crit}(S)$ constructed above is evident: it is given by the natural bundle isomorphism

$$(L \oplus L^![-3])^![-3] \cong L^![-3] \oplus L.$$

In other words, the pairing arises, by a shift, from the natural bundle map $L \otimes L^! \to \mathrm{Dens}_M$.

5.3.4.1 Lemma. *This pairing on* $\mathrm{Crit}(S)$ *is invariant.*

PROOF. The original L_∞ structure on $\mathcal{L} \oplus \mathcal{L}^![-3]$ (that is, the L_∞ structure not involving S) is easily seen to be invariant. We will verify that the deformation of this structure coming from S is also invariant.

We need to show that if

$$\alpha_1, \ldots, \alpha_{n+1} \in \mathcal{L}_c \oplus \mathcal{L}_c^![-3]$$

are compactly supported sections of $L \oplus L^1[-3]$, then

$$\langle l_n(\alpha_1, \ldots, \alpha_n), \alpha_{n+1} \rangle$$

is totally antisymmetric in the variables α_i. Now, the part of this expression that comes from S is just

$$\left(\frac{\partial}{\partial \alpha_1} \cdots \frac{\partial}{\partial \alpha_{n+1}}\right) S(0).$$

The fact that partial derivatives commute — combined with the shift in grading due to $C^*(\mathcal{L}_c) = \mathscr{O}(\mathcal{L}_c[1])$ — immediately implies that this term is totally antisymmetric. $\square$

Note that, although the local L_∞ algebra $\mathrm{Crit}(S)$ always has a symplectic form, it does not always define a classical field theory, in our sense. To be a classical field theory, we also require that the local L_∞ algebra $\mathrm{Crit}(S)$ is elliptic.

5.4. A succinct definition of a classical field theory

We defined a classical field theory to be a formal elliptic moduli problem equipped with a symplectic form of degree -1. In this section we will rewrite this definition in a more concise (but less conceptual) way. This version is included largely for consistency with [**Cos11b**] — where the language of elliptic moduli problems is not used — and for ease of reference when we discuss the quantum theory.

5.4.0.1 Definition. *Let E be a graded vector bundle on a manifold M. A* degree -1 symplectic structure *on E is an isomorphism of graded vector bundles*

$$\phi : E \cong E^![-1]$$

that is anti-symmetric, in the sense that $\phi^* = -\phi$ where ϕ^* is the formal adjoint of ϕ.

Note that if L is an elliptic L_∞ algebra on M with an invariant pairing of degree -3, then the graded vector bundle $L[1]$ on M has a -1 symplectic form. Indeed, by definition, L is equipped with a symmetric isomorphism $L \cong L^![-3]$, which becomes an antisymmetric isomorphism $L[1] \cong (L[1])^![-1]$.

Note also that the tangent space at the basepoint to the formal moduli problem $B\mathcal{L}$ associated to $\mathcal{L}$ is $\mathcal{L}[1]$ (equipped with the differential induced from that on $\mathcal{L}$). Thus, the algebra $C^*(\mathcal{L})$ of cochains of $\mathcal{L}$ is isomorphic, as a graded algebra without the differential, to the algebra $\mathscr{O}(\mathcal{L}[1])$ of functionals on $\mathcal{L}[1]$.

Now suppose that E is a graded vector bundle equipped with a -1 symplectic form. Let $\mathscr{O}_{loc}(\mathscr{E})$ denote the space of local functionals on $\mathscr{E}$, as defined in section 4.5.1.

5.4.0.2 Proposition. *For E a graded vector bundle equipped with a -1 symplectic form, let $\mathscr{O}_{loc}(\mathscr{E})$ denote the space of local functionals on $\mathscr{E}$. Then we have the following.*

(1) *The symplectic form on $\mathscr{E}$ induces a Poisson bracket on $\mathscr{O}_{loc}(\mathscr{E})$, of degree* $+1$.

(2) *Equipping* $E[-1]$ *with a local* L_∞ *algebra structure compatible with the given pairing on* $E[-1]$ *is equivalent to picking an element* $S \in \mathscr{O}_{loc}(\mathscr{E})$ *that has cohomological degree* 0*, is at least quadratic, and satisfies the classical master equation*

$$\{S, S\} = 0.$$

PROOF. Let $L = E[-1]$. Note that L is a local L_∞ algebra, with the zero differential and zero higher brackets (i.e., a totally abelian L_∞ algebra). We write $\mathscr{O}_{loc}(B\mathcal{L})$ or $C^*_{red,loc}(\mathcal{L})$ for the reduced local cochains of $\mathcal{L}$. This is a complex with zero differential which coincides with $\mathscr{O}_{loc}(\mathscr{E})$.

We have seen that the exterior derivative (section 5.3) gives a map

$$\mathrm{d} : \mathscr{O}_{loc}(\mathscr{E}) = \mathscr{O}_{loc}(B\mathcal{L}) \to C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]).$$

Note that the isomorphism

$$\mathcal{L} \cong \mathcal{L}^![-3]$$

gives an isomorphism

$$C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]) \cong C^*_{loc}(\mathcal{L}, \mathcal{L}[2]).$$

Finally, $C^*_{loc}(\mathcal{L}, \mathcal{L}[2])$ is the L_∞ algebra controlling deformations of $\mathcal{L}$ as a local L_∞ algebra. It thus remains to verify that $\mathscr{O}_{loc}(B\mathcal{L}) \subset C^*_{loc}(\mathcal{L}, \mathcal{L}[2])$ is a sub L_∞ algebra, which is straightforward. □

Note that the finite-dimensional analog of this statement is simply the fact that on a formal symplectic manifold, all symplectic derivations (which correspond, after a shift, to deformations of the formal symplectic manifold) are given by Hamiltonian functions, defined up to the addition of an additive constant. The additive constant is not mentioned in our formulation because $\mathscr{O}_{loc}(\mathscr{E})$, by definition, consists of functionals without a constant term.

Thus, we can make a concise definition of a field theory.

5.4.0.3 Definition. *A* pre-classical field theory *on a manifold* M *consists of a graded vector bundle* E *on* M*, equipped with a symplectic pairing of degree* -1*, and a local functional*

$$S \in \mathscr{O}_{loc}(\mathscr{E}_c(M))$$

of cohomological degree 0*, satisfying the following properties.*

(1) *S satisfies the classical master equation* $\{S, S\} = 0$.
(2) *S is at least quadratic (so that* $0 \in \mathscr{E}_c(M)$ *is a critical point of* S*).*

In this situation, we can write S as a sum (in a unique way)

$$S(e) = \langle e, Qe \rangle + I(e)$$

where $Q : \mathscr{E} \to \mathscr{E}$ is a skew self-adjoint differential operator of cohomological degree 1 and square zero.

5.4.0.4 Definition. *A pre-classical field is a* classical field theory *if the complex* $(\mathscr{E}, Q)$ *is elliptic.*

There is one more property we need of a classical field theories in order to be apply the quantization machinery of [**Cos11b**].

5.4.0.5 Definition. *A* gauge fixing operator *is a map*

$$Q^{GF} : \mathscr{E}(M) \to \mathscr{E}(M)$$

that is a differential operator of cohomological degree -1 *such that* $(Q^{GF})^2 = 0$ *and*

$$[Q, Q^{GF}] : \mathscr{E}(M) \to \mathscr{E}(M)$$

is a generalized Laplacian in the sense of [**BGV92**].

The only classical field theories we will try to quantize are those that admit a gauge fixing operator. Thus, we will only consider classical field theories which have a gauge fixing operator. An important point which will be discussed at length in the chapter on quantum field theory is the fact that the observables of the quantum field theory are independent (up to homotopy) of the choice of gauge fixing condition.

5.5. Examples of field theories from action functionals

Let us now give some basic examples of field theories arising as the derived critical locus of an action functional. We will only discuss scalar field theories in this section.

Let (M, g) be a Riemannian manifold. Let $\underline{\mathbb{R}}$ be the trivial line bundle on M and $Dens_M$ the density line bundle. Note that the volume form $\mathrm{d}Vol_g$ provides an isomorphism between these line bundles. Let

$$S(\phi) = \tfrac{1}{2} \int_M \phi \, \mathrm{D} \, \phi$$

denote the action functional for the free massless field theory on M. Here D is the Laplacian on M, viewed as a differential operator from $C^\infty(M)$ to $\mathrm{Dens}(M)$, so $D\phi = (\Delta_g \phi)\mathrm{d}Vol_g$.

The derived critical locus of S is described by the elliptic L_∞ algebra

$$\mathcal{L} = C^\infty(M)[-1] \xrightarrow{\mathrm{D}} \mathrm{Dens}(M)[-2]$$

where $\mathrm{Dens}(M)$ is the global sections of the bundle of densities on M. Thus, $C^\infty(M)$ is situated in degree 1, and the space $\mathrm{Dens}(M)$ is situated in degree 2. The pairing between $\mathrm{Dens}(M)$ and $C^\infty(M)$ gives the invariant pairing on $\mathcal{L}$, which is symmetric of degree -3 as desired.

5.5.1. Interacting scalar field theories. Next, let us write down the derived critical locus for a basic interacting scalar field theory, given by the action functional

$$S(\phi) = \frac{1}{2}\int_M \phi\, \mathrm{D}\,\phi + \frac{1}{4!}\int_M \phi^4.$$

The cochain complex underlying our elliptic L_∞ algebra is, as before,

$$\mathcal{L} = C^\infty(M)[-1] \xrightarrow{\mathrm{D}} \mathrm{Dens}(M)[-2].$$

The interacting term $\frac{1}{4!}\int_M \phi^4$ gives rise to a higher bracket l_3 on $\mathcal{L}$, defined by the map

$$\begin{aligned} C^\infty(M)^{\otimes 3} &\to \mathrm{Dens}(M) \\ \phi_1 \otimes \phi_2 \otimes \phi_3 &\mapsto \phi_1\phi_2\phi_3 \mathrm{d}Vol_g. \end{aligned}$$

Let (R, m) be a nilpotent Artinian ring, concentrated in degree 0. Then a section of $\phi \in C^\infty(M) \otimes m$ satisfies the Maurer-Cartan equation in this L_∞ algebra if and only if

$$\mathrm{D}\,\phi + \frac{1}{3!}\phi^3 \mathrm{d}Vol = 0.$$

Note that this is precisely the Euler-Lagrange equation for S. Thus, the formal moduli problem associated to $\mathcal{L}$ is, as desired, the derived version of the moduli of solutions to the Euler-Lagrange equations for S.

5.6. Cotangent field theories

We have defined a field theory to be a formal elliptic moduli problem equipped with a symplectic form of degree -1. In geometry, cotangent bundles are the basic examples of symplectic manifolds. We can apply this construction in our setting: given any elliptic moduli problem, we will produce a new elliptic moduli problem – its shifted cotangent bundle – that has a symplectic form of degree -1. We call the field theories that arise by this construction *cotangent field theories*. It turns out that a surprising number of field theories of interest in mathematics and physics arise as cotangent theories, including, for example, both the A- and the B-models of mirror symmetry and their half-twisted versions.

We should regard cotangent field theories as the simplest and most basic class of non-linear field theories, just as cotangent bundles are the simplest class of symplectic manifolds. One can show, for example, that the phase space of a cotangent field theory is always an (infinite-dimensional) cotangent bundle, whose classical Hamiltonian function is linear on the cotangent fibers.

5.6.1. The cotangent bundle to an elliptic moduli problem. Let $\mathcal{L}$ be an elliptic L_∞ algebra on a manifold X, and let $\mathcal{M}_{\mathcal{L}}$ be the associated elliptic moduli problem.

Let $L^!$ be the bundle $L^\vee \otimes \mathrm{Dens}(X)$. Note that there is a natural pairing between compactly supported sections of L and compactly supported sections of $L^!$.

Recall that we use the notation $\mathcal{L}$ to denote the space of sections of L. Likewise, we will let $\mathcal{L}^!$ denote the space of sections of $L^!$.

5.6.1.1 Definition. *Let $T^*[k]B\mathcal{L}$ denote the elliptic moduli problem associated to the elliptic L_∞ algebra $\mathcal{L} \oplus \mathcal{L}^![k-2]$.*

This elliptic L_∞ algebra has a pairing of cohomological degree $k-2$.

The L_∞ structure on the space $\mathcal{L} \oplus \mathcal{L}^![k-2]$ of sections of the direct sum bundle $L \oplus L^![k-2]$ arises from the natural $\mathcal{L}$-module structure on $\mathcal{L}^!$.

5.6.1.2 Definition. *Let $\mathcal{M} = B\mathcal{L}$ be an elliptic moduli problem corresponding to an elliptic L_∞ algebra $\mathcal{L}$. Then the* cotangent field theory *associated to $\mathcal{M}$ is the -1-symplectic elliptic moduli problem $T^*[-1]\mathcal{M}$, whose elliptic L_∞ algebra is $\mathcal{L} \oplus \mathcal{L}^![-3]$.*

5.6.2. Examples. In this section we will list some basic examples of cotangent theories, both gauge theories and nonlinear sigma models.

In order to make the discussion more transparent, we will not explicitly describe the elliptic L_∞ algebra related to every elliptic moduli problem we describe. Instead, we may simply define the elliptic moduli problem in terms of the geometric objects it classifies. In all examples, it is straightforward using the techniques we have discussed so far to write down the elliptic L_∞ algebra describing the formal neighborhood of a point in the elliptic moduli problems we will consider.

5.6.3. Self-dual Yang-Mills theory. Let X be an oriented 4-manifold equipped with a conformal class of a metric. Let G be a compact Lie group. Let $\mathcal{M}(X,G)$ denote the elliptic moduli problem parametrizing principal G-bundles on X with a connection whose curvature is self-dual.

Then we can consider the cotangent theory $T^*[-1]\mathcal{M}(X,G)$. This theory is known in the physics literature as *self-dual Yang-Mills theory*.

Let us describe the L_∞ algebra of this theory explicitly. Observe that the elliptic L_∞ algebra describing the completion of $\mathcal{M}(X,G)$ near a point (P,∇) is

$$\Omega^0(X,\mathfrak{g}_P) \xrightarrow{\mathrm{d}_\nabla} \Omega^1(X,\mathfrak{g}_P) \xrightarrow{\mathrm{d}_-} \Omega^2_-(X,\mathfrak{g}_P)$$

where $\mathfrak{g}_P$ is the adjoint bundle of Lie algebras associated to the principal G-bundle P. Here d_- denotes the connection followed by projection onto the anti-self-dual 2-forms.

Thus, the elliptic L_∞ algebra describing $T^*[-1]\mathcal{M}$ is given by the diagram

$$\begin{array}{ccccccc} \Omega^0(X,\mathfrak{g}_P) & \xrightarrow{\mathrm{d}_\nabla} & \Omega^1(X,\mathfrak{g}_P) & \xrightarrow{\mathrm{d}_-} & \Omega^2_-(X,\mathfrak{g}_P) & & \\ & & \oplus & & \oplus & & \\ & & \Omega^2_-(X,\mathfrak{g}_P) & \xrightarrow{\mathrm{d}_\nabla} & \Omega^3(X,\mathfrak{g}_P) & \xrightarrow{\mathrm{d}_\nabla} & \Omega^4(X,\mathfrak{g}_P) \end{array}$$

This is a standard presentation of the fields of self-dual Yang-Mills theory in the BV formalism (see [CCRF+98] and [Cos11b]). Note that it is, in fact, a dg Lie algebra, so there are no nontrivial higher brackets.

Ordinary Yang-Mills theory arises as a deformation of the self-dual theory. One simply deforms the differential in the diagram above by including a term that is the identity from $\Omega^2_-(X,\mathfrak{g}_P)$ in degree 1 to the copy of $\Omega^2_-(X,\mathfrak{g}_P)$ situated in degree 2.

5.6.4. The holomorphic σ-model. Let E be an elliptic curve and let X be a complex manifold. Let $\mathcal{M}(E,X)$ denote the elliptic moduli problem parametrizing holomorphic maps from $E \to X$. As before, there is an associated cotangent field theory $T^*[-1]\mathcal{M}(E,X)$. (In [Cos11a] it is explained how to describe the formal neighborhood of any point in this mapping space in terms of an elliptic L_∞ algebra on E.)

In [Cos10], this field theory was called a holomorphic Chern-Simons theory, because of the formal similarities between the action functional of this theory and that of the holomorphic Chern-Simons gauge theory. In the physics literature ([Wit05], [Kap05]) this theory is known as the twisted $(0,2)$ supersymmetric sigma model, or as the curved $\beta-\gamma$ system.

This theory has an interesting role in both mathematics and physics. For instance, it was shown in [Cos10, Cos11a] that the partition function of this theory (at least, the part which discards the contributions of non-constant maps to X) is the Witten genus of X.

5.6.5. Twisted supersymmetric gauge theories. Of course, there are many more examples of cotangent theories, as there are very many elliptic moduli problems. In [Cos13], it is shown how twisted versions of supersymmetric gauge theories can be written as cotangent theories. We will focus on holomorphic (or minimal) twists. Holomorphic twists are richer than the more well-studied topological twists, but contain less information than the full untwisted supersymmetric theory. As explained in [Cos13], one can obtain topological twists from holomorphic twists by applying a further twist.

The most basic example is the twisted $\mathcal{N}=1$ field theory. If X is a complex surface and G is a complex Lie group, then the $\mathcal{N}=1$ twisted theory is simply the cotangent theory to the elliptic moduli problem of holomorphic principal G-bundles on X. If we fix a principal G-bundle $P \to X$, then the elliptic L_∞ algebra describing this formal moduli problem near P is

$$\Omega^{0,*}(X, \mathfrak{g}_P),$$

where $\mathfrak{g}_P$ is the adjoint bundle of Lie algebras associated to P. It is a classic result of Kodaira and Spencer that this dg Lie algebra describes deformations of the holomorphic principal bundle P.

The cotangent theory to this elliptic moduli problem is thus described by the elliptic L_∞ algebra

$$\Omega^{0,*}(X, \mathfrak{g}_P \oplus \mathfrak{g}_P^\vee \otimes K_X[-1].).$$

Note that K_X denotes the canonical line bundle, which is the appropriate holomorphic substitute for the smooth density line bundle.

5.6.6. The twisted $\mathscr{N} = 2$ theory. Twisted versions of gauge theories with more supersymmetry have similar descriptions, as is explained in [**Cos13**]. The $\mathscr{N} = 2$ theory is the cotangent theory to the elliptic moduli problem for holomorphic G-bundles $P \to X$ together with a holomorphic section of the adjoint bundle $\mathfrak{g}_P$. The underlying elliptic L_∞ algebra describing this moduli problem is

$$\Omega^{0,*}(X, \mathfrak{g}_P + \mathfrak{g}_P[-1]).$$

Thus, the cotangent theory has

$$\Omega^{0,*}(X, \mathfrak{g}_P + \mathfrak{g}_P[-1] \oplus \mathfrak{g}_P^\vee \otimes K_X \oplus \mathfrak{g}_P^\vee \otimes K_X[-1])$$

for its elliptic L_∞ algebra.

5.6.7. The twisted $\mathscr{N} = 4$ theory. Finally, we will describe the twisted $\mathscr{N} = 4$ theory. There are two versions of this twisted theory: one used in the work of Vafa-Witten [**VW94**] on S-duality, and another by Kapustin-Witten [**KW06**] in their work on geometric Langlands. Here we will describe only the latter.

Let X again be a complex surface and G a complex Lie group. Then the twisted $\mathscr{N} = 4$ theory is the cotangent theory to the elliptic moduli problem describing principal G-bundles $P \to X$, together with a holomorphic section $\phi \in H^0(X, T^*X \otimes \mathfrak{g}_P)$ satisfying

$$[\phi, \phi] = 0 \in H^0(X, K_X \otimes g_P).$$

Here T^*X is the holomorphic cotangent bundle of X.

The elliptic L_∞ algebra describing this is

$$\Omega^{0,*}(X, \mathfrak{g}_P \oplus T^*X \otimes \mathfrak{g}_P[-1] \oplus K_X \otimes \mathfrak{g}_P[-2]).$$

Of course, this elliptic L_∞ algebra can be rewritten as

$$(\Omega^{*,*}(X, \mathfrak{g}_P), \bar{\partial}),$$

where the differential is just $\overline{\partial}$ and does not involve ∂. The Lie bracket arises from extending the Lie bracket on $\mathfrak{g}_P$ by tensoring with the commutative algebra structure on the algebra $\Omega^{*,*}(X)$ of forms on X.

Thus, the corresponding cotangent theory has

$$\Omega^{*,*}(X, \mathfrak{g}_P) \oplus \Omega^{*,*}(X, \mathfrak{g}_P)[1]$$

for its elliptic Lie algebra.

CHAPTER 6

The observables of a classical field theory

So far we have given a definition of a classical field theory, combining the ideas of derived deformation theory and the classical BV formalism. Our goal in this chapter is to show that the observables for such a theory do indeed form a commutative factorization algebra, denoted Obs^{cl}, and to explain how to equip it with a shifted Poisson bracket. The first part is straightforward — implicitly, we have already done it! — but the Poisson bracket is somewhat subtle, due to complications that arise when working with infinite-dimensional vector spaces. We will exhibit a sub-factorization algebra $\widetilde{\mathrm{Obs}}^{cl}$ of Obs^{cl} which is equipped with a commutative product and Poisson bracket, and such that the inclusion map $\widetilde{\mathrm{Obs}}^{cl} \to \mathrm{Obs}^{cl}$ is a quasi-isomorphism.

6.1. The factorization algebra of classical observables

We have given two descriptions of a classical field theory, and so we provide the two descriptions of the associated observables.

Let $\mathcal{L}$ be the elliptic L_∞ algebra of a classical field theory on a manifold M. Thus, the associated elliptic moduli problem is equipped with a symplectic form of cohomological degree -1.

6.1.0.1 Definition. *The* observables with support in the open subset U *is the commutative dg algebra*

$$\mathrm{Obs}^{cl}(U) = C^*(\mathcal{L}(U)).$$

The factorization algebra of observables *for this classical field theory, denoted* Obs^{cl}*, assigns the cochain complex* $\mathrm{Obs}^{cl}(U)$ *to the open* U*.*

The interpretation of this definition should be clear from the preceding chapters. The elliptic L_∞ algebra $\mathcal{L}$ encodes the space of solutions to the Euler-Lagrange equations for the theory (more accurately, the formal neighborhood of the solution given by the base-point of the formal moduli problem). Its Chevalley-Eilenberg cochains $C^*(\mathcal{L}(U))$ on the open U are interpreted as the algebra of functions on the space of solutions over the open U.

By the results of section **??**, we know that this construction is in fact a factorization algebra.

We often call Obs^{cl} simply the *classical observables*, in contrast to the factorization algebras of some quantization, which we will call the quantum observables.

Alternatively, let E be a graded vector bundle on M, equipped with a symplectic pairing of degree -1 and a local action functional S which satisfies the classical master equation. As we explained in section 5.4 this data is an alternative way of describing a classicla field theory. The bundle L whose sections are the local L_∞ algebra $\mathcal{L}$ is $E[-1]$.

6.1.0.2 Definition. *The* observables with support in the open subset U *is the commutative dg algebra*

$$\mathrm{Obs}^{cl}(U) = \mathscr{O}(\mathscr{E}(U)),$$

equipped with the differential $\{S, -\}$.

The factorization algebra of observables *for this classical field theory, denoted* Obs^{cl}*, assigns the cochain complex* $\mathrm{Obs}^{cl}(U)$ *to the open* U.

Recall that the operator $\{S, -\}$ is well-defined because the bracket with the local functional is always well-defined.

The underlying graded-commutative algebra of $\mathrm{Obs}^{cl}(U)$ is manifestly the functions on the fields $\mathscr{E}(U)$ over the open set U. The differential imposes the relations between observables arising from the Euler-Lagrange equations for S. In physical language, we are giving a cochain complex whose cohomology is the "functions on the fields that are on-shell."

It is easy to check that this definition of classical observables coincides with the one in terms of cochains of the sheaf of L_∞-algebras $\mathcal{L}(U)$.

6.2. The graded Poisson structure on classical observables

Recall the following definition.

6.2.0.1 Definition. *A P_0 algebra (in the category of cochain complexes) is a commutative differential graded algebra together with a Poisson bracket $\{-,-\}$ of cohomological degree* 1*, which satisfies the Jacobi identity and the Leibniz rule.*

The main result of this chapter is the following.

6.2.0.2 Theorem. *For any classical field theory (section 5.4) on M, there is a P_0 factorization algebra $\widetilde{\mathrm{Obs}}^{cl}$, together with a weak equivalence of commutative factorization algebras.*

$$\widetilde{\mathrm{Obs}}^{cl} \cong \mathrm{Obs}^{cl}.$$

Concretely, $\widetilde{\mathrm{Obs}}^{cl}(U)$ is built from functionals on the space of solutions to the Euler-Lagrange equations that have more regularity than the functionals in $\mathrm{Obs}^{cl}(U)$.

The idea of the definition of the P_0 structure is very simple. Let us start with a finite-dimensional model. Let $\mathfrak{g}$ be an L_∞ algebra equipped with an invariant antisymmetric element $P \in \mathfrak{g} \otimes \mathfrak{g}$ of cohomological degree 3. This element can be viewed (according to the correspondence between formal moduli problems and Lie algebras given in section 4.1) as a bivector on $B\mathfrak{g}$, and so it defines a Poisson bracket on $\mathscr{O}(B\mathfrak{g}) = C^*(\mathfrak{g})$. Concretely, this Poisson bracket is defined, on the generators $\mathfrak{g}^\vee[-1]$ of $C^*(\mathfrak{g})$, as the map

$$\mathfrak{g}^\vee \otimes \mathfrak{g}^\vee \to \mathbb{R}$$

determined by the tensor P.

Now let $\mathcal{L}$ be an elliptic L_∞ algebra describing a classical field theory. Then the kernel for the isomorphism $\mathcal{L}(U) \cong \mathcal{L}^!(U)[-3]$ is an element $P \in \overline{\mathcal{L}}(U) \otimes \overline{\mathcal{L}}(U)$, which is symmetric, invariant, and of degree 3.

We would like to use this idea to define the Poisson bracket on

$$\mathrm{Obs}^{cl}(U) = C^*(\mathcal{L}(U)).$$

As in the finite dimensional case, in order to define such a Poisson bracket, we would need an invariant tensor in $\mathcal{L}(U)^{\otimes 2}$. The tensor representing our pairing is instead in $\overline{\mathcal{L}}(U)^{\otimes 2}$, which contains $\mathcal{L}(U)^{\otimes 2}$ as a dense subspace. In other words, we run into a standard problem in analysis: our construction in finite-dimensional vector spaces does not port immediately to infinite-dimensional vector spaces.

We solve this problem by finding a subcomplex

$$\widetilde{\mathrm{Obs}}^{cl}(U) \subset \mathrm{Obs}^{cl}(U)$$

such that the Poisson bracket *is* well-defined on the subcomplex and the inclusion is a weak equivalence. Up to quasi-isomorphism, then, we have the desired Poisson structure.

6.3. The Poisson structure for free field theories

In this section, we will construct a P_0 structure on the factorization algebra of observables of a free field theory. More precisely, we will construct for every open subset U, a subcomplex

$$\widetilde{\mathrm{Obs}}^{cl}(U) \subset \mathrm{Obs}^{cl}(U)$$

of the complex of classical observables such that

(1) $\widetilde{\mathrm{Obs}}^{cl}$ forms a sub-commutative factorization algebra of Obs^{cl};
(2) the inclusion $\widetilde{\mathrm{Obs}}^{cl}(U) \subset \mathrm{Obs}^{cl}(U)$ is a weak equivalence of differentiable pro-cochain complexes for every open set U; and
(3) $\widetilde{\mathrm{Obs}}^{cl}$ has the structure of P_0 factorization algebra.

The complex $\mathrm{Obs}^{cl}(U)$ consists of a product over all n of certain distributional sections of a vector bundle on U^n. The complex $\widetilde{\mathrm{Obs}}^{cl}$ is defined by considering instead smooth sections on U^n of the same vector bundle.

Let us now make this definition more precise. Recall that a free field theory is a classical field theory associated to an elliptic L_∞ algebra $\mathcal{L}$ that is abelian, i.e., where all the brackets $\{l_n \mid n \geq 2\}$ vanish.

Thus, let L be the graded vector bundle associated to an abelian elliptic L_∞ algebra, and let $\mathcal{L}(U)$ be the elliptic complex of sections of L on U. To say that L defines a field theory means we have a symmetric isomorphism $\mathcal{L} \cong \mathcal{L}^![-3]$.

Recall (section **??**) that we use the notation $\overline{\mathcal{L}}(U)$ to denote the space of distributional sections of L on U. A lemma of Atiyah-Bott (section **??**) shows that the inclusion

$$\mathcal{L}(U) \hookrightarrow \overline{\mathcal{L}}(U)$$

is a continuous homotopy equivalence of topological cochain complexes.

It follows that the natural map

$$C^*(\overline{\mathcal{L}}(U)) \hookrightarrow C^*(\mathcal{L}(U))$$

is a cochain homotopy equivalence. Indeed, because we are dealing with an abelian L_∞ algebra, the Chevalley-Eilenberg cochains become quite simple:

$$C^*(\mathcal{L}(U)) = \widehat{\mathrm{Sym}}(\mathcal{L}(U)^\vee[-1]),$$
$$C^*(\overline{\mathcal{L}}(U)) = \widehat{\mathrm{Sym}}(\overline{\mathcal{L}}(U)^\vee[-1]),$$

where, as always, the symmetric algebra is defined using the completed tensor product. The differential is simply the differential on, for instance, $\mathcal{L}(U)^\vee$ extended as a derivation, so that we are simply taking the completed symmetric algebra of a complex. The complex $C^*(\mathcal{L}(U))$ is built from distributional sections of the bundle $(L^!)^{\boxtimes n}[-n]$ on U^n, and the complex $C^*(\overline{\mathcal{L}}(U))$ is built from smooth sections of the same bundle.

Note that

$$\mathcal{L}(U)^\vee = \overline{\mathcal{L}}_c^!(U) = \overline{\mathcal{L}}_c(U)[3].$$

Thus,

$$C^*(\mathcal{L}(U)) = \widehat{\mathrm{Sym}}(\overline{\mathcal{L}}_c(U)[2]),$$
$$C^*(\overline{\mathcal{L}}(U)) = \widehat{\mathrm{Sym}}(\mathcal{L}_c(U)[2]).$$

We can define a Poisson bracket of degree 1 on $C^*(\overline{\mathcal{L}}(U))$ as follows. On the generators $\mathcal{L}_c(U)[2]$, it is defined to be the given pairing

$$\langle -, - \rangle : \mathcal{L}_c(U) \times \mathcal{L}_c(U) \to \mathbb{R},$$

since we *can* pair smooth sections. This pairing extends uniquely, by the Leibniz rule, to continuous bilinear map

$$C^*(\overline{\mathcal{L}}(U)) \times C^*(\overline{\mathcal{L}}(U)) \to C^*(\overline{\mathcal{L}}(U)).$$

In particular, we see that $C^*(\overline{\mathcal{L}}(U))$ has the structure of a P_0 algebra in the multicategory of differentiable cochain complexes.

Let us define the modified observables in this theory by

$$\widetilde{\mathrm{Obs}}^{cl}(U) = C^*(\overline{\mathcal{L}}(U)).$$

We have seen that $\widetilde{\mathrm{Obs}}^{cl}(U)$ is homotopy equivalent to $\mathrm{Obs}^{cl}(U)$ and that $\widetilde{\mathrm{Obs}}^{cl}(U)$ has a P_0 structure.

6.3.0.1 Lemma. $\mathrm{Obs}^{cl}(U)$ *has the structure of a P_0 factorization algebra.*

PROOF. It remains to verify that if $U_1, \dots, U_n$ are disjoint open subsets of M, each contained in an open subset W, then the map

$$\widetilde{\mathrm{Obs}}^{cl}(U_1) \times \cdots \times \widetilde{\mathrm{Obs}}^{cl}(U_n) \to \widetilde{\mathrm{Obs}}^{cl}(W)$$

is compatible with the P_0 structures. This map automatically respects the commutative structure, so it suffices to verify that for $\alpha \in \widetilde{\mathrm{Obs}}^{cl}(U_i)$ and $\beta \in \widetilde{\mathrm{Obs}}^{cl}(U_j)$, where $i \neq j$, then

$$\{\alpha, \beta\} = 0 \in \widetilde{\mathrm{Obs}}^{cl}(W).$$

That this bracket vanishes follows from the fact that if two "linear observables" $\phi, \psi \in \mathcal{L}_c(W)$ have disjoint support, then

$$\langle \phi, \psi \rangle = 0.$$

Every Poisson bracket reduces to a sum of brackets between linear terms by applying the Leibniz rule repeatedly. □

6.4. The Poisson structure for a general classical field theory

In this section we will prove the following.

6.4.0.1 Theorem. *For any classical field theory (section 5.4) on M, there is a P_0 factorization algebra $\widetilde{\mathrm{Obs}}^{cl}$, together with a quasi-isomorphism of commutative factorization algebras*

$$\widetilde{\mathrm{Obs}}^{cl} \cong \mathrm{Obs}^{cl}.$$

6.4.1. Functionals with smooth first derivative. For a free field theory, we defined a subcomplex $\widetilde{\mathrm{Obs}}^{cl}$ of observables which are built from smooth sections of a vector bundle on U^n, instead of distributional sections as in the definition of Obs^{cl}. It turns out that, for an interacting field theory, this subcomplex of Obs^{cl} is not preserved by the differential. Instead, we have to find a subcomplex built from distributions on U^n which are not smooth but which satisfy a mild regularity condition. We will call also this complex $\widetilde{\mathrm{Obs}}^{cl}$ (thus introducing a conflict with the terminology introduced in the case of free field theories).

Let $\mathcal{L}$ be an elliptic L_∞ algebra on M that defines a classical field theory. Recall that the cochain complex of observables is

$$\mathrm{Obs}^{cl}(U) = C^*(\mathcal{L}(U)),$$

where $\mathcal{L}(U)$ is the L_∞ algebra of sections of L on U.

Recall that as a graded vector space, $C^*(\mathcal{L}(U))$ is the algebra of functionals $\mathscr{O}(\mathcal{L}(U)[1])$ on the graded vector space $\mathcal{L}(U)[1]$. In the appendix (section B.1), given any graded vector bundle E on M, we define a subspace

$$\mathscr{O}^{sm}(\mathscr{E}(U)) \subset \mathscr{O}(\mathscr{E}(U))$$

of functionals that have "smooth first derivative". A function $\Phi \in \mathscr{O}(\mathscr{E}(U))$ is in $\mathscr{O}^{sm}(\mathscr{E}(U))$ precisely if

$$\mathrm{d}\Phi \in \mathscr{O}(\mathscr{E}(U)) \otimes \mathscr{E}_c^!(U).$$

(The exterior derivative of a general function in $\mathscr{O}(\mathscr{E}(U))$ will lie *a priori* in the larger space $\mathscr{O}(\mathscr{E}(U)) \otimes \overline{\mathscr{E}}_c^!(U)$.) The space $\mathscr{O}^{sm}(\mathscr{E}(U))$ is a differentiable pro-vector space.

Recall that if $\mathfrak{g}$ is an L_∞ algebra, the exterior derivative maps $C^*(\mathfrak{g})$ to $C^*(\mathfrak{g}, \mathfrak{g}^\vee[-1])$. The complex $C^*_{sm}(\mathcal{L}(U))$ of cochains with smooth first derivative is thus defined to be the subcomplex of $C^*(\mathcal{L}(U))$ consisting of those cochains whose first derivative lies in $C^*(\mathcal{L}(U), \mathcal{L}_c^!(U)[-1])$, which is a subcomplex of $C^*(\mathcal{L}(U), \mathcal{L}(U)^\vee[-1])$.

In other words, $C^*_{sm}(\mathcal{L}(U))$ is defined by the fiber diagram

$$\begin{array}{ccc} C^*_{sm}(\mathcal{L}(U)) & \xrightarrow{\mathrm{d}} & C^*(\mathcal{L}(U), \mathcal{L}^!_c(U)[-1]) \\ \downarrow & & \downarrow \\ C^*(\mathcal{L}(U)) & \xrightarrow{\mathrm{d}} & C^*(\mathcal{L}(U), \overline{\mathcal{L}_c}^!(U)[-1]). \end{array}$$

(Note that differentiable pro-cochain complexes are closed under taking limits, so that this fiber product is again a differentiable pro-cochain complex; more details are provided in the appendix B.1).

Note that

$$C^*_{sm}(\mathcal{L}(U)) \subset C^*(\mathcal{L}(U))$$

is a sub-commutative dg algebra for every open U. Furthermore, as U varies, $C^*_{sm}(\mathcal{L}(U))$ defines a sub-commutative prefactorization algebra of the prefactorization algebra defined by $C^*(\mathcal{L}(U))$.

We define

$$\widetilde{\mathrm{Obs}}^{cl}(U) = C^*_{sm}(\mathcal{L}(U)) \subset C^*(\mathcal{L}(U)) = \mathrm{Obs}^{cl}(U).$$

The next step is to construct the Poisson bracket.

6.4.2. The Poisson bracket. Because the elliptic L_∞ algebra L defines a classical field theory, it is equipped with an isomorphism $L \cong L^![-3]$. Thus, we have an isomorphism

$$\Phi : C^*(\mathcal{L}(U), \mathcal{L}^!_c(U)[-1]) \cong C^*(\mathcal{L}(U), \mathcal{L}_c(U)[2]).$$

In the appendix (section B.2), we show that $C^*(\mathcal{L}(U), \mathcal{L}(U)[1])$ — which we think of as vector fields on the formal manifold $B\mathcal{L}(U)$ — has a natural structure of a dg Lie algebra in the multicategory of differentiable pro-cochain complexes. The bracket is, of course, a version of the bracket of vector fields. Further, $C^*(\mathcal{L}(U), \mathcal{L}(U)[1])$ acts on $C^*(\mathcal{L}(U))$ by derivations. This action is in the multicategory of differentiable pro-cochain complexes: the map

$$C^*(\mathcal{L}(U), \mathcal{L}(U)[1]) \times C^*(\mathcal{L}(U)) \to C^*(\mathcal{L}(U))$$

is a smooth bilinear cochain map. We will write $\mathrm{Der}(C^*(\mathcal{L}(U)))$ for this dg Lie algebra $C^*(\mathcal{L}(U), \mathcal{L}(U)[1])$.

Thus, composing the map Φ above with the exterior derivative d and with the inclusion $\mathcal{L}_c(U) \hookrightarrow \mathcal{L}(U)$, we find a cochain map

$$C^*_{sm}(\mathcal{L}(U)) \to C^*(\mathcal{L}(U), \mathcal{L}_c(U)[2]) \to \mathrm{Der}(C^*(\mathcal{L}(U)))[1].$$

If $f \in C^*_{sm}(\mathcal{L}(U))$, we will let $X_f \in \mathrm{Der}(C^*(\mathcal{L}(U)))$ denote the corresponding derivation. If f has cohomological degree k, then X_f has cohomological degree $k+1$.

If $f, g \in C^*_{sm}(\mathcal{L}(U)) = \widetilde{\mathrm{Obs}}^{cl}(U)$, we define

$$\{f, g\} = X_f g \in \widetilde{\mathrm{Obs}}^{cl}(U).$$

This bracket defines a bilinear map

$$\widetilde{\mathrm{Obs}}^{cl}(U) \times \widetilde{\mathrm{Obs}}^{cl}(U) \to \widetilde{\mathrm{Obs}}^{cl}(U).$$

Note that we are simply adopting the usual formulas to our setting.

6.4.2.1 Lemma. *This map is smooth, i.e., a bilinear map in the multicategory of differentiable pro-cochain complexes.*

PROOF. This follows from the fact that the map

$$\mathrm{d} : \widetilde{\mathrm{Obs}}^{cl}(U) \to \mathrm{Der}(C^*(\mathcal{L}(U)))[1]$$

is smooth, which is immediate from the definitions, and from the fact that the map

$$\mathrm{Der}(C^*(\mathcal{L}(U)) \times C^*(\mathcal{L}(U)) \to C^*(\mathcal{L}(U))$$

is smooth (which is proved in the appendix B.2). □

6.4.2.2 Lemma. *This bracket satisfies the Jacobi rule and the Leibniz rule. Further, for U, V disjoint subsets of M, both contained in W, and for any $f \in \widetilde{\mathrm{Obs}}^{cl}(U)$, $g \in \widetilde{\mathrm{Obs}}^{cl}(V)$, we have*

$$\{f, g\} = 0 \in \widetilde{\mathrm{Obs}}^{cl}(W).$$

PROOF. The proof is straightforward. □

Following the argument for lemma 6.3.0.1, we obtain a P_0 factorization algebra.

6.4.2.3 Corollary. $\widetilde{\mathrm{Obs}}^{cl}$ *defines a P_0 factorization algebra in the valued in the multicategory of differentiable pro-cochain complexes.*

The final thing we need to verify is the following.

6.4.2.4 Proposition. *For all open subset $U \subset M$, the map*

$$\widetilde{\mathrm{Obs}}^{cl}(U) \to \mathrm{Obs}^{cl}(U)$$

is a weak equivalence.

PROOF. It suffices to show that it is a weak equivalence on the associated graded for the natural filtration on both sides. Now, $\operatorname{Gr}^n \widetilde{\operatorname{Obs}}^{cl}(U)$ fits into a fiber diagram

$$\begin{tikzcd} \operatorname{Gr}^n \widetilde{\operatorname{Obs}}^{cl}(U) \arrow[r] \arrow[d] & \operatorname{Sym}^n(\overline{\mathcal{L}}^!_c(U)[-1]) \otimes \mathcal{L}^!_c(U) \arrow[d] \\ \operatorname{Gr}^n \operatorname{Obs}^{cl}(U) \arrow[r] & \operatorname{Sym}^n(\overline{\mathcal{L}}^!_c(U)[-1]) \otimes \overline{\mathcal{L}}^!_c(U). \end{tikzcd}$$

Note also that

$$\operatorname{Gr}^n \operatorname{Obs}^{cl}(U) = \operatorname{Sym}^n \overline{\mathcal{L}}^!_c(U).$$

The Atiyah-Bott lemma **??** shows that the inclusion

$$\mathcal{L}^!_c(U) \hookrightarrow \overline{\mathcal{L}}^!_c(U)$$

is a continuous cochain homotopy equivalence. We can thus choose a homotopy inverse

$$P : \overline{\mathcal{L}}^!_c(U) \to \mathcal{L}^!_c(U)$$

and a homotopy

$$H : \overline{\mathcal{L}}^!_c(U) \to \overline{\mathcal{L}}^!_c(U)$$

such that $[\mathrm{d}, H] = P - \mathrm{Id}$ and such that H preserves the subspace $\mathcal{L}^!_c(U)$.

Now,

$$\operatorname{Sym}^n \mathcal{L}^!_c(U) \subset \operatorname{Gr}^n \widetilde{\operatorname{Obs}}^{cl}(U) \subset \operatorname{Sym}^n \overline{\mathcal{L}}^!_c(U).$$

Using the projector P and the homotopy H, one can construct a projector

$$P_n = P^{\otimes n} : \overline{\mathcal{L}}^!_c(U)^{\otimes n} \to \mathcal{L}^!_c(U)^{\otimes n}.$$

We can also construct a homotopy

$$H_n : \overline{\mathcal{L}}^!_c(U)^{\otimes n} \to \mathcal{L}^!_c(U)^{\otimes n}.$$

The homotopy H_n is defined inductively by the formula

$$H_n = H \otimes P_{n-1} + 1 \otimes H_{n-1}.$$

This formula defines a homotopy because

$$[\mathrm{d}, H_n] = P \otimes P_{n-1} - 1 \otimes P_{n-1} + 1 \otimes P_{n-1} - 1 \otimes 1.$$

Notice that the homotopy H_n preserves all the subspaces of the form

$$\overline{\mathcal{L}}^!_c(U)^{\otimes k} \otimes \mathcal{L}^!_c(U) \otimes \overline{\mathcal{L}}^!_c(U)^{\otimes n-k-1}.$$

This will be important momentarily.

Next, let

$$\pi : \overline{\mathcal{L}}^!_c(U)^{\otimes n}[-n] \to \operatorname{Sym}^n(\overline{\mathcal{L}}^!_c(U)[-1])$$

be the projection, and let

$$\Gamma_n = \pi^{-1} \operatorname{Gr}^n \widetilde{\mathrm{Obs}}^{cl}(U).$$

Then Γ_n is acted on by the symmetric group S_n, and the S_n invariants are $\widetilde{\mathrm{Obs}^{cl}}(U)$.

Thus, it suffices to show that the inclusion

$$\Gamma_n \hookrightarrow \overline{\mathcal{L}}_c(U)^{\otimes n}$$

is a weak equivalence of differentiable spaces. We will show that it is continuous homotopy equivalence.

The definition of $\widetilde{\mathrm{Obs}}^{cl}(U)$ allows one to identify

$$\Gamma_n = \cap_{k=0}^{n-1} \overline{\mathcal{L}}^!_c(U)^{\otimes k} \otimes \mathcal{L}^!_c(U) \otimes \overline{\mathcal{L}}^!_c(U)^{\otimes n-k-1}.$$

The homotopy H_n preserves Γ_n, and the projector P_n maps

$$\overline{\mathcal{L}}^!_c(U)^{\otimes n} \to \mathcal{L}_c(U)^{\otimes n} \subset \Gamma_n.$$

Thus, P_n and H_n provide a continuous homotopy equivalence between $\overline{\mathcal{L}}^!_c(U)^{\otimes n}$ and Γ_n, as desired. □

Part 2

Quantum field theory

CHAPTER 7

Introduction to quantum field theory

As explained in the introduction, this book develops a version of deformation quantization for field theories, rather than mechanics. In the chapters on classical field theory, we showed that the observables of a classical BV theory naturally form a commutative factorization algebra, with a homotopical P_0 structure. In the following chapters, we will show that every quantization of a classical BV theory produces a factorization algebra (in Beilinson-Drinfeld algebras) that we call the quantum observables of the quantum field theory. To be precise, the main theorem of this part is the following.

7.0.0.1 Theorem. *Any quantum field theory on a manifold M, in the sense of* [**Cos11b**], *gives rise to a factorization algebra* Obs^q *on M of* quantum observables*. This is a factorization algebra over $\mathbb{C}[[\hbar]]$, valued in differentiable pro-cochain complexes, and it quantizes (in the weak sense of 1.3) the P_0 factorization algebra of classical observables of the corresponding classical field theory.*

For free field theories, this factorization algebra of quantum observables is essentially the same as the one discussed in Chapter **??**. (The only difference is that, when discussing free field theories, we normally set $\hbar = 1$ and took our observables to be polynomial functions of the fields. When we discuss interacting theories, we take our observables to be power series on the space of fields, and we take $\hbar$ to be a formal parameter).

Chapter 8 is thus devoted to reviewing the formalism of [**Cos11b**], stated in a form most suitable to our purposes here. It's important to note that, in contrast to the deformation quantization of Poisson manifolds, a classical BV theory may not possess any quantizations (i.e., quantization may be *obstructed*) or it may have many quantizations. A central result of [**Cos11b**], stated in section 8.5, is that there is a space of BV quantizations. Moreover, this space can be constructed as a tower of fibrations, where the fiber between any pair of successive layers is described by certain cohomology groups of local functionals. These cohomology groups can be computed just from the classical theory.

The machinery of [**Cos11b**] allows one to construct many examples of quantum field theories, by calculating the appropriate cohomology groups. For example, in [**Cos11b**], the quantum Yang-Mills gauge theory is constructed. Theorem 7.0.0.1, together with the results of [**Cos11b**], thus produces many interesting examples of factorization algebras.

Remark: We forewarn the reader that our definitions and constructions involve a heavy use of functional analysis and (perhaps more surprisingly) simplicial sets, which is our preferred way of describing a space of field theories. Making a quantum field theory typically requires many choices, and as mathematicians, we wish to pin down precisely how the quantum field theory depends on these choices. The machinery we use gives us very precise statements, but statements that can be forbidding at first sight. We encourage the reader, on a first pass through this material, to simply make all necessary choices (such as a parametrix) and focus on the output of our machine, namely the factorization algebra of quantum observables. Keeping track of the dependence on choices requires careful bookkeeping (aided by the machinery of simplicial sets) but is straightforward once the primary construction is understood. $\Diamond$

The remainder of this chapter consists of an introduction to the quantum BV formalism, building on our motivation for the classical BV formalism in section 5.1.

7.1. The quantum BV formalism in finite dimensions

In section 5.1, we motivated the classical BV formalism with a finite-dimensional toy model. To summarize, we described the *derived* critical locus of a function S on a smooth manifold M of dimension n. The functions on this derived space $\mathscr{O}(\mathrm{Crit}^h(S))$ form a commutative dg algebra,

$$\Gamma(M, \wedge^n TM) \xrightarrow{\vee \mathrm{d}S} \dots \xrightarrow{\vee \mathrm{d}S} \Gamma(M, \wedge^2 TM) \xrightarrow{\vee \mathrm{d}S} \Gamma(M, TM) \xrightarrow{\vee \mathrm{d}S} C^\infty(M),$$

the polyvector fields $PV(M)$ on M with the differential given by contraction with $\mathrm{d}S$. This complex remembers how $\mathrm{d}S$ vanishes and not just where it vanishes.

The quantum BV formalism uses a deformation of this *classical BV complex* to encode, in a homological way, oscillating integrals.

In finite dimensions, there already exists a homological approach to integration: the de Rham complex. For instance, on a compact, oriented n-manifold without boundary, M, we have the commuting diagram

$$\begin{tikzcd} \Omega^n(M) \arrow[rr, "\int_M"] \arrow[dr, "{[-]}"'] & & \mathbb{R} \\ & H^n(M) \arrow[ur, "{\langle [M], - \rangle}"'] & \end{tikzcd}$$

where $[\mu]$ denotes the cohomology class of the top form μ and $\langle [M], - \rangle$ denotes pairing the class with the fundamental class of M. Thus, integration factors through the de Rham cohomology.

Suppose μ is a smooth probability measure, so that $\int_M \mu = 1$ and μ is everywhere nonnegative (which depends on the choice of orientation). Then we can interpret the expected value of a function f on M — an "observable on the space of fields M" — as the cohomology class $[f\mu] \in H^n(M)$.

The BV formalism in finite dimensions secretly exploits this use of the de Rham complex, as we explain momentarily. For an infinite-dimensional manifold, though, the de Rham complex ceases to encode integration over the whole manifold because there are no top forms. In contrast, the BV version scales to the infinite-dimensional setting. Infinite dimensions, of course, introduces extra difficulties to do with the fact that integration in infinite dimensions is not well-defined. These difficulties manifest themselves as ultraviolte divergences of quantum field theory, and we deal with them using the techniques developed in [**Cos11b**].

In the classical BV formalism, we work with the polyvector fields rather than de Rham forms. A choice of probability measure μ, however, produces a map between these graded vector spaces

$$\begin{array}{ccccccc} \Gamma(M, \wedge^n TM) & \dots & \Gamma(M, \wedge^2 TM) & \Gamma(M, TM) & C^\infty(M) \\ \downarrow{\scriptstyle \vee\mu} & & \downarrow{\scriptstyle \vee\mu} & \downarrow{\scriptstyle \vee\mu} & \downarrow{\scriptstyle \vee\mu} \\ C^\infty(M) & \dots & \Omega^{n-2}(M) & \Omega^{n-1}(M) & \Omega^n(M) \end{array}$$

where $\vee\mu$ simply contracts a k-polyvector field with μ to get a $n-k$-form. When μ is nowhere-vanishing (i.e., when μ is a *volume form*), this map is an isomorphism and so we can "pull back" the exterior derivative to equip the polyvector fields with a differential. This differential is usually called the *divergence operator for μ*, so we denote it div_μ.

By the *divergence complex for μ*, we mean the polyvector fields (concentrated in nonpositive degrees) with differential div_μ. Its cohomology is isomorphic, by construction, to $H^*_{dR}(M)[n]$. In particular, given a function f on M, viewed as living in degree zero and providing an "observable," we see that its cohomology class $[f]$ in the divergence complex corresponds to the expected value of f against μ. More precisely, we can define the ratio $[f]/[1]$ as the expected value of f. Under the map $\vee\mu$, it goes to the usual expected value.

What we've done above is provide an alternative homological approach to integration. More accurately, we've shown how "integration against a volume form" can be encoded by an appropriate choice of differential on the polyvector fields. Cohomology classes in this divergence complex encode the expected values of functions against this measure. Of course, this is what we want from the path integral! The divergence complex is the motivating example for the quantum BV formalism, and so it is also called a *quantum BV complex*.

We can now explain why this approach to homological integration is more suitable to extension to infinite dimensions than the usual de Rham picture. Even for an infinite-dimensional manifold M, the polyvector fields are well-defined (although one must make choices in how to define them, depending on one's preferences with functional analysis). One can still try to construct a "divergence-type operator" and view it as the effective replacement for the probability measure. By taking cohomology classes, we compute the expected values of observables. The difficult part is making sense of the divergence operator; this is achieved through renormalization.

This vein of thought leads to a question: how to characterize, in an abstract fashion, the nature of a divergence operator? An answer leads, as we've shown, to a process for defining a homological path integral. Below, we'll describe one approach, but first we examine a simple case.

Remark: The cohomology of the complex (both in the finite and infinite dimensional settings) always makes sense, but H^0 is not always one-dimensional. For example, on a manifold X that is not closed, the de Rham cohomology often vanishes at the top. If the manifold is disconnected but closed, the top de Rham cohomology has dimension equal to the number of components of the manifold. In general, one must choose what class of functions to integrate against the volume form, and the cohomology depends on this choice (e.g., consider compactly supported de Rham cohomology).

Instead of computing expected values, the cohomology provides relations between expected values of observables. We will see how the cohomology encodes relations in the example below. In the setting of conformal field theory, for instance, one often uses such relations to obtain formulas for the operator product expansion. $\Diamond$

7.2. The "free scalar field" in finite dimensions

A concrete example is in order. We will work with a simple manifold, the real line, equipped with the Gaussian measure and recover the baby case of Wick's lemma. The generalization to a finite-dimensional vector space will be clear.

Remark: This example is especially pertinent to us because in this book we are working with perturbative quantum field theories. Hence, for us, there is always a free field theory — whose space of fields is a vector space equipped with some kind of Gaussian measure — that we've modified by adding an interaction to the action functional. The underlying vector space is equipped with a linear pairing that yields the BV Laplacian, as we work with it. As we will see in this example, the usual BV formalism relies upon the underlying "manifold" being linear in nature. To extend to a global nonlinear situation, on e needs to develop new techniques (see, for instance, [**Cos11a**]). $\Diamond$

Before we undertake the Gaussian measure, let's begin with the Lebesgue measure $\mathrm{d}x$ on $\mathbb{R}$. This is not a probability measure, but it is nowhere-vanishing, which is the only property necessary to construct a divergence operator. In this case, we compute

$$\mathrm{div}_{Leb} : f\frac{\partial}{\partial x} \mapsto \frac{\partial f}{\partial x}.$$

In one popular notion, we use ξ to denote the vector field $\partial/\partial x$, and the polyvector fields are then $C^\infty(\mathbb{R})[\xi]$, where ξ has cohomological degree -1. The divergence operator becomes

$$\mathrm{div}_{Leb} = \frac{\partial}{\partial x}\frac{\partial}{\partial \xi},$$

which is also the standard example of the BV Laplacian $\triangle$. (In short, the usual BV Laplacian on $\mathbb{R}^n$ is simply the divergence operator for the Lebesgue measure.) We will use $\triangle$ for it, as this notation will continue throughout the book.

It is easy to see, by direct computation or the Poincaré lemma, that the cohomology of the divergence complex for the Lebesgue measure is simply $H^{-1} \cong \mathbb{R}$ and $H^0 \cong \mathbb{R}$.

Let μ_b be the usual Gaussian probability measure on $\mathbb{R}$ with variance b:

$$\mu_b = \sqrt{\frac{1}{2\pi b}}e^{-x^2/2b}\mathrm{d}x.$$

As μ is a nowhere-vanishing probability measure, we obtain a divergence operator

$$\mathrm{div}_b : f\frac{\partial}{\partial x} \mapsto \frac{\partial f}{\partial x} - \frac{x}{b}f.$$

We have

$$\mathrm{div}_b = \triangle + \vee \mathrm{d}S$$

where $S = -x^2/2b$. Note that this complex is a deformation of the classical BV complex for S by adding the BV Laplacian $\triangle$.

This divergence operator preserves the subcomplex of polynomial polyvector fields. That is, a vector field with polynomial coefficient goes to a polynomial function.

Explicitly, we see

$$\mathrm{div}_b\left(x^n\frac{\partial}{\partial x}\right) = nx^{n-1} - \frac{1}{b}x^{n+1}.$$

Hence, at the level of cohomology, we see $[x^{n+1}] = bn[x^{n-1}]$. We have just obtained the following, by a purely cohomological process.

7.2.0.1 Lemma (Baby case of Wick's lemma). *The expected value of x^n with respect to the Gaussian measure is zero if n odd and $b^k(2k-1)(2k-3)\cdots 5\cdot 3$ if $n = 2k$.*

Since Wick's lemma appears by this method, it should be clear that one can recover the usual Feynman diagrammatic expansion. Indeed, the usual arguments with integration by parts are encoded here by the relations between cohomology classes.

Note that for any function $S : \mathbb{R} \to \mathbb{R}$, the volume form $e^S \mathrm{d}x$ has divergence operator

$$\mathrm{div}_S = \triangle + \frac{\partial S}{\partial x}\frac{\partial}{\partial x},$$

and using the Schouten bracket $\{-,-\}$ on polyvector fields, we can write it as

$$\mathrm{div}_S = \triangle + \{S, -\}.$$

The *quantum master equation* (QME) is the equation $\mathrm{div}_S^2 = 0$. The *classical master equation* (CME) is the equation $\{S, S\} = 0$, which just encodes the fact that the differential of the classical BV complex is square-zero. (In the examples we've discussed so far, this property is immediate, but in many contexts, such as gauge theories, finding such a function S can be a nontrivial process.)

7.3. An operadic description

Before we provide abstract properties that characterize a divergence operator, we should recall properties that characterize the classical BV complex. Of course, functions on the derived critical locus are a commutative dg algebra. Polyvector fields, however, also have the Schouten bracket — the natural extension of the Lie bracket of vector fields and functions — which is a Poisson bracket of cohomological degree 1 and which is compatible with the differential $\vee S = \{S, -\}$. Thus, we introduced the notion of a P_0 algebra, where P_0 stands for "Poisson-zero," in section 2.3. In chapter 6, we showed that the factorization algebra of observables for a classical BV theory have a lax P_0 structure.

Examining the divergence complex for a measure of the form $e^S \mathrm{d}x$ in the preceding section, we saw that the divergence operator was a deformation of $\{S, -\}$, the differential for the classical BV complex. Moreover, a simple computation shows that a divergence operator satisfies

$$\mathrm{div}(ab) = (\mathrm{div}\, a)b + (-1)^{|a|}a(\mathrm{div}\, b) + (-1)^{|a|}\{a, b\}$$

for any polyvector fields a and b. (This relation follows, under the polyvector-de Rham isomorphism given by the measure, from the fact that the exterior derivative is a derivation for the wedge product.) Axiomatizing these two properties, we obtain the notion of a Beilinson-Drinfeld algebra, discussed in section 2.4. The differential of a BD algebra possesses many of the essential properties of a divergence operator, and so we view a BD algebra as a homological way to encode integration on (a certain class of) derived spaces.

In short, the quantum BV formalism aims to find, for a P_0 algebra A^{cl}, a BD algebra A^q such that $A^{cl} = A^q \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R}[[\hbar]]/(\hbar)$. We view it as moving from studying functions on the derived critical locus of some action functional S to the divergence complex for $e^S \mathscr{D}\phi$.

This motivation for the definition of a BD algebra is complementary to our earlier motivation, which emphasizes the idea that we simply want to deform from a commutative factorization algebra to a "plain," or E_0, factorization algebra. It grows out of the path integral approach to quantum field theory, rather than extending to field theory the deformation quantization approach to mechanics.

For us, the basic situation is a formal moduli space $\mathcal{M}$ with -1-symplectic pairing. Its algebra of functions is a P_0 algebra. By a version of the Darboux lemma for formal moduli spaces, we can identify $\mathcal{M}$ with an L_∞ algebra $\mathfrak{g}$ equipped with an invariant symmetric pairing. Geometrically, this means the symplectic pairing is translation-invariant and all the nonlinearity is pushed into the brackets. As the differential d on $\mathscr{O}(\mathcal{M})$ respects the Poisson bracket, we view it as a symplectic vector field of cohomological degree 1, and in this formal situation, we can find a Hamiltonian function S such that $d = \{S, -\}$.

Comparing to our finite-dimensional example above, we are seeing the analog of the fact that any nowhere-vanishing volume form on $\mathbb{R}^n$ can be written as $e^S \mathrm{d}x_1 \cdots \mathrm{d}x_n$. The associated divergence operator looks like $\triangle + \{S, -\}$, where the BV Laplacian $\triangle$ is the divergence operator for Lebesgue measure.

The translation-invariant Poisson bracket on $\mathscr{O}(\mathcal{M})$ also produces a translation-invariant BV Laplacian $\triangle$. Quantizing then amounts to finding a function $I \in \hbar\mathscr{O}(\mathcal{M})[[\hbar]]$ such that

$$\{S, -\} + \{I, -\} + \hbar\triangle$$

is square-zero. In the BV formalism, we call I a "solution to the quantum master equation for the action S." As shown in chapter 6 of [**Cos11b**], we have the following relationship.

7.3.0.1 Proposition. *Let $\mathcal{M}$ be a formal moduli space with -1-symplectic structure. There is an equivalence of spaces*

$$\{\textit{solutions of the QME}\} \simeq \{\textit{BD quantizations}\}.$$

7.4. Equivariant BD quantization and volume forms

We now return to our discussion of volume forms and formulate a precise relationship with BD quantization. This relationship, first noted by Koszul [**Kos85**], generalizes naturally to the setting of cotangent field theories. In section 10.4, we explain how cotangent quantizations provide volume forms on elliptic moduli problems.

For a smooth manifold M, there is a special feature of a divergence complex that we have not yet discussed. Polyvector fields have a natural action of the multiplicative group

$\mathbb{G}_m$, where functions have weight zero, vector fields have weight -1, and k-vector fields have weight $-k$. This action arises because polyvector fields are functions on the shifted cotangent bundle $T^*[-1]M$, and there is always a scaling action on the cotangent fibers.

We can make the classical BV complex into a $\mathbb{G}_m$-equivariant P_0 algebra, as follows. Simply equip the Schouten bracket with weight 1 and the commutative product with weight zero. We now ask for a $\mathbb{G}_m$-equivariant BD quantization.

To make this question precise, we rephrase our observations operadically. Equip the operad P_0 with the $\mathbb{G}_m$ action where the commutative product is weight zero and the Poisson bracket is weight 1. An equivariant P_0 algebra is then a P_0 algebra with a $\mathbb{G}_m$ action such that the bracket has weight 1 and the product has weight zero. Similarly, equip the operad BD with the $\mathbb{G}_m$ action where $\hbar$ has weight -1, the product has weight zero, and the bracket has weight 1. A filtered BD algebra is a BD algebra with a $\mathbb{G}_m$ action with the same weights.

Given a volume form μ on M, the $\hbar$-weighted divergence complex

$$(PV(M)[[\hbar]], \hbar \operatorname{div}_\mu)$$

is a filtered BD algebra.

On an smooth manifold, we saw that each volume form μ produced a divergence operator div_μ, via "conjugating" the exterior derivative d by the isomorphism $\vee\mu$. In fact, any rescaling $c\mu$, with $c \in \mathbb{R}^\times$, produces the same divergence operator. Since we want to work with probability measures, this fact meshes well with our objectives: we would always divide by the integral $\int_X \mu$ anyway. In fact, one can show that every filtered BD quantization of the P_0 algebra $PV(M)$ arises in this way.

7.4.0.1 Proposition. *There is a bijection between projective volume forms on M, and filtered BV quantizations of $PV(M)$.*

See [**Cos11a**] for more details on this.

7.5. How renormalization group flow interlocks with the BV formalism

So far, we have introduced the quantum BV formalism in the finite dimensional setting and extracted the essential algebraic structures. Applying these ideas in the setting of field theories requires nontrivial work. Much of this work is similar in flavor to our construction of a lax P_0 structure on Obs^{cl}: issues with functional analysis block the most naive approach, but there are alternative approaches, often well-known in physics, that accomplish our goal, once suitably reinterpreted.

Here, we build on the approach of [Cos11b]. The book uses exact renormalization group flow to define the notion of effective field theory and develops an effective version of the BV formalism. In chapter 8, we review these ideas in detail. We will sketch how to apply the BV formalism to formal elliptic moduli problems $\mathcal{M}$ with -1-symplectic pairing.

The main problem here is the same as in defining a shifted Poisson structure on the classical observables: the putative Poisson bracket $\{-,-\}$, arising from the symplectic structure, is well-defined only on a subspace of all observables. As a result, the associated BV Laplacian $\triangle$ is also only partially-defined.

To work around this problem, we use the fact that every parametrix Φ for the elliptic complex underlying $\mathcal{M}$ yields a mollified version $\triangle_\Phi$ of the BV Laplacian, and hence a mollified bracket $\{-,-\}_\Phi$. An *effective field theory* consists of a BD algebra Obs_Φ for every parametrix and a homotopy equivalence for any two parametrices, $\mathrm{Obs}_\Phi \simeq \mathrm{Obs}_\Psi$, satisfying coherence relations. In other words, we get a family of BD algebras over the space of parametrices. The renormalization group (RG) flow provides the homotopy equivalences for any pair of parametrices. Modulo $\hbar$, we also get a family Obs^{cl}_Φ of P_0 algebras over the space of parametrices. The tree-level RG flow produces the homotopy equivalences modulo $\hbar$.

An effective field theory is a quantization of $\mathcal{M}$ if, in the limit as $\triangle_\Phi$ goes to $\triangle$, the P_0 algebra goes to the functions $\mathscr{O}(\mathcal{M})$ on the formal moduli problem.

The space of parametrices is contractible, so an effective field theory describes just one BD algebra, up to homotopy equivalence. From the perspective developed thus far, we interpret this BD algebra as encoding integration over $\mathcal{M}$.

There is another way to interpret this definition, though, that may be attractive. The RG flow amounts to a Feynman diagram expansion, and hence we can see it as a definition of functional integration (in particular, flowing from energy scale Λ to Λ' integrates over the space of functions with energies between those scales). In [Cos11b], the RG flow is extended to the setting where the underlying free theory is an elliptic complex, not just given by an elliptic operator.

7.6. Overview of the rest of this Part

Here is a detailed summary of the chapters on quantum field theory.

(1) In section 9.1 we recall the definition of a free theory in the BV formalism and construct the factorization algebra of quantum observables of a general free theory, using the factorization envelope construction of section **??** of Chapter **??**. This generalizes the discussion in chapter **??**.

(2) In sections **??** to 8.5 we give an overview of the definition of QFT developed in [**Cos11b**].
(3) In section 9.2 we show how the definition of a QFT leads immediately to a construction of a BD algebra of "global observables" on the manifold M, which we denote $\mathrm{Obs}^q_{\mathscr{P}}(M)$.
(4) In section 9.3 we start the construction of the factorization algebra associated to a QFT. We construct a cochain complex $\mathrm{Obs}^q(M)$ of global observables, which is quasi-isomorphic to (but much smaller than) the BD algebra $\mathrm{Obs}^q_{\mathscr{P}}(M)$.
(5) In section 9.5 we construct, for every open subset $U \subset M$, the subspace $\mathrm{Obs}^q(U) \subset \mathrm{Obs}^q(M)$ of observables supported on U.
(6) Section 9.6 accomplishes the primary aim of the chapter. In it, we prove that the cochain complexes $\mathrm{Obs}^q(U)$ form a factorization algebra. The proof of this result is the most technical part of the chapter.
(7) In section 10.1 we show that translation-invariant theories have translation-invariant factorization algebras of observables, and we treat the holomorphic situation as well.
(8) In section 10.4 we explain how to interpret our definition of a QFT in the special case of a cotangent theory: roughly speaking, a quantization of the cotangent theory to an elliptic moduli problem yields a locally-defined volume form on the moduli problem we start with.

CHAPTER 8

Effective field theories and Batalin-Vilkovisky quantization

In this chapter, we will give a summary of the definition of a QFT as developed in [**Cos11b**]. We will emphasize the aspects used in our construction of the factorization algebra associated to a QFT. This means that important aspects of the story there — such as the concept of renormalizability — will not be mentioned. The introductory chapter of [**Cos11b**] is a leisurely exposition of the main physical and mathematical ideas, and we encourage the reader to examine it before delving into what follows. The approach there is perturbative and hence has the flavor of formal geometry (that is, geometry with formal manifolds).

A perturbative field theory is defined to be a family of effective field theories parametrized by some notion of "scale." The notion of scale can be quite flexible; the simplest version is where the scale is a positive real number, the length. In this case, the effective theory at a length scale L is obtained from the effective theory at scale ε by integrating out over fields with length scale between ε and L. In order to construct factorization algebras, we need a more refined notion of "scale," where there is a scale for every parametrix Φ of a certain elliptic operator. We denote such a family of effective field theories by $\{I[\Phi]\}$, where $I[\Phi]$ is the "interaction term" in the action functional $S[\Phi]$ at "scale" Φ. We always study families with respect to a fixed free theory.

A local action functional (see section 8.1) S is a real-valued function on the space of fields such that $S(\phi)$ is given by integrating some function of the field and its derivatives over the base manifold (the "spacetime"). The main result of [**Cos11b**] states that the space of perturbative QFTs is the "same size" as the space of local action functionals. More precisely, the space of perturbative QFTs defined modulo $\hbar^{n+1}$ is a torsor over the space of QFTs defined modulo $\hbar^n$ for the abelian group of local action functionals. In consequence, the space of perturbative QFTs is non-canonically isomorphic to local action functionals with values in $\mathbb{R}[[\hbar]]$ (where the choice of isomorphism amounts to choosing a way to construct counterterms).

The starting point for many physical constructions — such as the path integral — is a local action functional. However, a naive application of these constructions to such an action functional yields a nonsensical answer. Many of these constructions *do* work if, instead of applying them to a local action functional, they are applied to a family $\{I[\Phi]\}$ of effective action functionals. Thus, one can view the family of effective action functionals

$\{I[\Phi]\}$ as a quantum version of the local action functional defining classical field theory. The results of [**Cos11b**] allow one to construct such families of action functionals. Many formal manipulations with path integrals in the physics literature apply rigorously to families $\{I[\Phi]\}$ of effective actions. Our strategy for constructing the factorization algebra of observables is to mimic path-integral definitions of observables one can find in the physics literature, but replacing local functionals by families of effective actions.

8.1. Local action functionals

In studying field theory, there is a special class of functions on the fields, known as local action functionals, that parametrize the possible classical physical systems. Let M be a smooth manifold. Let $\mathscr{E} = C^\infty(M, E)$ denote the smooth sections of a $\mathbb{Z}$-graded super vector bundle E on M, which has finite rank when all the graded components are included. We call $\mathscr{E}$ the *fields*.

Various spaces of functions on the space of fields are defined in the appendix B.1.

8.1.0.1 Definition. *A* functional *F is an element of*

$$\mathscr{O}(\mathscr{E}) = \prod_{n=0}^{\infty} \mathrm{Hom}_{DVS}(\mathscr{E}^{\times n}, \mathbb{R})^{S_n}.$$

This is also the completed symmetric algebra of $\mathscr{E}^\vee$, where the tensor product is the completed projective one.

Let $\mathscr{O}_{red}(\mathscr{E}) = \mathscr{O}(\mathscr{E})/\mathbb{C}$ be the space of functionals on $\mathscr{E}$ modulo constants.

Note that every element of $\mathscr{O}(\mathscr{E})$ has a Taylor expansion whose terms are smooth multilinear maps

$$\mathscr{E}^{\times n} \to \mathbb{C}.$$

Such smooth mulitilinear maps are the same as compactly-supported distributional sections of the bundle $(E^!)^{\boxtimes n}$ on M^n. Concretely, a functional is then an infinite sequence of vector-valued distributions on powers of M.

The local functionals depend only on the local behavior of a field, so that at each point of M, a local functional should only depend on the jet of the field at that point. In the Lagrangian formalism for field theory, their role is to describe the permitted actions, so we call them *local action functionals*. A local action functional is the essential datum of a *classical* field theory.

8.1.0.2 Definition. *A functional F is* local *if each homogeneous component F_n is a finite sum of terms of the form*

$$F_n(\phi) = \int_M (D_1\phi) \cdots (D_n\phi)\, d\mu,$$

where each D_i is a differential operator from $\mathscr{E}$ to $C^\infty(M)$ and $d\mu$ is a density on M.

We let

$$\mathscr{O}_{loc}(\mathscr{E}) \subset \mathscr{O}_{red}(\mathscr{E})$$

denote denote the space of local action functionals modulo constants.

As explained in section 5.4, a classical BV theory is a choice of local action functional S of cohomological degree 0 such that $\{S, S\} = 0$. That is, S must satisfy the classical master equation.

8.2. The definition of a quantum field theory

In this section, we will give the formal definition of a quantum field theory. The definition is a little long and somewhat technical. The reader should consult the first chapter of [**Cos11b**] for physical motivations for this definition. We will provide some justification for the definition from the point of view of homological algebra shortly (section 9.2).

8.2.1.

8.2.1.1 Definition. *A* free BV theory *on a manifold M consists of the following data:*

(1) a $\mathbb{Z}$-graded super vector bundle $\pi : E \to M$ that is of finite rank;
(2) a graded antisymmetric map of vector bundles $\langle -, - \rangle_{loc} : E \otimes E \to \mathrm{Dens}(M)$ of cohomological degree -1 that is fiberwise nondegenerate. It induces a graded antisymmetric pairing of degree -1 on compactly supported smooth sections $\mathscr{E}_c$ of E:

$$\langle \phi, \psi \rangle = \int_{x \in M} \langle \phi(x), \psi(x) \rangle_{loc};$$

(3) a square-zero differential operator $Q : \mathscr{E} \to \mathscr{E}$ of cohomological degree 1 that is skew self adjoint for the symplectic pairing.

In our constructions, we require the existence of a *gauge-fixing operator* $Q^{GF} : \mathscr{E} \to \mathscr{E}$ with the following properties:

(1) it is a square-zero differential operator of cohomological degree -1 ;
(2) it is self adjoint for the symplectic pairing;
(3) $D = [Q, Q^{GF}]$ is a generalized Laplacian on M, in the sense of [**BGV92**]. This means that D is an order 2 differential operator whose symbol $\sigma(D)$, which is an endomorphism of the pullback bundle p^*E on the cotangent bundle $p : T^*M \to M$, is

$$\sigma(D) = g \, \mathrm{Id}_{p^*E}$$

where g is some Riemannian metric on M, viewed as a function on T^*M.

All our constructions vary homotopically with the choice of gauge fixing operator. In practice, there is a natural contractible space of gauge fixing operators, so that our constructions are independent (up to contractible choice) of the choice of gauge fixing operator. (As an example of contractibility, if the complex $\mathscr{E}$ is simply the de Rham complex, each metric gives a gauge fixing operator d^*. The space of metrics is contractible.)

8.2.2. Operators and kernels. Let us recall the relationship between kernels and operators on $\mathscr{E}$. Any continuous linear map $F : \mathscr{E}_c \to \overline{\mathscr{E}}$ can be represented by a kernel

$$K_F \in \mathcal{D}(M^2, E \boxtimes E^!).$$

Here $\mathcal{D}(M, -)$ denotes distributional sections. We can also identify this space as

$$\begin{aligned} \mathcal{D}(M^2, E \boxtimes E^!) &= \mathrm{Hom}_{DVS}(\mathscr{E}_c^! \times \mathscr{E}_c, \mathbb{C}) \\ &= \mathrm{Hom}_{DVS}(\mathscr{E}_c, \overline{\mathscr{E}}) \\ &= \overline{\mathscr{E}} \widehat{\otimes}_\pi \overline{\mathscr{E}}^!. \end{aligned}$$

Here $\widehat{\otimes}_\pi$ denotes the completed projective tensor product.

The symplectic pairing on $\mathscr{E}$ gives an isomorphism between $\overline{\mathscr{E}}$ and $\overline{\mathscr{E}}^![-1]$. This allows us to view the kernel for any continuous linear map F as an element

$$K_F \in \overline{\mathscr{E}} \widehat{\otimes}_\pi \overline{\mathscr{E}} = \mathrm{Hom}_{DVS}(\mathscr{E}_c^! \times \mathscr{E}_c^!, \mathbb{C})$$

. If F is of cohomological degree k, then the kernel K_F is of cohomological degree $k+1$.

If the map $F : \mathscr{E}_c \to \overline{\mathscr{E}}$ has image in $\overline{\mathscr{E}}_c$ and extends to a continuous linear map $\mathscr{E} \to \overline{\mathscr{E}}_c$, then the kernel K_F has compact support. If F has image in $\mathscr{E}$ and extends to a continuous linear map $\overline{\mathscr{E}}_c \to \mathscr{E}$, then the kernel K_F is smooth.

Our conventions are such that the following hold.

(1) $K_{[Q,F]} = QK_F$, where Q is the total differential on $\overline{\mathscr{E}} \widehat{\otimes}_\pi \overline{\mathscr{E}}$.
(2) Suppose that $F : \mathscr{E}_c \to \mathscr{E}_c$ is skew-symmetric with respect to the degree -1 pairing on $\mathscr{E}_c$. Then K_F is symmetric. Similarly, if F is symmetric, then K_F is antisymmetric.

8.2.3. The heat kernel. In this section we will discuss heat kernels associated to the generalized Laplacian $D = [Q, Q^{GF}]$. These generalized heat kernels will not be essential to our story; most of our constructions will work with a general parametrix for the operator D, and the heat kernel simply provides a convenient example.

Suppose that we have a free BV theory with a gauge fixing operator Q^{GF}. As above, let $D = [Q, Q^{GF}]$. If our manifold M is compact, then this leads to a heat operator e^{-tD} acting on sections $\mathscr{E}$. The heat kernel K_t is the corresponding kernel, which is an element of $\overline{\mathscr{E}}\widehat{\otimes}_\pi\overline{\mathscr{E}}\widehat{\otimes}_\pi C^\infty(\mathbb{R}_{\geq 0})$. Further, if $t > 0$, the operator e^{-tD} is a smoothing operator, so that the kernel K_t is in $\mathscr{E}\widehat{\otimes}_\pi\mathscr{E}$. Since the operator e^{-tD} is skew symmetric for the symplectic pairing on $\mathscr{E}$, the kernel K_t is symmetric.

The kernel K_t is uniquely characterized by the following properties:

(1) The heat equation:
$$\frac{\mathrm{d}}{\mathrm{d}t}K_t + (D \otimes 1)K_t = 0.$$
(2) The initial condition that $K_0 \in \overline{\mathscr{E}}\widehat{\otimes}_\pi\overline{\mathscr{E}}$ is the kernel for the identity operator.

On a non-compact manifold M, there is more than one heat kernel satisfying these properties.

8.2.4. Parametrices.

In [Cos11b], two equivalent definitions of a field theory are given: one based on the heat kernel, and one based on a general parametrix. We will use exclusively the parametrix version in this book.

Before we define the notion of parametrix, we need a technical definition.

8.2.4.1 Definition. *If M is a manifold, a subset $V \subset M^n$ is* proper *if all of the projection maps $\pi_1, \ldots, \pi_n : V \to M$ are proper. We say that a function, distribution, etc. on M^n has proper support if its support is a proper subset of M^n.*

8.2.4.2 Definition. *A* parametrix Φ *is a distributional section*
$$\Phi \in \overline{\mathscr{E}}(M)\widehat{\otimes}_\pi\overline{\mathscr{E}}(M)$$
of the bundle $E \boxtimes E$ on $M \times M$ with the following properties.

(1) Φ is symmetric under the natural $\mathbb{Z}/2$ action on $\overline{\mathscr{E}}(M)\widehat{\otimes}_\pi\overline{\mathscr{E}}(M)$.
(2) Φ is of cohomological degree 1.
(3) Φ has proper support.
(4) Let $Q^{GF} : \mathscr{E} \to \mathscr{E}$ be the gauge fixing operator. We require that
$$([Q, Q^{GF}] \otimes 1)\Phi - K_{\mathrm{Id}}$$
is a smooth section of $E \boxtimes E$ on $M \times M$. Thus,
$$([Q, Q^{GF}] \otimes 1)\Phi - K_{\mathrm{Id}} \in \mathscr{E}(M)\widehat{\otimes}_\pi\mathscr{E}(M).$$
(Here K_{Id} is the kernel corresponding to the identity operator).

Remark: For clarity's sake, note that our definition depends on a choice of Q^{GF}. Thus, we are defining here parametrices for the generalized Laplacian $[Q, Q^{GF}]$, *not* general parametrices for the elliptic complex $\mathscr{E}$. $\Diamond$

Note that the parametrix Φ can be viewed (using the correspondence between kernels and operators described above) as a linear map $A_\Phi : \mathscr{E} \to \mathscr{E}$. This operator is of cohomological degree 0, and has the property that

$$A_\Phi[Q, Q^{GF}] = \mathrm{Id} + \text{a smoothing operator}$$
$$[Q, Q^{GF}]A_\Phi = \mathrm{Id} + \text{a smoothing operator.}$$

This property – being both a left and right inverse to the operator $[Q, Q^{GF}]$, up to a smoothing operator – is the standard definition of a parametrix.

An example of a parametrix is the following. For M compact, let $K_t \in \mathscr{E}\widehat{\otimes}_\pi\mathscr{E}$ be the heat kernel. Then, the kernel $\int_0^L K_t \mathrm{d}t$ is a parametrix, for any $L > 0$.

It is a standard result in the theory of pseudodifferential operators (see e.g. [Tar87]) that every elliptic operator admits a parametrix. Normally a parametrix is not assumed to have proper support; however, if Φ is a parametrix satisfying all conditions except that of proper support, and if $f \in C^\infty(M \times M)$ is a smooth function with proper support that is 1 in a neighborhood of the diagonal, then $f\Phi$ is a parametrix with proper support. This shows that parametrices with proper support always exist.

Let us now list some key properties of parametrices, all of which are consequences of elliptic regularity.

8.2.4.3 Lemma. *(1) If Φ, Ψ are parametrices, then the section $\Phi - \Psi$ of the bundle $E \boxtimes E$ on $M \times M$ is smooth.*
(2) Any parametrix Φ is smooth away from the diagonal in $M \times M$.
(3) Any parametrix Φ is such that $(Q \otimes 1 + 1 \otimes Q)\Phi$ is smooth on all of $M \times M$. (Note that $Q \otimes 1 + 1 \otimes Q$ is the natural differential on the space $\overline{\mathscr{E}}\widehat{\otimes}_\beta\overline{\mathscr{E}}$).

PROOF. We will let Q denote $Q \otimes 1 + 1 \otimes Q$, and similarly $Q^{GF} = Q^{GF} \otimes 1 + 1 \otimes Q^{GF}$, acting on the space $\overline{\mathscr{E}}\widehat{\otimes}_\beta\overline{\mathscr{E}}$. Note that

$$[Q, Q^{GF}] = [Q, Q^{GF}] \otimes 1 + 1 \otimes [Q, Q^{GF}].$$

(1) Since $[Q, Q^{GF}](\Phi - \Psi)$ is smooth, and the operator $[Q, Q^{GF}]$ is elliptic, this follows from elliptic regularity.
(2) Away from the diagonal, Φ is annihilated by the elliptic operator $[Q, Q^{GF}]$, and so is smooth.
(3) Note that

$$[Q, Q^{GF}]Q\Phi = Q[Q, Q^{GF}]\Phi$$

and that $[Q, Q^{GF}]\Phi - 2K_{Id}$ is smooth, where K_{Id} is the kernel for the identity operator. Since $QK_{Id} = 0$, the statement follows.

□

If Φ, Ψ are parametrices, we say that $\Phi < \Psi$ if the support of Φ is contained in the support of Ψ. In this way, parametrices acquire a partial order.

8.2.5. The propagator for a parametrix. In what follows, we will use the notation $Q, Q^{GF}, [Q, Q^{GF}]$ for the operators $Q \otimes 1 + 1 \otimes Q$, etc.

If Φ is a parametrix, we let

$$P(\Phi) = \tfrac{1}{2} Q^{GF} \Phi \in \overline{\mathscr{E}} \widehat{\otimes}_{\pi} \overline{\mathscr{E}}.$$

This is the propagator associated to Φ. We let

$$K_{\Phi} = K_{\mathrm{Id}} - QP(\Phi)..$$

Note that

$$\begin{aligned} QP(\Phi)) &= \tfrac{1}{2}[Q, Q^{GF}]\Phi - Q\Phi \\ &= K_{id} + \text{ smooth kernels} . \end{aligned}$$

Thus, K_{Φ} is smooth.

An important identity we will often use is that

$$K_{\Phi} - K_{\Psi} = QP(\Psi) - QP(\Phi).$$

To relate to section 8.2.3 and [**Cos11b**], we note that if M is a compact manifold and if

$$\Phi = \int_0^L K_t \mathrm{d}t$$

is the parametrix associated to the heat kernel, then

$$P(\Phi) = P(0, L) = \int_0^L (Q^{GF} \otimes 1) K_t \mathrm{d}t$$

and

$$K_{\Phi} = K_L.$$

8.2.6. Classes of functionals. In the appendix B.1 we define various classes of functions on the space $\mathscr{E}_c$ of compactly-supported fields. Here we give an overview of those classes. Many of the conditions seem somewhat technical at first, but they arise naturally as one attempts both to discuss the support of an observable and to extend the algebraic ideas of the BV formalism in this infinite-dimensional setting.

We are interested, firstly, in functions modulo constants, which we call $\mathscr{O}_{red}(\mathscr{E}_c)$. Every functional $F \in \mathscr{O}_{red}(\mathscr{E}_c)$ has a Taylor expansion in terms of symmetric smooth linear maps

$$F_k : \mathscr{E}_c^{\times k} \to \mathbb{C}$$

(for $k > 0$). Such linear maps are the same as distributional sections of the bundle $(E^!)^{\boxtimes k}$ on M^k. We say that F has *proper support* if the support of each F_k (as defined above) is a proper subset of M^k. The space of functionals with proper support is denoted $\mathscr{O}_P(\mathscr{E}_c)$ (as always in this section, we work with functionals modulo constants). This condition equivalently means that, when we think of F_k as an operator

$$\mathscr{E}_c^{\times k-1} \to \overline{\mathscr{E}}^!,$$

it extends to a smooth multilinear map

$$F_k : \mathscr{E}^{\times k-1} \to \overline{\mathscr{E}}^!.$$

At various points in this book, we will need to consider *functionals with smooth first derivative*, which are functionals satisfying a certain technical regularity constraint. Functionals with smooth first derivative are needed in two places in the text: when we define the Poisson bracket on classical observables, and when we give the definition of a quantum field theory. In terms of the Taylor components F_k, viewed as multilinear operators $\mathscr{E}_c^{\times k-1} \to \overline{\mathscr{E}}^!$, this condition means that the F_k has image in $\mathscr{E}^!$. (For more detail, see Appendix **??**, section B.1.)

We are interested in the functionals with smooth first derivative and with proper support. We denote this space by $\mathscr{O}_{P,sm}(\mathscr{E})$. These are the functionals with the property that the Taylor components F_k, when viewed as operators, give continuous linear maps

$$\mathscr{E}^{\times k-1} \to \mathscr{E}^!.$$

8.2.7. The renormalization group flow. Let Φ and Ψ be parametrices. Then $P(\Phi) - P(\Psi)$ is a smooth kernel with proper support.

Given any element

$$\alpha \in \mathscr{E} \widehat{\otimes}_\pi \mathscr{E} = C^\infty(M \times M, E \boxtimes E)$$

of cohomological degree 0, we define an operator

$$\partial_\alpha : \mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E}).$$

This map is an order 2 differential operator, which, on components, is the map given by contraction with α:

$$\alpha \vee - : \operatorname{Sym}^n \mathscr{E}^\vee \to \operatorname{Sym}^{n-2} \mathscr{E}^\vee.$$

The operator ∂_α is the unique order 2 differential operator that is given by pairing with α on $\operatorname{Sym}^2 \mathscr{E}^\vee$ and that is zero on $\operatorname{Sym}^{\leq 1} \mathscr{E}^\vee$.

We define a map

$$\begin{aligned} W(\alpha, -) : \mathscr{O}^+(\mathscr{E})[[\hbar]] &\to \mathscr{O}^+(\mathscr{E})[[\hbar]] \\ F &\mapsto \hbar \log \left(e^{\hbar \partial_\alpha} e^{F/\hbar} \right), \end{aligned}$$

known as the *renormalization group flow* with respect to α. (When $\alpha = P(\Phi) - P(\Psi)$, we call it the RG flow from Ψ to Φ.) This formula is a succinct way of summarizing a Feynman diagram expansion. In particular, $W(\alpha, F)$ can be written as a sum over Feynman diagrams with the Taylor components F_k of F labelling vertices of valence k, and with α as propagator. (All of this, and indeed everything else in this section, is explained in far greater detail in chapter 2 of [Cos11b].) For this map to be well-defined, the functional F must have only cubic and higher terms modulo $\hbar$. The notation $\mathscr{O}^+(\mathscr{E})[[\hbar]]$ denotes this restricted class of functionals.

If $\alpha \in \mathscr{E} \widehat{\otimes}_\pi \mathscr{E}$ has proper support, then the operator $W(\alpha, -)$ extends (uniquely, of course) to a continuous (or equivalently, smooth) operator

$$W(\alpha, -) : \mathscr{O}^+_{P,sm}(\mathscr{E}_c)[[\hbar]] \to \mathscr{O}^+_{P,sm}(\mathscr{E}_c)[[\hbar]].$$

Our philosophy is that a parametrix Φ is like a choice of "scale" for our field theory. The renormalization group flow relating the scale given by Φ and that given by Ψ is $W(P(\Phi) - P(\Psi), -)$.

Because $P(\Phi)$ is not a smooth kernel, the operator $W(P(\Phi), -)$ is not well-defined. This is just because the definition of $W(P(\Phi), -)$ involves multiplying distributions. In physics terms, the singularities that appear when one tries to define $W(P(\Phi), -)$ are called ultraviolet divergences.

However, if $I \in \mathscr{O}^+_{P,sm}(\mathscr{E})$, the tree level part

$$W_0(P(\Phi), I) = W((P(\Phi), I) \mod \hbar$$

is a well-defined element of $\mathscr{O}^+_{P,sm}(\mathscr{E})$. The $\hbar \to 0$ limit of $W(P(\Phi), I)$ is called the tree-level part because, whereas the whole object $W(P(\Phi), I)$ is defined as a sum over graphs, the $\hbar \to 0$ limit $W_0(P(\Phi), I)$ is defined as a sum over trees. It is straightforward to see that $W_0(P(\Phi), I)$ only involves multiplication of distributions with transverse singular support, and so is well defined.

8.2.8. The BD algebra structure associated to a parametrix. A parametrix also leads to a BV operator

$$\triangle_\Phi = \partial_{K_\Phi} : \mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E}).$$

Again, this operator preserves the subspace $\mathscr{O}_{P,sm}(\mathscr{E})$ of functions with proper support and smooth first derivative. The operator $\triangle_\Phi$ commutes with Q, and it satisfies $(\triangle_\Phi)^2 = 0$. In a standard way, we can use the BV operator $\triangle_\Phi$ to define a bracket on the space $\mathscr{O}(\mathscr{E})$, by

$$\{I, J\}_\Phi = \triangle_\Phi(IJ) - (\triangle_\Phi I)J - (-1)^{|I|} I \triangle_\Phi J.$$

This bracket is a Poisson bracket of cohomological degree 1. If we give the graded-commutative algebra $\mathscr{O}(\mathscr{E})[[\hbar]]$ the standard product, the Poisson bracket $\{-,-\}_\Phi$, and the differential $Q + \hbar\triangle_\Phi$, then it becomes a BD algebra.

The bracket $\{-,-\}_\Phi$ extends uniquely to a continuous linear map

$$\mathscr{O}_P(\mathscr{E}) \times \mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E}).$$

Further, the space $\mathscr{O}_{P,sm}(\mathscr{E})$ is closed under this bracket. (Note, however, that $\mathscr{O}_{P,sm}(\mathscr{E})$ is *not* a commutative algebra if M is not compact: the product of two functionals with proper support no longer has proper support.)

A functional $F \in \mathscr{O}(\mathscr{E})[[\hbar]]$ is said to satisfy the Φ-quantum master equation if

$$QF + \hbar\triangle_\Phi F + \tfrac{1}{2}\{F, F\}_\Phi = 0.$$

It is shown in [**Cos11b**] that if F satisfies the Φ-QME, and if Ψ is another parametrix, then $W(P(\Psi) - P(\Phi), F)$ satisfies the Ψ-QME. This follows from the identity

$$[Q, \partial_{P(\Phi)} - \partial_{P(\Psi)}] = \triangle_\Psi - \triangle_\Phi$$

of order 2 differential operators on $\mathscr{O}(\mathscr{E})$. This relationship between the renormalization group flow and the quantum master equation is a key part of the approach to QFT of [**Cos11b**].

8.2.9. The definition of a field theory. Our definition of a field theory is as follows.

8.2.9.1 Definition. *Let $(\mathscr{E}, Q, \langle -,- \rangle)$ be a free BV theory. Fix a gauge fixing condition Q^{GF}. Then a* quantum field theory *(with this space of fields) consists of the following data.*

(1) For all parametrices Φ, a functional

$$I[\Phi] \in \mathscr{O}^+_{P,sm}(\mathscr{E}_c)[[\hbar]]$$

that we call the scale Φ effective interaction*. As we explained above, the subscripts indicate that $I[\Phi]$ must have smooth first derivative and proper support. The superscript $+$ indicates that, modulo $\hbar$, $I[\Phi]$ must be at least cubic. Note that we work with functions modulo constants.*

(2) *For two parametrices Φ, Ψ, $I[\Phi]$ must be related by the renormalization group flow:*

$$I[\Phi] = W\left(P(\Phi) - P(\Psi), I[\Psi]\right).$$

(3) *Each $I[\Phi]$ must satisfy the* Φ-quantum master equation

$$(Q + \hbar \triangle_{\Phi}) e^{I[\Phi]/\hbar} = 0.$$

Equivalently,

$$Q I[\Phi] + \hbar \triangle_{\Phi} I[\Phi] + \tfrac{1}{2}\{I[\Phi], I[\Phi]\}_{\Phi}.$$

(4) *Finally, we require that $I[\Phi]$ satisfies a locality axiom. Let*

$$I_{i,k}[\Phi] : \mathscr{E}_c^{\times k} \to \mathbb{C}$$

be the kth Taylor component of the coefficient of $\hbar^i$ in $I[\Phi]$. We can view this as a distributional section of the bundle $(E^!)^{\boxtimes k}$ on M^k. Our locality axiom says that, as Φ tends to zero, the support of

$$I_{i,k}[\Phi]$$

becomes closer and closer to the small diagonal in M^k.

For the constructions in this book, it turns out to be useful to have precise bounds on the support of $I_{i,k}[\Phi]$. To give these bounds, we need some notation. Let $\operatorname{Supp}(\Phi) \subset M^2$ *be the support of the parametrix Φ, and let* $\operatorname{Supp}(\Phi)^n \subset M^2$ *be the subset obtained by convolving* $\operatorname{Supp}(\Phi)$ *with itself n times. (Thus, $(x, y) \in \operatorname{Supp}(\Phi)^n$ if there exists a sequence $x = x_0, x_1, \ldots, x_n = y$ such that $(x_i, x_{i+1}) \in \operatorname{Supp}(\Phi)$.)*

Our support condition is that, if $e_j \in \mathscr{E}_c$, then

$$I_{i,k}(e_1, \ldots, e_k) = 0$$

unless, for all $1 \leq r < s \leq k$,

$$\operatorname{Supp}(e_r) \times \operatorname{Supp}(e_s) \subset \operatorname{Supp}(\Phi)^{3i+k}.$$

Remark: (1) The locality axiom condition as presented here is a little unappealing. An equivalent axiom is that for all open subsets $U \subset M^k$ containing the small diagonal $M \subset M^k$, there exists a parametrix Φ_U such that

$$\operatorname{Supp} I_{i,k}[\Phi] \subset U \text{ for all } \Phi < \Phi_U.$$

In other words, by choosing a small parametrix Φ, we can make the support of $I_{i,k}[\Phi]$ as close as we like to the small diagonal on M^k.

We present the definition with a precise bound on the size of the support of $I_{i,k}[\Phi]$ because this bound will be important later in the construction of the factorization algebra. Note, however, that the precise exponent $3i + k$ which appears in the definition (in $\operatorname{Supp}(\Phi)^{3i+k}$) is not important. What is important is that we have some bound of this form.

(2) It is important to emphasize that the notion of quantum field theory is only defined once we have chosen a gauge fixing operator. Later, we will explain in detail how to understand the dependence on this choice. More precisely, we will construct a simplicial set of QFTs and show how this simplicial set only depends on the homotopy class of gauge fixing operator (in most examples, the space of natural gauge fixing operators is contractible).

$\Diamond$

Let $I_0 \in \mathscr{O}_{loc}(\mathscr{E})$ be a local functional (defined modulo constants) that satisfies the classical master equation

$$QI_0 + \tfrac{1}{2}\{I_0, I_0\} = 0.$$

Suppose that I_0 is at least cubic.

Then, as we have seen above, we can define a family of functionals

$$I_0[\Phi] = W_0\left(P(\Phi), I_0\right) \in \mathscr{O}_{P,sm}(\mathscr{E})$$

as the tree-level part of the renormalization group flow operator from scale 0 to the scale given by the parametrix Φ. The compatibility between this classical renormalization group flow and the classical master equation tells us that $I_0[\Phi]$ satisfies the Φ-classical master equation

$$QI_0[\Phi] + \tfrac{1}{2}\{I_0[\Phi], I_0[\Phi]\}_\Phi = 0.$$

8.2.9.2 Definition. *Let $I[\Phi] \in \mathscr{O}^+_{P,sm}(\mathscr{E})[[\hbar]]$ be the collection of effective interactions defining a quantum field theory. Let $I_0 \in \mathscr{O}_{loc}(\mathscr{E})$ be a local functional satisfying the classical master equation, and so defining a classical field theory. We say that the quantum field theory $\{I[\Phi]\}$ is a* quantization *of the classical field theory defined by I_0 if*

$$I[\Phi] = I_0[\Phi] \textit{ mod } \hbar,$$

or, equivalently, if

$$\lim_{\Phi \to 0} I[\Phi] - I_0 \textit{ mod } \hbar = 0.$$

8.3. Families of theories over nilpotent dg manifolds

Before discussing the interpretation of these axioms and also explaining the results of [Cos11b] that allow one to construct such quantum field theories, we will explain how to define families of quantum field theories over some base dg algebra. The fact that we can work in families in this way means that the moduli space of quantum field theories is something like a derived stack. For instance, by considering families over the base dg algebra of forms on the n-simplex, we see that the set of quantizations of a given classical field theory is a simplicial set.

One particularly important use of the families version of the theory is that it allows us to show that our constructions and results are independent, up to homotopy, of the choice of gauge fixing condition (provided one has a contractible — or at least connected — space of gauge fixing conditions, which happens in most examples).

In later sections, we will work implicitly over some base dg ring in the sense described here, although we will normally not mention this base ring explicitly.

8.3.0.1 Definition. *A* nilpotent dg manifold *is a manifold X (possibly with corners), equipped with a sheaf $\mathscr{A}$ of commutative differential graded algebras over the sheaf Ω^*_X, with the following properties.*

1. *$\mathscr{A}$ is concentrated in finitely many degrees.*
2. *Each $\mathscr{A}^i$ is a locally free sheaf of Ω^0_X-modules of finite rank. This means that $\mathscr{A}^i$ is the sheaf of sections of some finite rank vector bundle A^i on X.*
3. *We are given a map of dg Ω^*_X-algebras $\mathscr{A} \to C^\infty_X$.*
   We will let $\mathscr{I} \subset \mathscr{A}$ be the ideal which is the kernel of the map $\mathscr{A} \to C^\infty_X$: we require that $\mathscr{I}$, its powers $\mathscr{I}^k$, and each $\mathscr{A} / \mathscr{I}^k$ are locally free sheaves of C^∞_X-modules. Also, we require that $\mathscr{I}^k = 0$ for k sufficiently large.

Note that the differential d *on $\mathscr{A}$ is necessary a differential operator.*

We will use the notation $\mathscr{A}^\sharp$ to refer to the bundle of graded algebras on X whose smooth sections are $\mathscr{A}^\sharp$, the graded algebra underlying the dg algebra $\mathscr{A}$.

If $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ are nilpotent dg manifolds, a map $(Y, \mathscr{B}) \to (X, \mathscr{A})$ is a smooth map $f : Y \to X$ together with a map of dg $\Omega^*(X)$-algebras $\mathscr{A} \to \mathscr{B}$.

Here are some basic examples.

1. $\mathscr{A} = C^\infty(X)$ and $\mathscr{I} = 0$. This describes the smooth manifold X.
2. $\mathscr{A} = \Omega^*(X)$ and $\mathscr{I} = \Omega^{>0}(X)$. This equips X with its de Rham complex as a structure sheaf. (Informally, we can say that "constant functions are the only functions on a small open" so that this dg manifold is sensitive to topological rather than smooth structure.)
3. If R is a dg Artinian $\mathbb{C}$-algebra with maximal ideal m, then R can be viewed as giving the structure of nilpotent graded manifold on a point.
4. If again R is a dg Artinian algebra, then for any manifold $(X, R \otimes \Omega^*(X))$ is a nilpotent dg manifold.
5. If X is a complex manifold, then $\mathscr{A} = (\Omega^{0,*}(X), \overline{\partial})$ is a nilpotent dg manifold.

Remark: We study field theories in families over nilpotent dg manifolds for both practical and structural reasons. First, we certainly wish to discuss familes of field theories over smooth manifolds. However, we would also like to access a "derived moduli space" of field theories.

In derived algebraic geometry, one says that a derived stack is a functor from the category of non-positively graded dg rings to that of simplicial sets. Thus, such non-positively graded dg rings are the "test objects" one uses to define derived algebraic geometry. Our use of nilpotent dg manifolds mimics this story: we could say that a C^∞ derived stack is a functor from nilpotent dg manifolds to simplicial sets. The nilpotence hypothesis is not a great restriction, as the test objects used in derived algebraic geometry are naturally pro-nilpotent, where the pro-nilpotent ideal consists of the elements in degrees < 0.

Second, from a practical point of view, our arguments are tractable when working over nilpotent dg manifolds. This is related to the fact that we choose to encode the analytic structure on the vector spaces we consider using the language of differentiable vector spaces. Differentiable vector spaces are, by definition, objects where one can talk about smooth families of maps depending on a smooth manifold. In fact, the definition of differentiable vector space is strong enough that one can talk about smooth families of maps depending on nilpotent dg manifolds. ◊

We can now give a precise notion of "family of field theories." We will start with the case of a family of field theories parameterized by the nilpotent dg manifold $X = (X, C^\infty_X)$, i.e. the sheaf of dg rings on X is just the sheaf of smooth functions.

8.3.0.2 Definition. *Let M be a manifold and let $(X, \mathscr{A})$ be a nilpotent dg manifold. A family over $(X, \mathscr{A})$ of free BV theories is the following data.*

(1) *A graded bundle E on $M \times X$ of locally free $A^\sharp$-modules. We will refer to global sections of E as $\mathscr{E}$. The space of those sections $s \in \Gamma(M \times X, E)$ with the property that the map $\operatorname{Supp} s \to X$ is proper will be denoted $\mathscr{E}_c$. Similarly, we let $\overline{\mathscr{E}}$ denote the space of sections which are distributional on M and smooth on X, that is,*

$$\overline{\mathscr{E}} = \mathscr{E} \otimes_{C^\infty(M\times X)} \left(\mathcal{D}(M) \widehat{\otimes}_\pi C^\infty(X)\right).$$

(This is just the algebraic tensor product, which is reasonable as $\mathscr{E}$ is a finitely generated projective $C^\infty(M \times X)$-module).

As above, we let

$$E^! = \operatorname{Hom}_{A^\sharp}(E, A^\sharp) \otimes \operatorname{Dens}_M$$

denote the "dual" bundle. There is a natural $\mathscr{A}^\sharp$-valued pairing between $\mathscr{E}$ and $\mathscr{E}^!_c$.

(2) *A differential operator $Q : \mathscr{E} \to \mathscr{E}$, of cohomological degree 1 and square-zero, making $\mathscr{E}$ into a dg module over the dg algebra $\mathscr{A}$.*

(3) *A map*

$$E \otimes_{A^\sharp} E \to \operatorname{Dens}_M \otimes A^\sharp$$

which is of degree -1, anti-symmetric, and leads to an isomorphism

$$\mathrm{Hom}_{A^\sharp}(E, A^\sharp) \otimes \mathrm{Dens}_M \to E$$

of sheaves of $A^\sharp$-modules on $M \times X$.

This pairing leads to a degree -1 anti-symmetric $\mathscr{A}$-linear pairing

$$\langle -, - \rangle : \mathscr{E}_c \widehat{\otimes}_\pi \mathscr{E}_c \to \mathscr{A}.$$

We require it to be a cochain map. In other words, if $e, e' \in \mathscr{E}_c$,

$$\mathrm{d}_{\mathscr{A}} \langle e, e' \rangle = \langle Qe, e' \rangle + (-1)^{|e|} \langle e, Qe' \rangle .$$

8.3.0.3 Definition. *Let $(E, Q, \langle -, - \rangle)$ be a family of free BV theories on M parameterized by $\mathscr{A}$. A* gauge fixing condition *on $\mathscr{E}$ is an $\mathscr{A}$-linear differential operator*

$$Q^{GF} : \mathscr{E} \to \mathscr{E}$$

such that

$$\mathrm{D} = [Q, Q^{GF}] : \mathscr{E} \to \mathscr{E}$$

is a generalized Laplacian, in the following sense.

Note that D is an $\mathscr{A}$-linear cochain map. Thus, we can form

$$\mathrm{D}_0 : \mathscr{E} \otimes_{\mathscr{A}} C^\infty(X) \to \mathscr{E} \otimes_{\mathscr{A}} C^\infty(X)$$

by reducing modulo the maximal ideal $\mathscr{I}$ of $\mathscr{A}$.

Let $E_0 = E/I$ be the bundle on $M \times X$ obtained by reducing modulo the ideal I in the bundle of algebras A. Let

$$\sigma(\mathrm{D}_0) : \pi^* E_0 \to \pi^* E_0$$

be the symbol of the $C^\infty(X)$-linear operator D_0. Thus, $\sigma(\mathrm{D}_0)$ is an endomorphism of the bundle of $\pi^* E_0$ on $(T^*M) \times X$.

We require that $\sigma(\mathrm{D}_0)$ is the product of the identity on E_0 with a smooth family of metrics on M parameterized by X.

Throughout this section, we will fix a family of free theories on M, parameterized by $\mathscr{A}$. We will take $\mathscr{A}$ to be our base ring throughout, so that everything will be $\mathscr{A}$-linear. We would also like to take tensor products over $\mathscr{A}$. Since $\mathscr{A}$ is a topological dg ring and we are dealing with topological modules, the issue of tensor products is a little fraught. Instead of trying to define such things, we will use the following shorthand notations:

(1) $\mathscr{E} \otimes_{\mathscr{A}} \mathscr{E}$ is defined to be sections of the bundle

$$E \boxtimes_{A^\sharp} E = \pi_1^* E \otimes_{A^\sharp} \pi_2^* E$$

on $M \times M \times X$, with its natural differential which is a differential operator induced from the differentials on each copy of $\mathscr{E}$.

(2) $\overline{\mathscr{E}}$ is the space of sections of the bundle E on $M \times X$ which are smooth in the X-direction and distributional in the M-direction. Similarly for $\overline{\mathscr{E}}_c$, $\overline{\mathscr{E}}^!$, etc.
(3) $\overline{\mathscr{E}} \otimes_{\mathscr{A}} \overline{\mathscr{E}}$ is defined to be sections of the bundle $E \boxtimes_{A^\sharp} E$ on $M \times M \times X$, which are distributions in the M-directions and smooth as functions of X.
(4) If $x \in X$, let $\mathscr{E}_x$ denote the sections on M of the restriction of the bundle E on $M \times X$ to $M \times x$. Note that $\mathscr{E}_x$ is an $A^\sharp_x$-module. Then, we define $\mathscr{O}(\mathscr{E})$ to be the space of smooth sections of the bundle of topological (or differentiable) vector spaces on X whose fibre at x is

$$\mathscr{O}(\mathscr{E})_x = \prod_n \operatorname{Hom}_{DVS/A^\sharp_x}(\mathscr{E}_x^{\times n}, \mathscr{A}^\sharp_x)_{S_n}.$$

That is an element of $\mathscr{O}(\mathscr{E}_x)$ is something whose Taylor expansion is given by smooth $A^\sharp_x$-multilinear maps to $A^\sharp_x$.

If $F \in \mathscr{O}(\mathscr{E})$ is a smooth section of this bundle, then the Taylor terms of F are sections of the bundle $(E^!)^{\boxtimes_{A^\sharp} n}$ on $M^n \times X$ which are distributional in the M^n-directions, smooth in the X-directions, and whose support maps properly to X.

In other words: when we want to discuss spaces of functionals on $\mathscr{E}$, or tensor powers of $\mathscr{E}$ or its distributional completions, we just to everything we did before fibrewise on X and linear over the bundle of algebras $A^\sharp$. Then, we take sections of this bundle on X.

8.3.1. Now that we have defined free theories over a base ring $\mathscr{A}$, the definition of an interacting theory over $\mathscr{A}$ is very similar to the definition given when $\mathscr{A} = \mathbb{C}$. First, one defines a parametrix to be an element

$$\Phi \in \overline{\mathscr{E}} \otimes_{\mathscr{A}} \overline{\mathscr{E}}$$

with the same properties as before, but where now we take all tensor products (and so on) over $\mathscr{A}$. More precisely,

(1) Φ is symmetric under the natural $\mathbb{Z}/2$ action on $\overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$.
(2) Φ is of cohomological degree 1.
(3) Φ is closed under the differential on $\overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$.
(4) Φ has proper support: this means that the map $\operatorname{Supp}\Phi \to M \times X$ is proper.
(5) Let $Q^{GF} : \mathscr{E} \to \mathscr{E}$ be the gauge fixing operator. We require that

$$([Q, Q^{GF}] \otimes 1)\Phi - K_{\mathrm{Id}}$$

is an element of $\mathscr{E} \otimes \mathscr{E}$ (where, as before, $K_{\mathrm{Id}} \in \overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$ is the kernel for the identity map).

An interacting field theory is then defined to be a family of $\mathscr{A}$-linear functionals

$$I[\Phi] \in \mathscr{O}_{red}(\mathscr{E})[[\hbar]] = \prod_{n \geq 1} \operatorname{Hom}_{\mathscr{A}}(\mathscr{E}^{\otimes_{\mathscr{A}} n}, \mathscr{A})_{S_n}[[\hbar]]$$

satisfying the renormalization group flow equation, quantum master equation, and locality condition, just as before. In order for the RG flow to make sense, we require that each $I[\Phi]$ has proper support and smooth first derivative. In this context, this means the following. Let $I_{i,k}[\Phi] : \mathscr{E}^{\otimes k} \to \mathscr{A}$ be the kth Taylor component of the coefficient of $\hbar^i$ in $I_{i,k}[\Phi]$. Proper support means that any projection map

$$\operatorname{Supp} I_{i,k}[\Phi] \subset M^k \times X \to M \times X$$

is proper. Smooth first derivative means, as usual, that when we think of $I_{i,k}[\Phi]$ as an operator $\mathscr{E}^{\otimes k-1} \to \overline{\mathscr{E}}$, the image lies in $\mathscr{E}$.

If we have a family of theories over $(X, \mathscr{A})$, and a map

$$f : (Y, \mathscr{B}) \to (X, \mathscr{A})$$

of dg manifolds, then we can base change to get a family over $(Y, \mathscr{B})$. The bundle on Y of $B_x^\sharp$-modules of fields is defined, fibre by fibre, by

$$(f^* \mathscr{E})_y = \mathscr{E}_{f(y)} \otimes_{A^\sharp_{f(y)}} B^\sharp_y.$$

The gauge fixing operator

$$Q^{GF} : f^* \mathscr{E} \to f^* \mathscr{E}$$

is the $\mathscr{B}$-linear extension of the gauge fixing condition for the family of theories over $\mathscr{A}$.

If

$$\Phi \in \overline{\mathscr{E}} \otimes_{\mathscr{A}} \overline{\mathscr{E}} \subset f^* \overline{\mathscr{E}} \otimes_{\mathscr{B}} f^* \overline{\mathscr{E}}$$

is a parametrix for the family of free theories $\mathscr{E}$ over $\mathscr{A}$, then it defines a parametrix $f^*\Phi$ for the family of free theories $f^*\mathscr{E}$ over $\mathscr{B}$. For parametrices of this form, the effective action functionals

$$f^* I[f^*\Phi] \in \mathscr{O}^+_{sm,P}(f^*\mathscr{E})[[\hbar]] = \mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]] \otimes_{\mathscr{A}} \mathscr{B}$$

is simply the image of the original effective action functional

$$I[\Phi] \in \mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]] \subset \mathscr{O}^+_{sm,P}(f^*\mathscr{E})[[\hbar]].$$

For a general parametrix Ψ for $f^*\mathscr{E}$, the effective action functional is defined by the renormalization group equation

$$f^* I[\Psi] = W\left(P(\Psi) - P(f^*\Phi), f^* I[f^*\Phi]\right).$$

This is well-defined because

$$P(\Psi) - P(f^*\Phi) \in f^*\mathscr{E} \otimes_{\mathscr{B}} f^*\mathscr{E}$$

has no singularities.

The compatibility between the renormalization group equation and the quantum master equation guarantees that the effective action functionals $f^* I[\Psi]$ satisfy the QME for every parametrix Ψ. The locality axiom for the original family of effective action functionals

$I[\Phi]$ guarantees that the pulled-back family $f^* I[\Psi]$ satisfy the locality axiom necessary to define a family of theories over $\mathscr{B}$.

8.4. The simplicial set of theories

One of the main reasons for introducing theories over a nilpotent dg manifold $(X, \mathscr{A})$ is that this allows us to talk about the simplicial set of theories. This is essential, because the main result we will use from [Cos11b] is homotopical in nature: it relates the simplicial set of theories to the simplicial set of local functionals.

We introduce some useful notation. Let us fix a family of classical field theories on a manifold M over a nilpotent dg manifold $(X, \mathscr{A})$. As above, the fields of such a theory are a dg $\mathscr{A}$-module $\mathscr{E}$ equipped with an $\mathscr{A}$-linear local functional $I \in \mathscr{O}_{loc}(\mathscr{E})$ satisfying the classical master equation $QI + \frac{1}{2}\{I, I\} = 0$.

By pulling back along the projection map

$$(X \times \triangle^n, \mathscr{A} \otimes C^\infty(\triangle^n)) \to (X, \mathscr{A}),$$

we get a new family of classical theories over the dg base ring $\mathscr{A} \otimes C^\infty(\triangle^n)$, whose fields are $\mathscr{E} \otimes C^\infty(\triangle^n)$. We can then ask for a gauge fixing operator

$$Q^{GF} : \mathscr{E} \otimes C^\infty(\triangle^n) \to \mathscr{E} \otimes C^\infty(\triangle^n).$$

for this family of theories. This is the same thing as a smooth family of gauge fixing operators for the original theory depending on a point in the n-simplex.

8.4.0.1 Definition. *Let $(\mathscr{E}, I)$ denote the classical theory we start with over $\mathscr{A}$. Let $\mathscr{GF}(\mathscr{E}, I)$ denote the simplicial set whose n-simplices are such families of gauge fixing operators over $\mathscr{A} \otimes C^\infty(\triangle^n)$. If there is no ambiguity as to what classical theory we are considering, we will denote this simplicial set by $\mathscr{GF}$.*

Any such gauge fixing operator extends, by $\Omega^*(\triangle^n)$-linearity, to a linear map $\mathscr{E} \otimes \Omega^*(\triangle^n) \to \mathscr{E} \otimes \Omega^*(\triangle^n)$, which thus defines a gauge fixing operator for the family of theories over $\mathscr{A} \otimes \Omega^*(\triangle^n)$ pulled back via the projection

$$(X \times \triangle^n, \mathscr{A} \otimes \Omega^*(\triangle^n)) \to (X, \mathscr{A}).$$

(Note that $\Omega^*(\triangle^n)$ is equipped with the de Rham differential.)

Example: Suppose that $\mathscr{A} = \mathbb{C}$, and the classical theory we are considering is Chern-Simons theory on a 3-manifold M, where we perturb around the trivial bundle. Then, the space of fields is $\mathscr{E} = \Omega^*(M) \otimes \mathfrak{g}[1]$ and $Q = d_{dR}$. For every Riemannian metric on M, we find a gauge fixing operator $Q^{GF} = d^*$. More generally, if we have a smooth family

$$\{g_\sigma \mid \sigma \in \triangle^n\}$$

of Riemannian metrics on M, depending on the point σ in the n-simplex, we get an n-simplex of the simplicial set $\mathscr{GF}$ of gauge fixing operators.

Thus, if $\mathrm{Met}(M)$ denotes the simplicial set whose n-simplices are the set of Riemannian metrics on the fibers of the submersion $M \times \triangle^n \to \triangle^n$, then we have a map of simplicial sets

$$\mathrm{Met}(M) \to \mathscr{GF}.$$

Note that the simplicial set $\mathrm{Met}(M)$ is (weakly) contractible (which follows from the familiar fact that, as a topological space, the space of metrics on M is contractible).

A similar remark holds for almost all theories we consider. For example, suppose we have a theory where the space of fields

$$\mathscr{E} = \Omega^{0,*}(M, V)$$

is the Dolbeault complex on some complex manifold M with coefficients in some holomorphic vector bundle V. Suppose that the linear operator $Q : \mathscr{E} \to \mathscr{E}$ is the $\overline{\partial}$-operator. The natural gauge fixing operators are of the form $\overline{\partial}^*$. Thus, we get a gauge fixing operator for each choice of Hermitian metric on M together with a Hermitian metric on the fibers of V. This simplicial set is again contractible.

It is in this sense that we mean that, in most examples, there is a natural contractible space of gauge fixing operators. $\lozenge$

8.4.1. We will use the shorthand notation $(\mathscr{E}, I)$ to denote the classical field theory over $\mathscr{A}$ that we start with; and we will use the notation $(\mathscr{E}_{\triangle^n}, I_{\triangle^n})$ to refer to the family of classical field theories over $\mathscr{A} \otimes \Omega^*(\triangle^n)$ obtained by base-change along the projection $(X \times \triangle^n, \mathscr{A} \otimes \Omega^*(\triangle^n)) \to (X, \mathscr{A})$.

8.4.1.1 Definition. *We let $\mathscr{T}^{(n)}$ denote the simplicial set whose k-simplices consist of the following data.*

1. *A k-simplex $Q^{GF}_{\triangle^k} \in \mathscr{GF}[k]$, defining a gauge-fixing operator for the family of theories $(\mathscr{E}_{\triangle^k}, I_{\triangle^k})$ over $\mathscr{A} \otimes \Omega^*(\triangle^k)$.*
2. *A quantization of the family of classical theories with gauge fixing operator $(\mathscr{E}_{\triangle^k}, I_{\triangle^k}, Q^{GF}_{\triangle^k})$, defined modulo $\hbar^{n+1}$.*

We let $\mathscr{T}^{(\infty)}$ denote the corresponding simplicial set where the quantizations are defined to all orders in $\hbar$.

Note that there are natural maps of simplicial sets $\mathscr{T}^{(n)} \to \mathscr{T}^{(m)}$, and that $\mathscr{T}^{(\infty)} = \varprojlim \mathscr{T}^{(n)}$. Further, there are natural maps $\mathscr{T}^{(n)} \to \mathscr{GF}$.

Note further that $\mathscr{T}^{(0)} = \mathscr{G}\mathscr{F}$.

This definition describes the most sophisticated version of the set of theories we will consider. Let us briefly explain how to interpret this simplicial set of theories.

Suppose for simplicity that our base ring $\mathscr{A}$ is just $\mathbb{C}$. Then, a 0-simplex of $\mathscr{T}^{(0)}$ is simply a gauge-fixing operator for our theory. A 0-simplex of $\mathscr{T}^{(n)}$ is a gauge fixing operator, together with a quantization (defined with respect to that gauge-fixing operator) to order n in $\hbar$.

A 1-simplex of $\mathscr{T}^{(0)}$ is a homotopy between two gauge fixing operators. Suppose that we fix a 0-simplex of $\mathscr{T}^{(0)}$, and consider a 1-simplex of $\mathscr{T}^{(\infty)}$ in the fiber over this 0-simplex. Such a 1-simplex is given by a collection of effective action functionals

$$I[\Phi] \in \mathscr{O}^+_{P,sm}(\mathscr{E}) \otimes \Omega^*([0,1])[[\hbar]]$$

one for each parametrix Φ, which satisfy a version of the QME and the RG flow, as explained above.

We explain in some more detail how one should interpret such a 1-simplex in the space of theories. Let us fix a parametrix Φ on $\mathscr{E}$ and extend it to a parametrix for the family of theories over $\Omega^*([0,1])$. We can then expand our effective interaction $I[\Phi]$ as

$$I[\Phi] = J[\Phi](t) + J'[\Phi](t)\mathrm{d}t$$

where $J[\Phi](t), J'[\Phi](t)$ are elements

$$J[\Phi](t), J'[\Phi](t) \in \mathscr{O}^+_{P,sm}(\mathscr{E}) \otimes C^\infty([0,1])[[\hbar]].$$

Here t is the coordinate on the interval $[0,1]$.

The quantum master equation implies that the following two equations hold, for each value of $t \in [0,1]$,

$$QJ[\Phi](t) + \tfrac{1}{2}\{J[\Phi](t), J[\Phi](t)\}_\Phi + \hbar\triangle_\Phi J[\Phi](t) = 0,$$
$$\frac{\partial}{\partial t}J[\Phi](t) + QJ'[\Phi](t) + \{J[\Phi](t), J'[\Phi](t)\}_\Phi + \hbar\triangle_\Phi J'[\Phi](t) = 0.$$

The first equation tells us that for each value of t, $J[\Phi](t)$ is a solution of the quantum master equation. The second equation tells us that the t-derivative of $J[\Phi](t)$ is homotopically trivial as a deformation of the solution to the QME $J[\Phi](t)$.

In general, if I is a solution to some quantum master equation, a transformation of the form

$$I \mapsto I + \varepsilon J = I + \varepsilon QI' + \{I, I'\} + \hbar\triangle I'$$

is often called a “BV canonical transformation” in the physics literature. In the physics literature, solutions of the QME related by a canonical transformation are regarded as equivalent: the canonical transformation can be viewed as a change of coordinates on the space of fields.

For us, this interpretation is not so important. If we have a family of theories over $\Omega^*([0,1])$, given by a 1-simplex in $\mathscr{T}^{(\infty)}$, then the factorization algebra we will construct from this family of theories will be defined over the dg base ring $\Omega^*([0,1])$. This implies that the factorization algebras obtained by restricting to 0 and 1 are quasi-isomorphic.

8.4.2. Generalizations. We will shortly state the theorem which allows us to construct such quantum field theories. Let us first, however, briefly introduce a slightly more general notion of "theory."

We work over a nilpotent dg manifold $(X, \mathscr{A})$. Recall that part of the data of such a manifold is a differential ideal $I \subset \mathscr{A}$ whose quotient is $C^\infty(X)$. In the above discussion, we assumed that our classical action functional S was at least quadratic; we then split S as

$$S = \langle e, Qe \rangle + I(e)$$

into kinetic and interacting terms.

We can generalize this to the situation where S contains linear terms, as long as they are accompanied by elements of the ideal $\mathscr{I} \subset \mathscr{A}$. In this situation, we also have some freedom in the splitting of S into kinetic and interacting terms; we require only that linear and quadratic terms in the interaction I are weighted by elements of the nilpotent ideal $\mathscr{I}$.

In this more general situation, the classical master equation $\{S, S\} = 0$ does not imply that $Q^2 = 0$, only that $Q^2 = 0$ modulo the ideal $\mathscr{I}$. However, this does not lead to any problems; the definition of quantum theory given above can be easily modified to deal with this more general situation.

In the L_∞ language used in Chapter 4, this more general situation describes a family of curved L_∞ algebras over the base dg ring $\mathscr{A}$ with the property that the curving vanishes modulo the nilpotent ideal $\mathscr{I}$.

Recall that ordinary (not curved) L_∞ algebras correspond to formal pointed moduli problems. These curved L_∞ algebras correspond to families of formal moduli problems over $\mathscr{A}$ which are pointed modulo $\mathscr{I}$.

8.5. The theorem on quantization

Let M be a manifold, and suppose we have a family of classical BV theories on M over a nilpotent dg manifold $(X, \mathscr{A})$. Suppose that the space of fields on M is the $\mathscr{A}$-module $\mathscr{E}$. Let $\mathscr{O}_{loc}(\mathscr{E})$ be the dg $\mathscr{A}$-module of local functionals with differential $Q + \{I, -\}$.

Given a cochain complex C, we denote the Dold-Kan simplicial set associated to C by $\mathrm{DK}(C)$. Its n-simplices are the closed, degree 0 elements of $C \otimes \Omega^*(\triangle^n)$.

8.5.0.1 Theorem. *All of the simplicial sets $\mathscr{T}^{(n)}(\mathscr{E}, I)$ are Kan complexes and $\mathscr{T}^{(\infty)}(\mathscr{E}, I)$. The maps $p : \mathscr{T}^{(n+1)}(\mathscr{E}, I) \to \mathscr{T}^{(n)}(\mathscr{E}, I)$ are Kan fibrations.*

Further, there is a homotopy fiber diagram of simplicial sets

$$\begin{tikzcd} \mathscr{T}^{(n+1)}(\mathscr{E}, I) \arrow[r] \arrow[d, "p"'] & 0 \arrow[d] \\ \mathscr{T}^{(n)}(\mathscr{E}, I) \arrow[r, "O"] & \mathrm{DK}(\mathscr{O}_{loc}(\mathscr{E})[1], Q + \{I, -\}) \end{tikzcd}$$

where O is the "obstruction map."

In more prosaic terms, the second part of the theorem says the following. If $\alpha \in \mathscr{T}^{(n)}(\mathscr{E}, I)[0]$ is a zero-simplex of $\mathscr{T}^{(n)}(\mathscr{E}, I)$, then there is an obstruction $O(\alpha) \in \mathscr{O}_{loc}(\mathscr{E})$. This obstruction is a closed degree 1 element. The simplicial set $p^{-1}(\alpha) \in \mathscr{T}^{(n+1)}(\mathscr{E}, I)$ of extensions of α to the next order in $\hbar$ is homotopy equivalent to the simplicial set of ways of making $O(\alpha)$ exact. In particular, if the cohomology class $[O(\alpha)] \in H^1(\mathscr{O}_{loc}(E), Q + \{I, -\})$ is non-zero, then α does not admit a lift to the next order in $\hbar$. If this cohomology class is zero, then the simplicial set of possible lifts is a torsor for the simplicial Abelian group $\mathrm{DK}(\mathscr{O}_{loc}(\mathscr{E}))[1]$.

Note also that a first order deformation of the classical field theory $(\mathscr{E}, Q, I)$ is given by a closed degree 0 element of $\mathscr{O}_{loc}(\mathscr{E})$. Further, two such first order deformations are equivalent if they are cohomologous. Thus, this theorem tells us that the moduli space of QFTs is "the same size" as the moduli space of classical field theories: at each order in $\hbar$, the data needed to describe a QFT is a local action functional.

The first part of the theorem says can be interpreted as follows. A Kan simplicial set can be thought of as an "infinity-groupoid." Since we can consider families of theories over arbitrary nilpotent dg manifolds, we can consider $\mathscr{T}^{\infty}(\mathscr{E}, I)$ as a functor from the category of nilpotent dg manifolds to that of Kan complexes, or infinity-groupoids. Thus, the space of theories forms something like a "derived stack" [**Toë06**, **Lur11**].

This theorem also tells us in what sense the notion of "theory" is independent of the choice of gauge fixing operator. The simplicial set $\mathscr{T}^{(0)}(\mathscr{E}, I)$ is the simplicial set $\mathscr{G}\mathscr{F}$ of gauge fixing operators. Since the map

$$\mathscr{T}^{(\infty)}(\mathscr{E}, I) \to \mathscr{T}^{(0)}(\mathscr{E}, I) = \mathscr{G}\mathscr{F}$$

is a fibration, a path between two gauge fixing conditions Q_0^{GF} and Q_1^{GF} leads to a homotopy between the corresponding fibers, and thus to an equivalence between the ∞-groupoids of theories defined using Q_0^{GF} and Q_1^{GF}.

As we mentioned several times, there is often a natural contractible simplicial set mapping to the simplicial set $\mathscr{G}\mathscr{F}$ of gauge fixing operators. Thus, $\mathscr{G}\mathscr{F}$ often has a canonical "homotopy point". From the homotopical point of view, having a homotopy point is just as good as having an actual point: if $S \to \mathscr{G}\mathscr{F}$ is a map out of a contractible simplicial set, then the fibers in $\mathscr{T}^{(\infty)}$ above any point in S are canonically homotopy equivalent.

CHAPTER 9

The observables of a quantum field theory

9.1. Free fields

Before we give our general construction of the factorization algebra associated to a quantum field theory, we will give the much easier construction of the factorization algebra for a free field theory.

Let us recall the definition of a free BV theory.

9.1.0.1 Definition. *A free BV theory on a manifold M consists of the following data:*

(1) *a $\mathbb{Z}$-graded super vector bundle $\pi : E \to M$ that has finite rank;*
(2) *an antisymmetric map of vector bundles $\langle -, - \rangle_{loc} : E \otimes E \to \mathrm{Dens}(M)$ of degree -1 that is fiberwise nondegenerate. It induces a symplectic pairing on compactly supported smooth sections $\mathscr{E}_c$ of E:*

$$\langle \phi, \psi \rangle = \int_{x \in M} \langle \phi(x), \psi(x) \rangle_{loc};$$

(3) *a square-zero differential operator $Q : \mathscr{E} \to \mathscr{E}$ of cohomological degree 1 that is skew self adjoint for the symplectic pairing.*

Remark: When we consider deforming free theories into interacting theories, we will need to assume the existence of a "gauge fixing operator": this is a degree -1 operator $Q^{GF} : \mathscr{E} \to E$ such that $[Q, Q^{GF}]$ is a generalized Laplacian in the sense of [**BGV92**]. $\Diamond$

On any open set $U \subset M$, the commutative dg algebra of classical observables supported in U is

$$\mathrm{Obs}^{cl}(U) = (\widehat{\mathrm{Sym}}(\mathscr{E}^{\vee}(U)), Q),$$

where

$$\mathscr{E}^{\vee}(U) = \overline{\mathscr{E}}^{!}_{c}(U)$$

denotes the distributions dual to $\mathscr{E}$ with compact support in U and Q is the derivation given by extending the natural action of Q on the distributions.

In section 6.3 we constructed a sub-factorization algebra

$$\widetilde{\mathrm{Obs}}^{cl}(U) = (\widehat{\mathrm{Sym}}(\mathscr{E}_c^!(U)), Q)$$

defined as the symmetric algebra on the compactly-supported smooth (rather than distributional) sections of the bundle $E^!$. We showed that the inclusion $\widetilde{\mathrm{Obs}}^{cl}(U) \to \mathrm{Obs}^{cl}(U)$ is a weak equivalence of factorization algebras. Further, $\widetilde{\mathrm{Obs}}^{cl}(U)$ has a Poisson bracket of cohomological degree 1, defined on the generators by the natural pairing

$$\mathscr{E}_c^!(U)\widehat{\otimes}_\pi \mathscr{E}_c^!(U) \to \mathbb{R},$$

which arises from the dual pairing on $\mathscr{E}_c(U)$. In this section we will show how to construct a quantization of the P_0 factorization algebra $\widetilde{\mathrm{Obs}}^{cl}$.

9.1.1. The Heisenberg algebra construction. Our quantum observables on an open set U will be built from a certain Heisenberg Lie algebra.

Recall the usual construction of a Heisenberg algebra. If V is a symplectic vector space, viewed as an abelian Lie algebra, then the Heisenberg algebra $\mathrm{Heis}(V)$ is the central extension

$$0 \to \mathbb{C}\cdot\hbar \to \mathrm{Heis}(V) \to V$$

whose bracket is $[x,y] = \hbar\langle x,y\rangle$.

Since the element $\hbar \in \mathrm{Heis}(V)$ is central, the algebra $\widehat{U}(\mathrm{Heis}(V))$ is an algebra over $\mathbb{C}[[\hbar]]$, the completed universal enveloping algebra of the Abelian Lie algebra $\mathbb{C}\cdot\hbar$.

In quantum mechanics, this Heisenberg construction typically appears in the study of systems with quadratic Hamiltonians. In this context, the space V can be viewed in two ways. Either it is the space of solutions to the equations of motion, which is a linear space because we are dealing with a free field theory; or it is the space of linear observables dual to the space of solutions to the equations of motion. The natural symplectic pairing on V gives an isomorphism between these descriptions. The algebra $\widehat{U}(\mathrm{Heis}(V))$ is then the algebra of non-linear observables.

Our construction of the quantum observables of a free field theory will be formally very similar. We will start with a space of linear observables, which (after a shift) is a cochain complex with a symplectic pairing of cohomological degree 1. Then, instead of applying the usual universal enveloping algebra construction, we will take Chevalley-Eilenberg chain complex, whose cohomology is the Lie algebra homology.[1] This fits with our operadic philosophy: Chevalley-Eilenberg chains are the E_0 analog of the universal enveloping algebra.

[1] As usual, we always use gradings such that the differential has degree $+1$.

9.1.2. The basic homological construction. Let us start with a 0-dimensional free field theory. Thus, let V be a cochain complex equipped with a symplectic pairing of cohomological degree -1. We will think of V as the space of fields of our theory. The space of linear observables of our theory is $V^\vee$; the Poisson bracket on $\mathscr{O}(V)$ induces a symmetric pairing of degree 1 on $V^\vee$. We will construct the space of all observables from a Heisenberg Lie algebra built on $V^\vee[-1]$, which has a symplectic pairing $\langle -, - \rangle$ of degree -1. Note that there is an isomorphism $V \cong V^\vee[-1]$ compatible with the pairings on both sides.

9.1.2.1 Definition. *The Heisenberg algebra* $\operatorname{Heis}(V)$ *is the Lie algebra central extension*

$$0 \to \mathbb{C} \cdot \hbar[-1] \to \operatorname{Heis}(V) \to V^\vee[-1] \to 0$$

whose bracket is

$$[v + \hbar a, w + \hbar b] = \hbar \langle v, w \rangle$$

The element $\hbar$ *labels the basis element of the center* $\mathbb{C}[-1]$.

Putting the center in degree 1 may look strange, but it is necessary to do this in order to get a Lie bracket of cohomological degree 0.

Let $\widehat{C}_*(\operatorname{Heis}(V))$ denote the completion[2] of the Lie algebra chain complex of $\operatorname{Heis}(V)$, defined by the product of the spaces $\operatorname{Sym}^n \operatorname{Heis}(V)$, instead of their sum.

In this zero-dimensional toy model, the classical observables are

$$\operatorname{Obs}^{cl} = \mathscr{O}(V) = \prod_n \operatorname{Sym}^n(V^\vee).$$

This is a commutative dg algebra equipped with the Poisson bracket of degree 1 arising from the pairing on V. Thus, $\mathscr{O}(V)$ is a P_0 algebra.

9.1.2.2 Lemma. *The completed Chevalley-Eilenberg chain complex* $\widehat{C}_*(\operatorname{Heis}(V))$ *is a BD algebra (section 2.4) which is a quantization of the* P_0 *algebra* $\mathscr{O}(V)$.

PROOF. The completed Chevalley-Eilenberg complex for $\operatorname{Heis}(V)$ has the completed symmetric algebra $\widehat{\operatorname{Sym}}(\operatorname{Heis}(V)[1])$ as its underlying graded vector space. Note that

$$\widehat{\operatorname{Sym}}(\operatorname{Heis}(V)[1]) = \operatorname{Sym}(V^\vee \oplus \mathbb{C} \cdot \hbar) = \widehat{\operatorname{Sym}}(V^\vee)[[\hbar]],$$

so that $\widehat{C}_*(\operatorname{Heis}(V))$ is a flat $\mathbb{C}[[\hbar]]$ module which reduces to $\widehat{\operatorname{Sym}}(V^\vee)$ modulo $\hbar$. The Chevalley-Eilenberg chain complex $\widehat{C}_*(\operatorname{Heis}(V))$ inherits a product, corresponding to the natural product on the symmetric algebra $\widehat{\operatorname{Sym}}(\operatorname{Heis}(V)[1])$. Further, it has a natural Poisson bracket of cohomological degree 1 arising from the Lie bracket on $\operatorname{Heis}(V)$, extended

[2] One doesn't need to take the completed Lie algebra chain complex. We do this to be consistent with our discussion of the observables of interacting field theories, where it is essential to complete.

to be a derivation of $\widehat{C}_*(\mathrm{Heis}(V))$. Note that, since $\mathbb{C} \cdot \hbar[-1]$ is central in $\mathrm{Heis}(V)$, this Poisson bracket reduces to the given Poisson bracket on $\widehat{\mathrm{Sym}}(V^\vee)$ modulo $\hbar$.

In order to prove that we have a BD quantization, it remains to verify that, although the commutative product on $\widehat{C}_*(\mathrm{Heis}(V))$ is not compatible with the product, it satisfies the BD axiom:

$$\mathrm{d}(a \cdot b) = (\mathrm{d}a) \cdot b + (-1)^{|a|} a \cdot (\mathrm{d}b) + \hbar\{a, b\}.$$

This follows by definition. □

9.1.3. Cosheaves of Heisenberg algebras. Next, let us give the analog of this construction for a general free BV theory E on a manifold M. As above, our classical observables are defined by

$$\widetilde{\mathrm{Obs}}^{cl}(U) = \widehat{\mathrm{Sym}}\, \mathscr{E}_c^!(U)$$

which has a Poisson bracket arising from the pairing on $\mathscr{E}_c^!(U)$. Recall that this is a factorization algebra.

To construct the quantum theory, we define, as above, a Heisenberg algebra $\mathrm{Heis}(U)$ as a central extension

$$0 \to \mathbb{C}[-1] \cdot \hbar \to \mathrm{Heis}(U) \to \mathscr{E}_c^!(U)[-1] \to 0.$$

Note that $\mathrm{Heis}(U)$ is a pre-cosheaf of Lie algebras. The bracket in this Heisenberg algebra arises from the pairing on $\mathscr{E}_c^!(U)$.

We then define the quantum observables by

$$\mathrm{Obs}^q(U) = \widehat{C}_*(\mathrm{Heis}(U)).$$

The underlying cochain complex is, as before,

$$\widehat{\mathrm{Sym}}(\mathrm{Heis}(U)[1])$$

where the completed symmetric algebra is defined (as always) using the completed tensor product.

9.1.3.1 Proposition. *Sending U to $\mathrm{Obs}^q(U)$ defines a BD factorization algebra in the category of differentiable pro-cochain complexes over $\mathbb{R}[[\hbar]]$, which quantizes $\mathrm{Obs}^{cl}(U)$.*

PROOF. First, we need to define the filtration on $\mathrm{Obs}^q(U)$ making it into a differentiable pro-cochain complex. The filtration is defined, in the identification

$$\mathrm{Obs}^q(U) = \widehat{\mathrm{Sym}}\, \mathscr{E}_c^!(U)[[\hbar]]$$

by saying

$$F^n \mathrm{Obs}^q(U) = \prod_k \hbar^k \mathrm{Sym}^{\geq n-2k} \mathscr{E}_c^!(U).$$

This filtration is engineered so that the $F^n \mathrm{Obs}^q(U)$ is a subcomplex of $\mathrm{Obs}^q(U)$.

It is immediate that Obs^q is a BD pre-factorization algebra quantizing $\mathrm{Obs}^{cl}(U)$. The fact that it is a factorization algebra follows from the fact that $\mathrm{Obs}^{cl}(U)$ is a factorization algebra, and then a simple spectral sequence argument. (A more sophisticated version of this spectral sequence argument, for interacting theories, is given in section 9.6.) □

9.2. The BD algebra of global observables

In this section, we will try to motivate our definition of a quantum field theory from the point of view of homological algebra. All of the constructions we will explain will work over an arbitrary nilpotent dg manifold $(X, \mathscr{A})$, but to keep the notation simple we will not normally mention the base ring $\mathscr{A}$.

Thus, suppose that $(\mathscr{E}, I, Q, \langle -, - \rangle)$ is a classical field theory on a manifold M. We have seen (Chapter 6, section 6.2) how such a classical field theory gives immediately a commutative factorization algebra whose value on an open subset is

$$\mathrm{Obs}^{cl}(U) = (\mathscr{O}(\mathscr{E}(U)), Q + \{I, -\}).$$

Further, we saw that there is a P_0 sub-factorization algebra

$$\widetilde{\mathrm{Obs}}^{cl}(U) = (\mathscr{O}_{sm}(\mathscr{E}(U)), Q + \{I, -\}).$$

In particular, we have a P_0 algebra $\widetilde{\mathrm{Obs}}^{cl}(M)$ of global sections of this P_0 algebra. We can think of $\widetilde{\mathrm{Obs}}^{cl}(M)$ as the algebra of functions on the derived space of solutions to the Euler-Lagrange equations.

In this section we will explain how a quantization of this classical field theory will give a quantization (in a homotopical sense) of the P_0 algebra $\widetilde{\mathrm{Obs}}^{cl}(M)$ into a BD algebra $\mathrm{Obs}^q(M)$ of global observables. This BD algebra has some locality properties, which we will exploit later to show that $\mathrm{Obs}^q(M)$ is indeed the global sections of a factorization algebra of quantum observables.

In the case when the classical theory is the cotangent theory to some formal elliptic moduli problem $B\mathcal{L}$ on M (encoded in an elliptic L_∞ algebra $\mathcal{L}$ on M), there is a particularly nice class of quantizations, which we call cotangent quantizations. Cotangent quantizations have a very clear geometric interpretation: they are locally-defined volume forms on the sheaf of formal moduli problems defined by $\mathcal{L}$.

9.2.1. The BD algebra associated to a parametrix. Suppose we have a quantization of our classical field theory (defined with respect to some gauge fixing condition, or family of gauge fixing conditions). Then, for every parametrix Φ, we have seen how to construct a cohomological degree 1 operator

$$\triangle_\Phi : \mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E})$$

and a Poisson bracket

$$\{-,-\}_\Phi : \mathscr{O}(\mathscr{E}) \times \mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E})$$

such that $\mathscr{O}(\mathscr{E})[[\hbar]]$, with the usual product, with bracket $\{-,-\}_\Phi$ and with differential $Q + \hbar\triangle_\Phi$, forms a BD algebra.

Further, since the effective interaction $I[\Phi]$ satisfies the quantum master equation, we can form a new BD algebra by adding $\{I[\Phi],-\}_\Phi$ to the differential of $\mathscr{O}(\mathscr{E})[[\hbar]]$.

9.2.1.1 Definition. *Let* $\mathrm{Obs}^q_\Phi(M)$ *denote the BD algebra*

$$\mathrm{Obs}^q_\Phi(M) = (\mathscr{O}(\mathscr{E})[[\hbar]], Q + \hbar\triangle_\Phi + \{I[\Phi],-\}_\Phi),$$

with bracket $\{-,-\}_\Phi$ *and the usual product.*

Remark: Note that $I[\Phi]$ is not in $\mathscr{O}(\mathscr{E})[[\hbar]]$, but rather in $\mathscr{O}^+_{P,sm}(\mathscr{E})[[\hbar]]$. However, as we remarked earlier in 8.2.8, the bracket

$$\{I[\Phi],-\}_\Phi : \mathscr{O}(\mathscr{E})[[\hbar]] \to \mathscr{O}(\mathscr{E})[[\hbar]]$$

is well-defined. $\Diamond$

Remark: Note that we consider $\mathrm{Obs}^q_\Phi(M)$ as a BD algebra valued in the multicategory of differentiable pro-cochain complexes (see Appendix **??**). This structure includes a filtration on $\mathrm{Obs}^q_\Phi(M) = \mathscr{O}(\mathscr{E})[[\hbar]]$. The filtration is defined by saying that

$$F^n\mathscr{O}(\mathscr{E})[[\hbar]] = \prod_i \hbar^i \,\mathrm{Sym}^{\geq(n-2i)}(\mathscr{E}^\vee);$$

it is easily seen that the differential $Q + \hbar\triangle_\Phi + \{I[\Phi],-\}_\Phi$ preserves this filtration. $\Diamond$

We will show that for varying Φ, the BD algebras $\mathrm{Obs}^q_\Phi(M)$ are canonically weakly equivalent. Moreover, we will show that there is a canonical weak equivalence of P_0 algebras

$$\mathrm{Obs}^q_\Phi(M) \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C} \simeq \widetilde{\mathrm{Obs}}^{cl}(M).$$

To show this, we will construct a family of BD algebras over the dg base ring of forms on a certain contractible simplicial set of parametrices that restricts to $\mathrm{Obs}^q_\Phi(M)$ at each vertex.

Before we get into the details of the construction, however, let us say something about how this result allows us to interpret the definition of a quantum field theory.

A quantum field theory gives a BD algebra for each parametrix. These BD algebras are all canonically equivalent. Thus, at first glance, one might think that the data of a QFT is entirely encoded in the BD algebra for a single parametrix. However, this does not take account of a key part of our definition of a field theory, that of *locality*.

The BD algebra associated to a parametrix Φ has underlying commutative algebra $\mathscr{O}(\mathscr{E})[[\hbar]]$, equipped with a differential which we temporarily denote

$$\mathrm{d}_\Phi = Q + \hbar\triangle_\Phi + \{I[\Phi], -\}_\Phi.$$

If $K \subset M$ is a closed subset, we have a restriction map

$$\mathscr{E} = \mathscr{E}(M) \to \mathscr{E}(K),$$

where $\mathscr{E}(K)$ denotes germs of smooth sections of the bundle E on K. There is a dual map on functionals $\mathscr{O}(\mathscr{E}(K)) \to \mathscr{O}(\mathscr{E})$. We say a functional $f \in \mathscr{O}(\mathscr{E})[[\hbar]]$ is *supported on K* if it is in the image of this map.

As $\Phi \to 0$, the effective interaction $I[\Phi]$ and the BV Laplacian $\triangle_\Phi$ become more and more local (i.e., their support gets closer to the small diagonal). This tells us that, for very small Φ, the operator d_Φ only increases the support of a functional in $\mathscr{O}(\mathscr{E})[[\hbar]]$ by a small amount. Further, by choosing Φ to be small enough, we can increase the support by an arbitrarily small amount.

Thus, a quantum field theory is

(1) A family of BD algebra structures on $\mathscr{O}(\mathscr{E})[[\hbar]]$, one for each parametrix, which are all homotopic (and which all have the same underlying graded commutative algebra).
(2) The differential d_Φ defining the BD structure for a parametrix Φ increases support by a small amount if Φ is small.

This property of d_Φ for small Φ is what will allow us to construct a factorization algebra of quantum observables. If d_Φ did not increase the support of a functional $f \in \mathscr{O}(\mathscr{E})[[\hbar]]$ at all, the factorization algebra would be easy to define: we would just set $\mathrm{Obs}^q(U) = \mathscr{O}(\mathscr{E}(U))[[\hbar]]$, with differential d_Φ. However, because d_Φ does increase support by some amount (which we can take to be arbitrarily small), it takes a little work to push this idea through.

Remark: The precise meaning of the statement that d_Φ increases support by an arbitrarily small amount is a little delicate. Let us explain what we mean. A functional $f \in \mathscr{O}(\mathscr{E})[[\hbar]]$ has an infinite Taylor expansion of the form $f = \sum \hbar^i f_{i,k}$, where $f_{i,k} : \mathscr{E}^{\widehat{\otimes}_\pi k} \to \mathbb{C}$ is a symmetric linear map. We let $\mathrm{Supp}_{\leq(i,k)} f$ be the unions of the supports of $f_{r,s}$ where $(r,s) \leq (i,k)$ in the lexicographical ordering. If $K \subset M$ is a subset, let $\Phi^n(K)$ denote the subset obtained by convolving n times with $\mathrm{Supp}\,\Phi \subset M^2$. The differential d_Φ has the following property: there are constants $c_{i,k} \in \mathbb{Z}_{>0}$ of a purely combinatorial nature (independent of the theory we are considering) such that, for all $f \in \mathscr{O}(\mathscr{E})[[\hbar]]$,

$$\mathrm{Supp}_{\leq(i,k)}\, \mathrm{d}_\Phi f \subset \Phi^{c_{i,k}}(\mathrm{Supp}_{\leq(i,k)} f).$$

Thus, we could say that d_Φ increase support by an amount linear in $\mathrm{Supp}\,\Phi$. We will use this concept in the main theorem of this chapter. $\Diamond$

9.2.2. Let us now turn to the construction of the equivalences between $\mathrm{Obs}^q_\Phi(M)$ for varying parametrices Φ. The first step is to construct the simplicial set $\mathscr{P}$ of parametrices; we will then construct a BD algebra $\mathrm{Obs}^q_{\mathscr{P}}(M)$ over the base dg ring $\Omega^*(\mathscr{P})$, which we define below.

Let
$$V \subset C^\infty(M \times M, E \boxtimes E) = \mathscr{E} \widehat{\otimes}_\pi \mathscr{E}$$
denote the subspace of those elements which are cohomologically closed and of degree 1, symmetric, and have proper support.

Note that the set of parametrices has the structure of an affine space for V: if Φ, Ψ are parametrices, then
$$\Phi - \Psi \in V$$
and, conversely, if Φ is a parametrix and $A \in V$, then $\Phi + A$ is a new parametrix.

Let $\mathscr{P}$ denote the simplicial set whose n-simplices are affine-linear maps from $\triangle^n$ to the affine space of parametrices. It is clear that $\mathscr{P}$ is contractible.

For any vector space V, let $V_\triangle$ denote the simplicial set whose k-simplices are affine linear maps $\triangle^k \to V$. For any convex subset $U \subset V$, there is a sub-simplicial set $U_\triangle \subset V_\triangle$ whose k-simplices are affine linear maps $\triangle^k \to U$. Note that $\mathscr{P}$ is a sub-simplicial set of $\overline{\mathscr{E}}_\triangle^{\widehat{\otimes}_\pi 2}$, corresponding to the convex subset of parametrices inside $\overline{\mathscr{E}}^{\widehat{\otimes}_\pi 2}$.

Let $\mathscr{C}\mathscr{P}[0] \subset \overline{\mathscr{E}}^{\widehat{\otimes}_\pi 2}$ denote the cone on the affine subspace of parametrices, with vertex the origin $\overline{0}$. An element of $\mathscr{C}\mathscr{P}[0]$ is an element of $\overline{\mathscr{E}}^{\widehat{\otimes}_\pi 2}$ of the form $t\Phi$, where Φ is a parametrix and $t \in [0,1]$. Let $\mathscr{C}\mathscr{P}$ denote the simplicial set whose k-simplices are affine linear maps to $\mathscr{C}\mathscr{P}[0]$.

Recall that the simplicial de Rham algebra $\Omega^*_\triangle(S)$ of a simplicial set S is defined as follows. Any element $\omega \in \Omega^i_\triangle(S)$ consists of an i-form
$$\omega(\phi) \in \Omega^i(\triangle^k)$$
for each k-simplex $\phi : \triangle^k \to S$. If $f : \triangle^k \to \triangle^l$ is a face or degeneracy map, then we require that
$$f^*\omega(\phi) = \omega(\phi \circ f).$$
The main results of this section are as follows.

9.2.2.1 Theorem. *There is a BD algebra* $\mathrm{Obs}^q_{\mathscr{P}}(M)$ *over* $\Omega^*(\mathscr{P})$ *which, at each* 0*-simplex* Φ*, is the BD algebra* $\mathrm{Obs}^q_\Phi(M)$ *discussed above.*

The underlying graded commutative algebra of $\mathrm{Obs}^q_{\mathscr{P}}(M)$ *is* $\mathscr{O}(\mathscr{E}) \otimes \Omega^*(\mathscr{P})[[\hbar]]$.

For every open subset $U \subset M \times M$, let $\mathscr{P}_U$ denote the parametrices whose support is in U. Let $\mathrm{Obs}^q_{\mathscr{P}_U}(M)$ denote the restriction of $\mathrm{Obs}^q_{\mathscr{P}}(M)$ to U. The differential on $\mathrm{Obs}^q_{\mathscr{P}_U}(M)$ increases support by an amount linear in U (in the sense explained precisely in the remark above).

The bracket $\{-,-\}_{\mathscr{P}_U}$ on $\mathrm{Obs}^q_{\mathscr{P}_U}(M)$ is also approximately local, in the following sense. If $O_1, O_2 \in \mathrm{Obs}^q_{\mathscr{P}_U}(M)$ have the property that

$$\mathrm{Supp}\, O_1 \times \mathrm{Supp}\, O_2 \cap U = \emptyset \in M \times M,$$

then $\{O_1, O_2\}_{\mathscr{P}_U} = 0$.

Further, there is a P_0 algebra $\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)$ over $\Omega^*(\mathscr{C}\mathscr{P})$ equipped with a quasi-isomorphism of P_0 algebras over $\Omega^*(\mathscr{P})$,

$$\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)\Big|_{\mathscr{P}} \simeq \mathrm{Obs}^q_{\mathscr{P}}(M) \text{ modulo } \hbar,$$

and with an isomorphism of P_0 algebras,

$$\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)\Big|_{\bar{0}} \cong \widetilde{\mathrm{Obs}}^{cl}(M),$$

where $\widetilde{\mathrm{Obs}}^{cl}(M)$ is the P_0 algebra constructed in Chapter 6.

The underlying commutative algebra of $\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)$ is $\widetilde{\mathrm{Obs}}^{cl}(M) \otimes \Omega^*(\mathscr{C}\mathscr{P})$, the differential on $\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)$ increases support by an arbitrarily small amount, and the Poisson bracket on $\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M)$ is approximately local in the same sense as above.

PROOF. We need to construct, for each k-simplex $\phi : \triangle^k \to \mathscr{P}$, a BD algebra $\mathrm{Obs}^q_\phi(M)$ over $\Omega^*(\triangle^k)$. We view the k-simplex as a subset of $\mathbb{R}^{k+1}$ by

$$\triangle^k := \left\{ (\lambda_0, \dots, \lambda_k) \subset [0,1]^{k+1} \ : \ \sum_i \lambda_i = 1 \right\}.$$

Since simplices in $\mathscr{P}$ are affine linear maps to the space of parametrices, the simplex ϕ is determined by $k+1$ parametrices $\Phi_0, \dots, \Phi_k$, with

$$\phi(\lambda_0, \dots, \lambda_k) = \sum_i \lambda_i \Phi_i$$

for $\lambda_i \in [0,1]$ and $\sum \lambda_i = 1$.

The graded vector space underlying our BD algebra is

$$\mathrm{Obs}^q_\phi(M) = \mathscr{O}(\mathscr{E})[[\hbar]] \otimes \Omega^*(\triangle^k).$$

The structure as a BD algebra will be encoded by an order two, $\Omega^*(\triangle^k)$-linear differential operator

$$\triangle_\phi : \mathrm{Obs}^q_\phi(M) \to \mathrm{Obs}^q_\phi(M).$$

We need to recall some notation in order to define this operator. Each parametrix Φ provides an order two differential operator $\triangle_\Phi$ on $\mathscr{O}(\mathscr{E})$, the BV Laplacian corresponding to Φ. Further, if Φ, Ψ are two parametrices, then the difference between the propagators $P(\Phi) - P(\Psi)$ is an element of $\mathscr{E} \otimes \mathscr{E}$, so that contracting with $P(\Phi) - P(\Psi)$ defines an order two differential operator $\partial_{P(\Phi)} - \partial_{P(\Psi)}$ on $\mathscr{O}(\mathscr{E})$. (This operator defines the infinitesimal version of the renormalization group flow from Ψ to Φ.) We have the equation

$$[Q, \partial_{P(\Phi)} - \partial_{P(\Psi)}] = -\triangle_\Phi + \triangle_\Psi.$$

Note that although the operator $\partial_{P(\Phi)}$ is only defined on the smaller subspace $\mathscr{O}(\overline{\mathscr{E}})$, because $P(\Phi) \in \overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$, the difference $\partial_{P(\Phi)}$ and $\partial_{P(\Psi)}$ is nonetheless well-defined on $\mathscr{O}(\mathscr{E})$ because $P(\Phi) - P(\Psi) \in \mathscr{E} \otimes \mathscr{E}$.

The BV Laplacian $\triangle_\phi$ associated to the k-simplex $\phi : \triangle^k \to \mathscr{P}$ is defined by the formula

$$\triangle_\phi = \sum_{i=0}^{k} \lambda_i \triangle_{\Phi_i} - \sum_{i=0}^{k} \mathrm{d}\lambda_i \partial_{P(\Phi_i)},$$

where the $\lambda_i \in [0,1]$ are the coordinates on the simplex $\triangle^k$ and, as above, the Φ_i are the parametrices associated to the vertices of the simplex ϕ.

It is not entirely obvious that this operator makes sense as a linear map $\mathscr{O}(\mathscr{E}) \to \mathscr{O}(\mathscr{E}) \otimes \Omega^*(\triangle^k)$, because the operators $\partial_{P(\Phi)}$ are only defined on the smaller subspace $\mathscr{O}(\overline{\mathscr{E}})$. However, since $\sum \mathrm{d}\lambda_i = 0$, we have

$$\sum \mathrm{d}\lambda_i \partial_{P(\Phi_i)} = \sum \mathrm{d}\lambda_i (\partial_{P(\Phi_i)} - \partial_{P(\Phi_0)}),$$

and the right hand side is well defined.

It is immediate that $\triangle_\phi^2 = 0$. If we denote the differential on the classical observables $\mathscr{O}(\mathscr{E}) \otimes \Omega^*(\triangle^n)$ by $Q + \mathrm{d}_{dR}$, we have

$$[Q + \mathrm{d}_{dR}, \triangle_\phi] = 0.$$

To see this, note that

$$\begin{aligned}
[Q + \mathrm{d}_{dR}, \triangle_\phi] &= \sum \mathrm{d}\lambda_i \triangle_{\Phi_i} + \sum \mathrm{d}\lambda_i [Q, \partial_{\Phi_i} - \partial_{\Phi_0}] \\
&= \sum \mathrm{d}\lambda_i \triangle_{\Phi_i} - \sum \mathrm{d}\lambda_i (\triangle_{\Phi_i} - \triangle_{\Phi_0}) \\
&= \sum \mathrm{d}\lambda_i \triangle_{\Phi_0} \\
&= 0,
\end{aligned}$$

where we use various identities from earlier.

The operator $\triangle_\phi$ defines, in the usual way, an $\Omega^*(\triangle^k)$-linear Poisson bracket $\{-,-\}_\phi$ on $\mathscr{O}(\mathscr{E}) \otimes \Omega^*(\triangle^k)$.

We have effective action functionals $I[\Psi] \in \mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]]$ for each parametrix Ψ. Let

$$I[\phi] = I[\sum \lambda_i \Phi_i] \in \mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]] \otimes C^\infty(\triangle^k).$$

The renormalization group equation tells us that $I[\sum \lambda_i \Phi_i]$ is smooth (actually polynomial) in the λ_i.

We define the structure of BD algebra on the graded vector space

$$\mathrm{Obs}^q_\phi(M) = \mathscr{O}(\mathscr{E})[[\hbar]] \otimes \Omega^*(\triangle^k)$$

as follows. The product is the usual one; the bracket is $\{-,-\}_\phi$, as above; and the differential is

$$Q + \mathrm{d}_{dR} + \hbar \triangle_\phi + \{I[\phi], -\}_\phi.$$

We need to check that this differential squares to zero. This is equivalent to the quantum master equation

$$(Q + \mathrm{d}_{dR} + \hbar \triangle_\phi) e^{I[\phi]/\hbar} = 0.$$

This holds as a consequence of the quantum master equation and renormalization group equation satisfied by $I[\phi]$. Indeed, the renormalization group equation tells us that

$$e^{I[\phi]/\hbar} = \exp\left(\hbar \sum \lambda_i \left(\partial_{P\Phi_i)} - \partial_{P(\Phi_0)}\right)\right) e^{I[\Phi_0]/\hbar}.$$

Thus,

$$\mathrm{d}_{dR} e^{I[\phi]/\hbar} = \hbar \sum \mathrm{d}\lambda_i \partial_{P(\Phi_i)} e^{I[\phi]/\hbar}$$

The QME for each $I[\sum \lambda_i \Phi_i]$ tells us that

$$(Q + \hbar \sum \lambda_i \triangle_{\Phi_i}) e^{I[\phi]/\hbar} = 0.$$

Putting these equations together with the definition of $\triangle_\phi$ shows that $I[\phi]$ satisfies the QME.

Thus, we have constructed a BD algebra $\mathrm{Obs}^q_\phi(M)$ over $\Omega^*(\triangle^k)$ for every simplex $\phi : \triangle^k \to \mathscr{P}$. It is evident that these BD algebras are compatible with face and degeneracy maps, and so glue together to define a BD algebra over the simplicial de Rham complex $\Omega^*_\triangle(\mathscr{P})$ of $\mathscr{P}$.

Let ϕ be a k-simplex of $\mathscr{P}$, and let

$$\mathrm{Supp}(\phi) = \cup_{\lambda \in \triangle^k} \mathrm{Supp}(\sum \lambda_i \Phi_i).$$

We need to check that the bracket $\{O_1, O_2\}_\phi$ vanishes for observables O_1, O_2 such that $(\mathrm{Supp}\, O_1 \times \mathrm{Supp}) O_2 \cap \mathrm{Supp}\, \phi = \emptyset$. This is immediate, because the bracket is defined by contracting with tensors in $\mathscr{E} \otimes \mathscr{E}$ whose supports sit inside $\mathrm{Supp}\, \phi$.

Next, we need to verify that, on a k-simplex ϕ of $\mathscr{P}$, the differential $Q + \{I[\phi], -\}_\phi$ increases support by an amount linear in $\mathrm{Supp}(\phi)$. This follows from the support properties satisfied by $I[\Phi]$ (which are detailed in the definition of a quantum field theory, definition 8.2.9.1).

It remains to construct the P_0 algebra over $\Omega^*(\mathscr{C}\mathscr{P})$. The construction is almost identical, so we will not give all details. A zero-simplex of $\mathscr{C}\mathscr{P}$ is an element of $\overline{\mathscr{E}} \otimes \overline{\mathscr{E}}$ of the form $\Psi = t\Phi$, where Φ is a parametrix. We can use the same formulae we used for parametrices to construct a propagator $P(\Psi)$ and Poisson bracket $\{-,-\}_\Psi$ for each $\Psi \in \mathscr{C}\mathscr{P}$. The kernel defining the Poisson bracket $\{-,-\}_\Psi$ need not be smooth. This means that the bracket $\{-,-\}_\Psi$ is only defined on the subspace $\mathscr{O}_{sm}(\mathscr{E})$ of functionals with smooth first derivative. In particular, if $\Psi = 0$ is the vertex of the cone $\mathscr{C}\mathscr{P}$, then $\{-,-\}_0$ is the Poisson bracket defined in Chapter 6 on $\widetilde{\mathrm{Obs}}^{cl}(M) = \mathscr{O}_{sm}(\mathscr{E})$.

For each $\Psi \in \mathscr{C}\mathscr{P}$, we can form a tree-level effective interaction

$$I_0[\Psi] = W_0\left(P(\Psi), I\right) \in \mathscr{O}_{sm,P}(\mathscr{E}),$$

where $I \in \mathscr{O}_{loc}(\mathscr{E})$ is the classical action functional we start with. There are no difficulties defining this expression because we are working at tree-level and using functionals with smooth first derivative. If $\Psi = 0$, then $I_0[0] = I$.

The P_0 algebra over $\Omega^*(\mathscr{C}\mathscr{P})$ is defined in almost exactly the same way as we defined the BD algebra over $\Omega^*_{\mathscr{P}}$. The underlying commutative algebra is $\mathscr{O}_{sm}(\mathscr{E}) \otimes \Omega^*(\mathscr{C}\mathscr{P})$. On a k-simplex ψ with vertices $\Psi_0, \dots, \Psi_k$, the Poisson bracket is

$$\{-,-\}_\psi = \sum \lambda_i \{-,-\}_{\Psi_i} + \sum \mathrm{d}\lambda_i \{-,-\}_{P(\Psi_i)},$$

where $\{-,-\}_{P(\Psi_i)}$ is the Poisson bracket of cohomological degree 0 defined using the propagator $P(\Psi_i) \in \overline{\mathscr{E}} \widehat{\otimes}_\pi \overline{\mathscr{E}}$ as a kernel. If we let $I_0[\psi] = I_0[\sum \lambda_i \Psi_i]$, then the differential is

$$\mathrm{d}_\psi = Q + \{I_0[\psi], -\}_\psi.$$

The renormalization group equation and classical master equation satisfied by the $I_0[\Psi]$ imply that $\mathrm{d}_\psi^2 = 0$. If $\Psi = 0$, this P_0 algebra is clearly the P_0 algebra $\widetilde{\mathrm{Obs}}^{cl}(M)$ constructed in Chapter 6. When restricted to $\mathscr{P} \subset \mathscr{C}\mathscr{P}$, this P_0 algebra is the sub P_0 algebra of $\mathrm{Obs}^q_{\mathscr{P}}(M)/\hbar$ obtained by restricting to functionals with smooth first derivative; the inclusion

$$\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{C}\mathscr{P}}(M) \mid_{\mathscr{P}} \hookrightarrow \mathrm{Obs}^q_{\mathscr{P}}(M)/\hbar$$

is thus a quasi-isomorphism, using proposition 6.4.2.4 of Chapter 6. □

9.3. Global observables

In the next few sections, we will prove the first version (section 1.3) of our quantization theorem. Our proof is by construction, associating a factorization algebra on M to a quantum field theory on M, in the sense of [**Cos11b**]. This is a quantization (in the weak sense) of the P_0 factorization algebra associated to the corresponding classical field theory.

More precisely, we will show the following.

9.3.0.1 Theorem. *For any quantum field theory on a manifold M over a nilpotent dg manifold $(X, \mathscr{A})$, there is a factorization algebra Obs^q on M, valued in the multicategory of differentiable pro-cochain complexes flat over $\mathscr{A}[[\hbar]]$.*

There is an isomorphism of factorization algebras

$$\mathrm{Obs}^q \otimes_{\mathscr{A}[[\hbar]]} \mathscr{A} \cong \mathrm{Obs}^{cl}$$

between Obs^q modulo $\hbar$ and the commutative factorization algebra Obs^{cl}.

Further, Obs^q is a weak quantization (in the sense of Chapter **??***, section 1.3) of the P_0 factorization algebra Obs^{cl} of classical observables.*

9.3.1. So far we have constructed a BD algebra $\mathrm{Obs}^q_\Phi(M)$ for each parametrix Φ; these BD algebras are all weakly equivalent to each other. In this section we will define a cochain complex $\mathrm{Obs}^q(M)$ of global observables which is independent of the choice of parametrix. For every open subset $U \subset M$, we will construct a subcomplex $\mathrm{Obs}^q(U) \subset \mathrm{Obs}^q(M)$ of observables supported on U. The complexes $\mathrm{Obs}^q(U)$ will form our factorization algebra.

Thus, suppose we have a quantum field theory on M, with space of fields $\mathscr{E}$ and effective action functionals $\{I[\Phi]\}$, one for each parametrix (as explained in section 8.2).

An *observable* for a quantum field theory (that is, an element of the cochain complex $\mathrm{Obs}^q(M)$) is simply a first-order deformation $\{I[\Phi] + \delta O[\Phi]\}$ of the family of effective action functionals $I[\Phi]$, which satisfies a renormalization group equation but does not necessarily satisfy the locality axiom in the definition of a quantum field theory. Definition 9.3.1.3 makes this idea precise.

Remark: This definition is motivated by a formal argument with the path integral. Let $S(\phi)$ be the action functional for a field ϕ, and let $O(\phi)$ be another function of the field, describing a measurement that one could make. Heuristically, the expectation value of the observable is

$$\langle O \rangle = \frac{1}{Z_S} \int O(\phi) e^{-S(\phi)/\hbar} \mathscr{D}\phi,$$

where Z_S denotes the partition function, simply the integral without O. A formal manipulation shows that

$$\langle O \rangle = \frac{d}{d\delta} \frac{1}{Z_S} \int e^{(-S(\phi)+\hbar\delta O(\phi))/\hbar} \mathscr{D}\phi.$$

In other words, we can view O as a first-order deformation of the action functional S and compute the expectation value as the change in the partition function. Because the book [**Cos11b**] gives an approach to the path integral that incorporates the BV formalism, we can define and compute expectation values of observables by exploiting the second description of $\langle O \rangle$ given above. $\Diamond$

Earlier we defined cochain complexes $\mathrm{Obs}^q_\Phi(M)$ for each parametrix. The underlying graded vector space of $\mathrm{Obs}^q_\Phi(M)$ is $\mathscr{O}(\mathscr{E})[[\hbar]]$; the differential on $\mathrm{Obs}^q_\Phi(M)$ is

$$\widehat{Q}_\Phi = Q + \{I[\Phi], -\}_\Phi + \hbar \triangle_\Phi.$$

9.3.1.1 Definition. *Define a linear map*

$$W^\Phi_\Psi : \mathscr{O}(\mathscr{E})[[\hbar]] \to \mathscr{O}(\mathscr{E})[[\hbar]]$$

by requiring that, for an element $f \in \mathscr{O}(\mathscr{E})[[\hbar]]$ of cohomological degree i,

$$I[\Phi] + \delta W^\Phi_\Psi(f) = W\left(P(\Phi) - P(\Psi), I[\Psi] + \delta f\right)$$

where δ is a square-zero parameter of cohomological degree $-i$.

9.3.1.2 Lemma. *The linear map*

$$W^\Phi_\Psi : \mathrm{Obs}^q_\Psi(M) \to \mathrm{Obs}^q_\Phi(M)$$

is an isomorphism of differentiable pro-cochain complexes.

PROOF. The fact that W^Φ_Ψ intertwines the differentials $\widehat{Q}_\Phi$ and $\widehat{Q}_\Psi$ follows from the compatibility between the quantum master equation and the renormalization group equation described in [Cos11b], Chapter 5 and summarized in section 8.2. It is not hard to verify that W^Φ_Ψ is a map of differentiable pro-cochain complexes. The inverse to W^Φ_Ψ is W^Ψ_Φ. $\square$

9.3.1.3 Definition. *A global observable O of cohomological degree i is an assignment to every parametrix Φ of an element*

$$O[\Phi] \in \mathrm{Obs}^q_\Phi(M) = \mathscr{O}(\mathcal{E})[[\hbar]]$$

of cohomological degree i such that

$$W^\Phi_\Psi O[\Psi] = O[\Phi].$$

If O is an observable of cohomological degree i, we let $\widehat{Q}O$ be defined by

$$\widehat{Q}(O)[\Phi] = \widehat{Q}_\Phi(O[\Phi]) = QO[\Phi] + \{I[\Phi], O[\Phi]\}_\Phi + \hbar \triangle_\Phi O[\Phi].$$

This makes the space of observables into a differentiable pro-cochain complex, which we call $\mathrm{Obs}^q(M)$.

Thus, if $O \in \mathrm{Obs}^q(M)$ is an observable of cohomological degree i, and if δ is a square-zero parameter of cohomological degree $-i$, then the collection of effective interactions $\{I[\Phi] + \delta O[\Phi]\}$ satisfy most of the axioms needed to define a family of quantum field theories over the base ring $\mathbb{C}[\delta]/\delta^2$. The only axiom which is not satisfied is the locality axiom: we have not imposed any constraints on the behavior of the $O[\Phi]$ as $\Phi \to 0$.

9.4. Local observables

So far, we have defined a cochain complex $\mathrm{Obs}^q(M)$ of global observables on the whole manifold M. If $U \subset M$ is an open subset of M, we would like to isolate those observables which are "supported on U".

The idea is to say that an observable $O \in \mathrm{Obs}^q(M)$ is supported on U if, for sufficiently small parametrices, $O[\Phi]$ is supported on U. The precise definition is as follows.

9.4.0.1 Definition. *An observable $O \in \mathrm{Obs}^q(M)$ is supported on U if, for each $(i,k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, there exists a compact subset $K \subset U^k$ and a parametrix Φ, such that for all parametrices $\Psi \leq \Phi$*

$$\operatorname{Supp} O_{i,k}[\Psi] \subset K.$$

Remark: Recall that $O_{i,k}[\Phi] : \mathscr{E}^{\otimes k} \to \mathbb{C}$ is the kth term in the Taylor expansion of the coefficient of $\hbar^i$ of the functional $O[\Phi] \in \mathscr{O}(\mathscr{E})[[\hbar]]$. $\Diamond$

Remark: As always, the definition works over an arbitrary nilpotent dg manifold $(X, \mathscr{A})$, even though we suppress this from the notation. In this generality, instead of a compact subset $K \subset U^k$ we require $K \subset U^k \times X$ to be a set such that the map $K \to X$ is proper. $\Diamond$

We let $\mathrm{Obs}^q(U) \subset \mathrm{Obs}^q(M)$ be the sub-graded vector space of observables supported on U.

9.4.0.2 Lemma. *$\mathrm{Obs}^q(U)$ is a sub-cochain complex of $\mathrm{Obs}^q(M)$. In other words, if $O \in \mathrm{Obs}^q(U)$, then so is $\widehat{Q}O$.*

PROOF. The only thing that needs to be checked is the support condition. We need to check that, for each (i,k), there exists a compact subset K of U^k such that, for all sufficiently small Φ, $\widehat{Q}O_{i,k}[\Phi]$ is supported on K.

Note that we can write

$$\widehat{Q}O_{i,k}[\Phi] = QO_{i,k}[\Phi] + \sum_{\substack{a+b=i \\ r+s=k+2}} \{I_{a,r}[\Phi], O_{b,s}[\Phi]\}_\Phi + \Delta_\Phi O_{i-1,k+2}[\Phi].$$

We now find a compact subset K for $\widehat{Q}O_{i,k}[\Phi]$. We know that, for each (i,k) and for all sufficiently small Φ, $O_{i,k}[\Phi]$ is supported on $\tilde{K}$, where $\tilde{K}$ is some compact subset of U^k. It follows that $QO_{i,k}[\Phi]$ is supported on $\tilde{K}$.

By making $\tilde{K}$ bigger, we can assume that for sufficiently small Φ, $O_{i-1,k+2}[\Phi]$ is supported on L, where L is a compact subset of U^{k+2} whose image in U^k, under every projection map, is in $\tilde{K}$. This implies that $\Delta_\Phi O_{i-1,k+2}[\Phi]$ is supported on $\tilde{K}$.

The locality condition for the effective actions $I[\Phi]$ implies that, by choosing Φ to be sufficiently small, we can make $I_{i,k}[\Phi]$ supported as close as we like to the small diagonal in M^k. It follows that, by choosing Φ to be sufficiently small, the support of $\{I_{a,r}[\Phi], O_{b,s}[\Phi]\}_\Phi$ can be taken to be a compact subset of U^k. Since there are only a finite number of terms like $\{I_{a,r}[\Phi], O_{b,s}[\Phi]\}_\Phi$ in the expression for $(\widehat{Q}O)_{i,k}[\Phi]$, we see that for Φ sufficiently small, $(\widehat{Q}O)_{i,k}[\Phi]$ is supported on a compact subset K of U^k, as desired. □

9.4.0.3 Lemma. $\mathrm{Obs}^q(U)$ *has a natural structure of differentiable pro-cochain complex space.*

PROOF. Our general strategy for showing that something is a differentiable vector space is to ensure that everything works in families over an arbitrary nilpotent dg manifold $(X, \mathscr{A})$. Thus, suppose that the theory we are working with is defined over $(X, \mathscr{A})$. If Y is a smooth manifold, we say a smooth map $Y \to \mathrm{Obs}^q(U)$ is an observable for the family of theories over $(X \times Y, \mathscr{A} \widehat{\otimes}_\pi C^\infty(Y))$ obtained by base-change along the map $X \times Y \to X$ (so this family of theories is constant over Y).

The filtration on $\mathrm{Obs}^q(U)$ (giving it the structure of pro-differentiable vector space) is inherited from that on $\mathrm{Obs}^q(M)$. Precisely, an observable $O \in \mathrm{Obs}^q(U)$ is in $F^k \mathrm{Obs}^q(U)$ if, for all parametrices Φ,

$$O[\Phi] \in \prod \hbar^i \,\mathrm{Sym}^{\geq (2k-i)} \mathscr{E}^\vee.$$

The renormalization group flow W_Φ^Ψ preserves this filtration.

So far we have verified that $\mathrm{Obs}^q(U)$ is a pro-object in the category of pre-differentiable cochain complexes. The remaining structure we need is a flat connection

$$\nabla : C^\infty(Y, \mathrm{Obs}^q(U)) \to \Omega^1(Y, \mathrm{Obs}^q(U))$$

for each manifold Y, where $C^\infty(Y, \mathrm{Obs}^q(U))$ is the space of smooth maps $Y \to \mathrm{Obs}^q(U)$.

This flat connection is equivalent to giving a differential on

$$\Omega^*(Y, \mathrm{Obs}^q(U)) = C^\infty(Y, \mathrm{Obs}^q(U)) \otimes_{C^\infty(Y)} \Omega^*(Y)$$

making it into a dg module for the dg algebra $\Omega^*(Y)$. Such a differential is provided by considering observables for the family of theories over the nilpotent dg manifold $(X \times Y, \mathscr{A} \widehat{\otimes}_\pi \Omega^*(Y))$, pulled back via the projection map $X \times Y \to Y$. □

9.5. Local observables form a prefactorization algebra

At this point, we have constructed the cochain complex $\mathrm{Obs}^q(M)$ of global observables of our factorization algebra. We have also constructed, for every open subset $U \subset M$, a sub-cochain complex $\mathrm{Obs}^q(U)$ of observables supported on U.

In this section we will see that the local quantum observables $\mathrm{Obs}^q(U)$ of a quantum field on a manifold M form a prefactorization algebra.

The definition of local observables makes it clear that they form a pre-cosheaf: there are natural injective maps of cochain complexes

$$\mathrm{Obs}^q(U) \to \mathrm{Obs}^q(U')$$

if $U \subset U'$ is an open subset.

Let U, V be disjoint open subsets of M. The structure of prefactorization algebra on the local observables is specified by the pre-cosheaf structure mentioned above, and a bilinear cochain map

$$\mathrm{Obs}^q(U) \times \mathrm{Obs}^q(V) \to \mathrm{Obs}^q(U \amalg V).$$

These product maps need to be maps of cochain complexes which are compatible with the pre-cosheaf structure and with reordering of the disjoint opens. Further, they need to satisfy a certain associativity condition which we will verify.

9.5.1. Defining the product map. Suppose that $O \in \mathrm{Obs}^q(U)$ and $O' \in \mathrm{Obs}^q(V)$ are observables on U and V respectively. Note that $O[\Phi]$ and $O'[\Phi]$ are elements of the cochain complex

$$\mathrm{Obs}^q_\Phi(M) = \left(\mathscr{O}(\mathscr{E})[[\hbar]], \widehat{Q}_\Phi \right)$$

which is a BD algebra and so a commutative algebra (ignoring the differential, of course). (The commutative product is simply the usual product of functions on $\mathscr{E}$.) In the definition of the prefactorization product, we will use the product of $O[\Phi]$ and $O'[\Phi]$ taken in the commutative algebra $\mathcal{O}(\mathcal{E})$. This product will be denoted $O[\Phi] * O'[\Phi] \in \mathcal{O}(\mathcal{E})$.

Recall (see definition 9.3.1.1) that we defined a linear renormalization group flow operator W_Φ^Ψ, which is an isomorphism between the cochain complexes $\mathrm{Obs}^q_\Phi(M)$ and $\mathrm{Obs}^q_\Psi(M)$.

The main result of this section is the following.

9.5.1.1 Theorem. *For all observables $O \in \mathrm{Obs}^q(U)$, $O' \in \mathrm{Obs}^q(V)$, where U and V are disjoint, the limit*

$$\lim_{\Psi \to 0} W_\Psi^\Phi \left(O[\Psi] * O'[\Psi] \right) \in \mathscr{O}(\mathscr{E})[[\hbar]]$$

exists. Further, this limit satisfies the renormalization group equation, so that we can define an observable $m(O, O')$ by

$$m(O, O')[\Phi] = \lim_{\Psi \to 0} W_\Psi^\Phi \left(O[\Psi] * O'[\Psi] \right).$$

The map

$$\begin{aligned}\mathrm{Obs}^q(U) \times \mathrm{Obs}^q(V) &\mapsto \mathrm{Obs}^q(U \amalg V) \\ O \times O' &\mapsto m(O, O')\end{aligned}$$

is a smooth bilinear cochain map, and it makes Obs^q *into a prefactorization algebra in the multicategory of differentiable pro-cochain complexes.*

PROOF. We will show that, for each i, k, the Taylor term

$$W_{\Phi}^{\Psi}(O[\Phi] * O'[\Phi])_{i,k} : \mathscr{E}^{\otimes k} \to \mathbb{C}$$

is independent of Φ for Φ sufficiently small. This will show that the limit exists.

Note that

$$W_{\Gamma}^{\Psi}\left(W_{\Phi}^{\Gamma}\left(O[\Phi] * O'[\Phi]\right)\right) = W_{\Phi}^{\Psi}\left(O[\Phi] * O'[\Phi]\right).$$

Thus, to show that the limit $\lim_{\Phi \to 0} W_{\Phi}^{\Psi}\left(O[\Phi] * O'[\Phi]\right)$ is eventually constant, it suffices to show that, for all sufficiently small Φ, Γ satisfying $\Phi < \Gamma$,

$$W_{\Phi}^{\Gamma}(O[\Phi] * O'[\Phi])_{i,k} = (O[\Gamma] * O'[\Gamma])_{i,k}.$$

This turns out to be an exercise in the manipulation of Feynman diagrams. In order to prove this, we need to recall a little about the Feynman diagram expansion of $W_{\Phi}^{\Gamma}(O[\Phi])$. (Feynman diagram expansions of the renormalization group flow are discussed extensively in [Cos11b].)

We have a sum of the form

$$W_{\Phi}^{\Gamma}(O[\Phi])_{i,k} = \sum_{G} \frac{1}{|\mathrm{Aut}(G)|} w_G\left(O[\Phi]; I[\Phi]; P(\Gamma) - P(\Phi)\right).$$

The sum is over all connected graphs G with the following decorations and properties.

(1) The vertices v of G are labelled by an integer $g(v) \in \mathbb{Z}_{\geq 0}$, which we call the genus of the vertex.
(2) The first Betti number of G, plus the sum of over all vertices of the genus $g(v)$, must be i (the "total genus").
(3) G has one special vertex.
(4) G has k tails (or external edges).

The weight $w_G\left(O[\Phi]; I[\Phi]; P(\Gamma) - P(\Phi)\right)$ is computed by the contraction of a collection of symmetric tensors. One places $O[\Phi]_{r,s}$ at the special vertex, when that vertex has genus r and valency s; places $I[\Phi]_{g,v}$ at every other vertex of genus g and valency v; and puts the propagator $P(\Gamma) - P(\Phi)$ on each edge.

Let us now consider $W_{\Phi}^{\Gamma}(O[\Phi] * O'[\Phi])$. Here, we a sum over graphs with one special vertex, labelled by $O[\Phi] * O'[\Phi]$. This is the same as having two special vertices, one of which is labelled by $O[\Phi]$ and the other by $O'[\Phi]$. Diagrammatically, it looks like we have split the special vertex into two pieces. When we make this maneuver, we introduce possibly disconnected graphs; however, each connected component must contain at least one of the two special vertices.

Let us now compare this to the graphical expansion of

$$O[\Gamma] * O'[\Gamma] = W_\Phi^\Gamma(O[\Phi]) * W_\Phi^\Gamma(O'[\Phi]).$$

The Feynman diagram expansion of the right hand side of this expression consists of graphs with two special vertices, labelled by $O[\Phi]$ and $O'[\Phi]$ respectively (and, of course, any number of other vertices, labelled by $I[\Phi]$, and the propagator $P(\Gamma) - P(\Phi)$ labelling each edge). Further, the relevant graphs have precisely two connected components, each of which contains one of the special vertices.

Thus, we see that

$$W_\Phi^\Gamma(O[\Phi] * O'[\Phi]) - W_\Phi^\Gamma(O[\Phi]) * W_\Phi^\Gamma(O'[\Phi]).$$

is a sum over *connected* graphs, with two special vertices, one labelled by $O[\Phi]$ and the other by $O'[\Phi]$. We need to show that the weight of such graphs vanish for Φ, Γ sufficiently small, with $\Phi < \Gamma$.

Graphs with one connected component must have a chain of edges connecting the two special vertices. (A chain is a path in the graph with no repeated vertices or edges.) For a graph G with "total genus" i and k tails, the length of any such chain is bounded by $2i + k$.

It is important to note here that we require a non-special vertex of genus zero to have valence at least three and a vertex of genus one to have valence at least one. See **[Cos11b]** for more discussion. If we are considering a family of theories over some dg ring, we do allow bivalent vertices to be accompanied by nilpotent parameters in the base ring; nilpotence of the parameter forces there to be a global upper bound on the number of bivalent vertices that can appear. The argument we are presenting works with minor modifications in this case too.

Each step along a chain of edges involves a tensor with some support that depends on the choice of parametrices *Phi* and Γ. As we move from the special vertex O toward the other O', we extend the support, and our aim is to show that we can choose Φ and Γ to be small enough so that the support of the chain, excluding $O'[\Phi]$, is disjoint from the support of $O'[\Phi]$. The contraction of a distribution and function with disjoint supports is zero, so that the weight will vanish. We now make this idea precise.

Let us choose arbitrarily a metric on M. By taking Φ and Γ to be sufficiently small, we can assume that the support of the propagator on each edge is within ε of the diagonal in this metric, and ε can be taken to be as small as we like. Similarly, the support of the $I_{r,s}[\Gamma]$

labelling a vertex of genus r and valency s can be taken to be within $c_{r,s}\varepsilon$ of the diagonal, where $c_{r,s}$ is a combinatorial constant depending only on r and s. In addition, by choosing Φ to be small enough we can ensure that the supports of $O[\Phi]$ and $O'[\Phi]$ are disjoint.

Now let G' denote the graph G with the special vertex for O' removed. This graph corresponds to a symmetric tensor whose support is within some distance $C_G\varepsilon$ of the small diagonal, where C_G is a combinatorial constant depending on the graph G'. As the supports K and K' (of O and O', respectively) have a finite distance d between them, we can choose ε small enough that $C_G\varepsilon < d$. It follows that, by choosing Φ and Γ to be sufficiently small, the weight of any connected graph is obtained by contracting a distribution and a function which have disjoint support. The graph hence has weight zero.

As there are finitely many such graphs with total genus i and k tails, we see that we can choose Γ small enough that for any $\Phi < \Gamma$, the weight of all such graphs vanishes.

Thus we have proved the first part of the theorem and have produced a bilinear map

$$\mathrm{Obs}^q(U) \times \mathrm{Obs}^q(V) \to \mathrm{Obs}^q(U \amalg V).$$

It is a straightforward to show that this is a cochain map and satisfies the associativity and commutativity properties necessary to define a prefactorization algebra. The fact that this is a smooth map of differentiable pro-vector spaces follows from the fact that this construction works for families of theories over an arbitrary nilpotent dg manifold $(X, \mathscr{A})$. □

9.6. Local observables form a factorization algebra

We have seen how to define a prefactorization algebra Obs^q of observables for our quantum field theory. In this section we will show that this prefactorization algebra is in fact a factorization algebra. In the course of the proof, we show that modulo $\hbar$, this factorization algebra is isomorphic to Obs^{cl}.

9.6.0.1 Theorem. *(1) The prefactorization algebra* Obs^q *of quantum observables is, in fact, a factorization algebra.*
(2) Further, there is an isomorphism

$$\mathrm{Obs}^q \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C} \cong \mathrm{Obs}^{cl}$$

between the reduction of the factorization algebra of quantum observables modulo $\hbar$, and the factorization algebra of classical observables.

9.6.1. Proof of the theorem. This theorem will be a corollary of a more technical proposition.

9.6.1.1 Proposition. *For any open subset $U \subset M$, filter $\operatorname{Obs}^q(U)$ by saying that the k-th filtered piece $G^k \operatorname{Obs}^q(U)$ is the sub $\mathbb{C}[[\hbar]]$-module consisting of those observables which are zero modulo $\hbar^k$. Note that this is a filtration by sub prefactorization algebras over the ring $\mathbb{C}[[\hbar]]$.*

Then, there is an isomorphism of prefactorization algebras (in differentiable pro-cochain complexes)

$$\operatorname{Gr} \operatorname{Obs}^q \simeq \operatorname{Obs}^{cl} \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]].$$

This isomorphism makes $\operatorname{Gr} \operatorname{Obs}^q$ into a factorization algebra.

Remark: We can give $G^k \operatorname{Obs}^q(U)$ the structure of a pro-differentiable cochain complex, as follows. The filtration on $G^k \operatorname{Obs}^q(U)$ that defines the pro-structure is obtained by intersecting $G^k \operatorname{Obs}^q(U)$ with the filtration on $\operatorname{Obs}^q(U)$ defining the pro-structure. Then the inclusion $G^k \operatorname{Obs}^q(U) \hookrightarrow \operatorname{Obs}^q(U)$ is a cofibration of differentiable pro-vector spaces (see definition **??**). $\Diamond$

PROOF OF THE THEOREM, ASSUMING THE PROPOSITION. We need to show that for every open U and for every Weiss cover $\mathfrak{U}$, the natural map

$$\check{C}(\mathfrak{U}, \operatorname{Obs}^q) \to \operatorname{Obs}^q(U) \tag{†}$$

is a quasi-isomorphism of differentiable pro-cochain complexes.

The basic idea is that the filtration induces a spectral sequence for both $\check{C}(\mathfrak{U}, \operatorname{Obs}^q)$ and $\operatorname{Obs}^q(U)$, and we will show that the induced map of spectral sequences is an isomorphism on the first page. Because we are working with differentiable pro-cochain complexes, this is a little subtle. The relevant statements about spectral sequences in this context are developed in this context in Appendix **??**.

Note that $\check{C}(\mathfrak{U}, \operatorname{Obs}^q)$ is filtered by $\check{C}(\mathfrak{U}, G^k \operatorname{Obs}^q)$. The map (†) preserves the filtrations. Thus, we have a maps of inverse systems

$$\check{C}(\mathfrak{U}, \operatorname{Obs}^q / G^k \operatorname{Obs}^q) \to \operatorname{Obs}^q(U) / G^k \operatorname{Obs}^q(U).$$

These inverse systems satisfy the properties of Appedix **??**, lemma **??**. Further, it is clear that

$$\operatorname{Obs}^q(U) = \varprojlim \operatorname{Obs}^q(U) / G^k \operatorname{Obs}^q(U).$$

We also have

$$\check{C}(\mathfrak{U}, \operatorname{Obs}^q) = \varprojlim \check{C}(\mathfrak{U}, \operatorname{Obs}^q / G^k \operatorname{Obs}^q).$$

This equality is less obvious, and uses the fact that the Čech complex is defined using the completed direct sum as described in Appendix **??**, section **??**.

Using lemma **??**, we need to verify that the map

$$\check{C}(\mathfrak{U}, \operatorname{Gr} \operatorname{Obs}^q) \to \operatorname{Gr} \operatorname{Obs}^q(U)$$

is an equivalence. This follows from the proposition because $\mathrm{Gr}\,\mathrm{Obs}^q$ is a factorization algebra. □

PROOF OF THE PROPOSITION. The first step in the proof of the proposition is the following lemma.

9.6.1.2 Lemma. *Let* $\mathrm{Obs}^q_{(0)}$ *denote the prefactorization algebra of observables which are only defined modulo* $\hbar$*. Then there is an isomorphism*

$$\mathrm{Obs}^q_{(0)} \simeq \mathrm{Obs}^{cl}$$

of differential graded prefactorization algebras.

PROOF OF LEMMA. Let $O \in \mathrm{Obs}^{cl}(U)$ be a classical observable. Thus, O is an element of the cochain complex $\mathscr{O}(\mathscr{E}(U))$ of functionals on the space of fields on U. We need to produce an element of $\mathrm{Obs}^q_{(0)}$ from O. An element of $\mathrm{Obs}^q_{(0)}$ is a collection of functionals $O[\Phi] \in \mathscr{O}(\mathscr{E})$, one for every parametrix Φ, satisfying a classical version of the renormalization group equation and an axiom saying that $O[\Phi]$ is supported on U for sufficiently small Φ.

Given an element

$$O \in \mathrm{Obs}^{cl}(U) = \mathscr{O}(\mathscr{E}(U)),$$

we define an element

$$\{O[\Phi]\} \in \mathrm{Obs}^q_{(0)}$$

by the formula

$$O[\Phi] = \lim_{\Gamma \to 0} W^{\Phi}_{\Gamma}(O) \text{ modulo } \hbar.$$

The Feynman diagram expansion of the right hand side only involves trees, since we are working modulo $\hbar$. As we are only using trees, the limit exists. The limit is defined by a sum over trees with one special vertex, where each edge is labelled by the propagator $P(\Phi)$, the special vertex is labelled by O, and every other vertex is labelled by the classical interaction $I_0 \in \mathscr{O}_{loc}(\mathscr{E})$ of our theory.

The map

$$\mathrm{Obs}^{cl}(U) \to \mathrm{Obs}^q_{(0)}(U)$$

we have constructed is easily seen to be a map of cochain complexes, compatible with the structure of prefactorization algebra present on both sides. (The proof is a variation on the argument in section 11, chapter 5 of [Cos11b], about the scale 0 limit of a deformation of the effective interaction I modulo $\hbar$.)

A simple inductive argument on the degree shows this map is an isomorphism.

Because the construction works over an arbitrary nilpotent dg manifold, it is clear that these maps are maps of differentiable cochain complexes. □

The next (and most difficult) step in the proof of the proposition is the following lemma. We use it to work inductively with the filtration of quantum observables.

Let $\mathrm{Obs}^q_{(k)}$ denote the prefactorization algebra of observables defined modulo $\hbar^{k+1}$.

9.6.1.3 Lemma. *For all open subsets $U \subset M$, the natural quotient map of differentiable pro-cochain complexes*

$$\mathrm{Obs}^q_{(k+1)}(U) \to \mathrm{Obs}^q_{(k)}(U)$$

is a fibration of differentiable pro-cochain complexes (see Appendix **??***, Definition* **??** *for the definition of a fibration). The fiber is isomorphic to* $\mathrm{Obs}^{cl}(U)$*.*

PROOF OF LEMMA. We give the set $(i,k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ the lexicographical ordering, so that $(i,k) > (r,s)$ if $i > r$ or if $i = r$ and $k > s$.

We will let $\mathrm{Obs}^q_{\leq (i,k)}(U)$ be the quotient of $\mathrm{Obs}^q_{(i)}$ consisting of functionals

$$O[\Phi] = \sum_{(r,s) \leq (i,k)} \hbar^r O_{(r,s)}[\Phi]$$

satisfying the renormalization group equation and locality axiom as before, but where $O_{(r,s)}[\Phi]$ is only defined for $(r,s) \leq (i,k)$. Similarly, we will let $\mathrm{Obs}^q_{<(i,k)}(U)$ be the quotient where the $O_{(r,s)}[\Phi]$ are only defined for $(r,s) < (i,k)$.

We will show that the quotient map

$$\mathsf{q} : \mathrm{Obs}^q_{\leq (i,k)}(U) \to \mathrm{Obs}^q_{<(i,k)}(U)$$

is a fibration. The result will follow.

Recall what it means for a map $f : V \to W$ of differentiable cochain complexes to be a fibration. For X a manifold, let $C^\infty_X(V)$ denote the sheaf of cochain complexes on X of smooth maps to V. We say f is a fibration if for every manifold X, the induced map of sheaves $C^\infty_X(V) \to C^\infty_X(W)$ is surjective in each degree. Equivalently, we require that for all smooth manifolds X, every smooth map $X \to W$ lifts locally on X to a map to V.

Now, by definition, a smooth map from X to $\mathrm{Obs}^q(U)$ is an observable for the constant family of theories over the nilpotent dg manifold $(X, C^\infty(X))$. Thus, in order to show q is a fibration, it suffices to show the following. For any family of theories over a nilpotent dg manifold $(X, \mathscr{A})$, any open subset $U \subset M$, and any observable α in the $\mathscr{A}$-module $\mathrm{Obs}^q_{<(i,k)}(U)$, we can lift α to an element of $\mathrm{Obs}^q_{\leq (i,k)}(U)$ locally on X.

To prove this, we will first define, for every parametrix Φ, a map

$$L_\Phi : \mathrm{Obs}^q_{<(i,k)}(U) \to \mathrm{Obs}^q_{\leq(i,k)}(M)$$

with the property that the composed map

$$\mathrm{Obs}^q_{<(i,k)}(U) \xrightarrow{L_\Phi} \mathrm{Obs}^q_{\leq(i,k)}(M) \to \mathrm{Obs}^q_{<(i,k)}(M)$$

is the natural inclusion map. Then, for every observable $O \in \mathrm{Obs}^q_{<(i,k)}(U)$, we will show that $L_\Phi(O)$ is supported on U, for sufficiently small parametrices Φ, so that $L_\Phi(O)$ provides the desired lift.

For

$$O \in \mathrm{Obs}^q_{<(i,k)}(U),$$

we define

$$L_\Phi(O) \in \mathrm{Obs}^q_{\leq(i,k)}(M)$$

by

$$L_\Phi(O)_{r,s}[\Phi] = \begin{cases} O_{r,s}[\Phi] & \text{if } (r,s) < (i,k) \\ 0 & \text{if } (r,s) = (i,k) \end{cases}.$$

For $\Psi \neq \Phi$, we obtain $L_\Phi(O)_{r,s}[\Psi]$ by the renormalization group flow from $L_\Phi(O)_{r,s}[\Phi]$. The RG flow equation tells us that if $(r,s) < (i,k)$, then

$$L_\Phi(O)_{r,s}[\Psi] = O_{r,s}[\Psi].$$

However, the RG equation for $L_\Phi(O)_{r,s}$ is non-trivial and tells us that

$$I_{i,k}[\Psi] + \delta\left(L_\Phi(O)_{i,k}[\Psi]\right) = W_{i,k}\left(P(\Psi) - P(\Phi), I[\Phi] + \delta O[\Phi]\right)$$

for δ a square-zero parameter of cohomological degree opposite to that of O.

To complete the proof of this lemma, we prove the required local lifting property in the sublemma below. □

9.6.1.4 Sub-lemma. *For each $O \in \mathrm{Obs}^q_{<(i,k)}(U)$, we can find a parametrix Φ — locally over the parametrizing manifold X — so that $L_\Phi O$ lies in $\mathrm{Obs}^q_{\leq(i,k)}(U) \subset \mathrm{Obs}^q_{\leq(i,k)}(M)$.*

PROOF. Although the observables Obs^q form a factorization algebra on the manifold M, they also form a sheaf on the parametrizing base manifold X. That is, for every open subset $V \subset X$, let $\mathrm{Obs}^q(U)\mid_V$ denote the observables for our family of theories restricted to V. In other words, $\mathrm{Obs}^q(U)\mid_V$ denotes the sections of this sheaf $\mathrm{Obs}^q(U)$ on V.

The map L_Φ constructed above is then a map of sheaves on X.

For every observable $O \in \mathrm{Obs}^q_{<(i,k)}(U)$, we need to find an open cover

$$X = \bigcup_{\alpha} Y_\alpha$$

of X, and on each Y_α a parametrix Φ_α (for the restriction of the family of theories to Y_α) such that

$$L_{\Phi_\alpha}(O \mid_{Y_\alpha}) \in \mathrm{Obs}^q_{\leq(i,k)}(U) \mid_{Y_\alpha}.$$

More informally, we need to show that locally in X, we can find a parametrix Φ such that for all sufficiently small Ψ, the support of $L_\Phi(O)_{(i,k)}[\Psi]$ is in a subset of $U^k \times X$ which maps properly to X.

This argument resembles previous support arguments (e.g., the product lemma from section 9.5). The proof involves an analysis of the Feynman diagrams appearing in the expression

$$(\star) \qquad L_\Phi(O)_{i,k}[\Psi] = \sum_\gamma \frac{1}{|\mathrm{Aut}(\gamma)|} w_\gamma \left(O[\Phi]; I[\Phi]; P(\Psi) - P(\Phi)\right).$$

The sum is over all connected Feynman diagrams of genus i with k tails. The edges are labelled by $P(\Psi) - P(\Phi)$. Each graph has one special vertex, where $O[\Phi]$ appears. More explicitly, if this vertex is of genus r and valency s, it is labelled by $O_{r,s}[\Phi]$. Each non-special vertex is labelled by $I_{a,b}[\Phi]$, where a is the genus and b the valency of the vertex. Note that only a finite number of graphs appear in this sum.

By assumption, O is supported on U. This means that there exists some parametrix Φ_0 and a subset $K \subset U \times X$ mapping properly to X such that for all $\Phi < \Phi_0$, $O_{r,s}[\Phi]$ is supported on K^s. (Here by $K^s \subset U^s \times X$ we mean the fibre product over X.)

Further, each $I_{a,b}[\Phi]$ is supported as close as we like to the small diagonal $M \times X$ in $M^k \times X$. We can find precise bounds on the support of $I_{a,b}[\Phi]$, as explained in section 8.2. To describe these bounds, let us choose metrics for X and M. For a parametrix Φ supported within ε of the diagonal $M \times X$ in $M \times M \times X$, the effective interaction $I_{a,b}[\Phi]$ is supported within $(2a + b)\varepsilon$ of the diagonal.

(In general, if $A \subset M^n \times X$, the ball of radius ε around A is defined to be the union of the balls of radius ε around each fibre A_x of $A \to X$. It is in this sense that we mean that $I_{a,b}[\Phi]$ is supported within $(2a + b)\varepsilon$ of the diagonal.)

Similarly, for every parametrix Ψ with $\Psi < \Phi$, the propagator $P(\Psi) - P(\Phi)$ is supported within ε of the diagonal.

In sum, there exists a set $K \subset U \times X$, mapping properly to X, such that for all $\varepsilon > 0$, there exists a parametrix Φ_ε, such that

(1) $O[\Phi_\varepsilon]_{r,s}$ is supported on K^s for all $(r, s) < (i, k)$.

(2) $I_{a,b}[\Phi_\varepsilon]$ is supported within $(2a+b)\varepsilon$ of the small diagonal.
(3) For all $\Psi < \Phi_\varepsilon$, $P(\Psi) - P(\Phi_\varepsilon)$ is supported within ε of the small diagonal.

The weight w_γ of a graph in the graphical expansion of the expression $(\star)$ above (using the parametrices Φ_ε and any $\Psi < \Phi_\varepsilon$) is thus supported in the ball of radius $c\varepsilon$ around K^k (where c is some combinatorial constant, depending on the number of edges and vertices in γ). There are a finite number of such graphs in the sum, so we can choose the combinatorial constant c uniformly over the graphs.

Since $K \subset U \times X$ maps properly to X, locally on X, we can find an ε so that the closed ball of radius $c\varepsilon$ is still inside $U^k \times X$. This completes the proof. □

□

9.7. The map from theories to factorization algebras is a map of presheaves

In [Cos11b], it is shown how to restrict a quantum field theory on a manifold M to any open subset U of M. Factorization algebras also form a presheaf in an obvious way. In this section, we will prove the following result.

9.7.0.1 Theorem. *The map from the simplicial set of theories on a manifold M to the ∞-groupoid of factorization algebras on M extends to a map of simplicial presheaves.*

The proof of this will rely on the results we have already proved, and in particular on thefact that observables form a factorization algebra.

As a corollary, we have the following very useful result.

9.7.0.2 Corollary. *For every open subset $U \subset M$, there is an isomorphism of graded differentiable vector spaces*

$$\mathrm{Obs}^q(U) \cong \mathrm{Obs}^{cl}(U)[[\hbar]].$$

Note that what we have proved already is that there is a filtration on $\mathrm{Obs}^q(U)$ whose associated graded is $\mathrm{Obs}^{cl}(U)[[\hbar]]$. This result shows that this filtration is split as a filtration of differentiable vector spaces.

Proof. By the theorem, $\mathrm{Obs}^q(U)$ can be viewed as global observables for the field theory obtained by restricting our field theory on M to one on U. Choosing a parametrix on U allows one to identify global observables with $\mathrm{Obs}^{cl}(U)[[\hbar]]$, with differential $\mathrm{d} + \{I[\Phi], -\}_\Phi + \hbar\triangle_\Phi$. This is an isomorphism of differentiable vector spaces. □

The proof of this theorem is a little technical, and uses the same techniques we have discussed so far. Before we explain the proof of the theorem, we need to explain how to restrict theories to open subsets.

Let $\mathscr{E}(M)$ denote the space of fields for a field theory on M. In order to relate field theories on U and on M, we need to relate parametrices on U and on M. If

$$\Phi \in \overline{\mathscr{E}}(M)\widehat{\otimes}_{\beta}\overline{\mathscr{E}}(M)$$

is a parametrix on M (with proper support as always), then the restriction

$$\Phi \mid_U \in \overline{\mathscr{E}}(U)\widehat{\otimes}_{\beta}\overline{\mathscr{E}}(U)$$

of Φ to U may no longer be a parametrix. It will satisfy all the conditions required to be a parametrix except that it will typically not have proper support.

We can modify $\Phi \mid_U$ so that it has proper support, as follows. Let $K \subset U$ be a compact set, and let f be a smooth function on $U \times U$ with the following properties:

(1) f is 1 on $K \times K$.
(2) f is 1 on a neighbourhood of the diagonal.
(3) f has proper support.

Then, $f\Phi \mid_U$ does have proper support, and further, $f\Phi \mid_U$ is equal to Φ on $K \times K$.

Conversely, given any parametrix Φ on U, there exists a parametrix $\widetilde{\Phi}$ on M such that Φ and $\widetilde{\Phi}$ agree on K. One can construct $\widetilde{\Phi}$ by taking any parametrix Ψ on M, and observing that, when restricted to U, Ψ and Φ differ by a smooth section of the bundle $E \boxtimes E$ on $U \times U$.

We can then choose a smooth section σ of this bundle on $U \times U$ such that f has compact support and $\sigma = \Psi - \Phi$ on $K \times K$. Then, we let $\widetilde{\Phi} = \Psi - f$.

Let us now explain what it means to restrict a theory on M to one on U. Then we will state the theorem that there exists a unique such restriction.

Fix a parametrix Φ on U. Let $K \subset U$ be a compact set, and consider the compact set

$$L_n = (\operatorname{Supp} \Phi *)^n K \subset U.$$

Here we are using the convolution construction discussed earlier, whereby the collection of proper subsets of $U \times U$ acts on that of compact sets in U by convolution. Thus, L_n is the set of those $x \in U$ such that there exists a sequence $(y_0, \dots, y_n)$ where (y_i, y_{i+1}) is in $\operatorname{Supp} \Phi$, $y_n \in K$ and $y_0 = x$.

9.7.0.3 Definition. *Fix a theory on M, specified by a collection $\{I[\Psi]\}$ of effective interactions. Then a restriction of $\{I[\Psi]\}$ to U consists of a collection of effective interactions $\{I^U[\Phi]\}$ with the* *following propery. For every parametrix Φ on U, and for all compact sets $K \subset U$, let $L_n \subset U$ be as above.*

Let $\widetilde{\Phi}_n$ be a parametrix on M with the property that

$$\widetilde{\Phi}_n = \Phi \text{ on } L_n \times L_n.$$

Then we require that

$$I^U_{i,k}[\Phi](e_1, \dots, e_k) = I_{i,k}[\widetilde{\Phi}_n](e_1, \dots, e_k)$$

where $e_i \in \mathscr{E}_c(U)$ have support on K, and where $n \geq 2i + k$.

This definition makes sense in families with obvious modifications.

9.7.0.4 Theorem. *Any theory $\{I[\Psi]\}$ on M has a unique restriction on U.*

This restriction map works in families, and so defines a map of simplicial sets from the simplicial set of theories on M to that on U.

In this way, we have a simplicial presheaf $\mathcal{T}$ on M whose value on U is the simplicial set of theories on U (quantizing a given classical theory). This simplicial presheaf is a homotopy sheaf, meaning that it satisfies Čech descent.

PROOF. It is obvious that the restriction, it if exists, is unique. Indeed, we have specified each $I^U_{i,k}[\Phi]$ for every Φ and for every compact subset $K \subset U$. Since each $I^U_{i,k}[\Phi]$ must have compact support on U^k, it is determined by its behaviour on compact sets of the form K^k.

In [**Cos11b**], a different definition of restriction was given, defined not in terms of general parametrices but in terms of those defined by the heat kernel. One therefore needs to check that the notion of restriction defined in [**Cos11b**] coincides with the one discussed in this theorem. This is easy to see by a Feynman diagram argument similar to the ones we discussed earlier. The statement that the simplicial presheaf of theories satisfies Čech descent is proved in [**Cos11b**]. □

Now here is the main theorem in this section.

9.7.0.5 Theorem. *The map which assigns to a field theory the corresponding factorization algebra is a map of presheaves. Further, the map which assigns to an n-simplex in the simplicial set of theories, a factorization algebra over $\Omega^*(\triangle^n)$, is also a map of presheaves.*

Let us explain what this means concretely. Consider a theory on M and let Obs^q_M denote the corresponding factorization algebra. Let Obs^q_U denote the factorization algebra for the theory restricted to U, and let $Obs_M^q \mid_U$ denote the factorization algebra Obs_M^q restricted to U (that is, we only consider open subsets contained in M). Then there is a canonical isomorphism of factorization algebras on U,

$$\mathrm{Obs}_U^q \cong \mathrm{Obs}_M^q \mid_U .$$

In addition, this construction works in families, and in particular in families over $\Omega^*(\triangle^n)$.

PROOF. Let $V \subset U$ be an open set whose closure in U is compact. We will first construct an isomorphism of differentiable cochain complexes

$$\mathrm{Obs}_M^q(V) \cong \mathrm{Obs}_U^q(V).$$

Later we will check that this isomorphism is compatible with the product structures. Finally, we will use the codescent properties for factorization algebras to extend to an isomorphism of factorization algebras defined on all open subsets $V \subset U$, and not just those whose closure is compact.

Thus, let $V \subset U$ have compact closure, and let $O \in \mathrm{Obs}_M^q(V)$. Thus, O is something which assigns to every parametrix Φ on M a collection of functionals $O_{i,k}[\Phi]$ satisfying the renormalization group equation and a locality axiom stating that for each i, k, there exists a parametrix Φ_0 such that $O_{i,k}[\Phi]$ is supported on V for $\Phi \leq \Phi_0$.

We want to construct from such an observable a collection of functionals $\rho(O)_{i,k}[\Psi]$, one for each parametrix Ψ on U, satisfying the RG flow on U and the same locality axiom. It suffices to do this for a collection of parametrices which include parametrices which are arbitrarily small (that is, with support contained in an arbitrarily small neighbourhood of the diagonal in $U \times U$).

Let $L \subset U$ be a compact subset with the property that $\overline{V} \subset \operatorname{Int} L$. Choose a function f on $U \times U$ which is 1 on a neighbourhood of the diagonal, 1 on $L \times L$, and has proper support. If Ψ is a parametrix on M, we let Ψ^f be the parametrix on U obtained by multiplying the restriction of Ψ to $U \times U$ by f. Note that the support of Ψ^f is a subset of that of Ψ.

The construction is as follows. Choose (i, k). We define

$$\rho(O)_{r,s}[\Psi^f] = O_{r,s}[\Psi]$$

for all $(r, s) \leq (i, k)$ and all Ψ sufficiently small. We will not spell out what we mean by sufficiently small, except that it in particular means it is small enough so that $O_{r,s}[\Psi]$ is supported on V for all $(r, s) \leq (i, k)$. The value of $\rho(O)_{r,s}$ for other parametrices is determined by the RG flow.

To check that this construction is well-defined, we need to check that if we take some parametrix $\widetilde{\Psi}$ on M which is also sufficiently small, then the $\rho(O)_{r,s}[\Psi^f]$ and $\rho(O)_{r,s}[\widetilde{\Psi}^f]$

are related by the RG flow for observables for the theory on U. This RG flow equation relating these two quantities is a sum over connected graphs, with one vertex labelled by $\rho(O)[\Psi^f]$, all other vertices labelled by $I^U[\Psi^f]$, and all internal edges labelled by $P(\widetilde{\Psi}^f) - P(\Psi^f)$. Since we are only considering $(r,s) \leq (i,k)$ only finitely many graphs can appear, and the number of internal edges of these graphs is bounded by $2i+k$. We are assuming that both Ψ and $\widetilde{\Psi}$ are sufficiently small so that $O_{r,s}[\Psi]$ and $O_{r,s}[\widetilde{\Psi}]$ have compact support on V. Also, by taking Ψ sufficiently small, we can assume that $I^U[\Psi]$ has support arbitrarily close to the diagonal. It follows that, if we choose both Ψ and $\widetilde{\Psi}$ to be sufficiently small, there is a compact set $L' \subset U$ containing V such that the weight of each graph appearing in the RG flow is zero if one of the inputs (attached to the tails) has support on the complement of L'. Further, by taking Ψ and $\widetilde{\Psi}$ sufficiently small, we can arrange so that L' is as small as we like, and in particular, we can assume that $L' \subset \operatorname{Int} L$ (where L is the compact set chosen above).

Recall that the weight of a Feynman diagram involves pairing quantities attached to edges with multilinear functionals attached to vertices. A similar combinatorial analysis tells us that, for each vertex in each graph appearing in this sum, the inputs to the multilinear functional attached to the vertex are all supported in L'.

Now, for Ψ sufficiently small, we have

$$I^U_{r,s}[\Psi^f](e_1,\dots,e_s) = I_{r,s}[\Psi](e_1,\dots,e_s)$$

if all of the e_i are supported in L'. (This follows from the definition of the restriction of a theory. Recall that I^U indicates the theory on U and I indicates the theory on M).

It follows that, in the sum over diagrams computing the RG flow, we get the same answer if we label the vertices by $I[\Psi]$ instead of $I^U[\Psi^f]$. The RG flow equation now follows from that for the original observable $O[\Psi]$ on M.

The same kind of argument tells us that if we change the choice of compact set $L \subset U$ with $\overline{V} \subset \operatorname{Int} L$, and if we change the bumb function f we chose, the map

$$\rho : \mathrm{Obs}^q_M(V) \to \mathrm{Obs}^q_U(V)$$

does not change.

A very similar argument also tells us that this map is a cochain map. It is immediate that ρ is an isomorphism, and that it commutes with the maps arising from inclusions $V \subset V'$.

We next need to verify that this map respects the product structure. Recall that the product of two observables O, O' in V, V' is defined by saying that $([\Psi]O'[\Psi])_{r,s}$ is simply the naive product in the symmetric algebra $\operatorname{Sym}^* \mathscr{E}_c^!(V \amalg V')$ for $(r,s) \leq (i,k)$ (some fixed (i,k)) and for Ψ sufficiently small.

Since, for $(r,s) \leq (i,k)$ and for Ψ sufficiently small, we defined

$$\rho(O)_{r,s}[\Psi^f] = O_{r,s}[\Psi],$$

we see immediately that ρ respects products.

Thus, we have constructed an isomorphism

$$\mathrm{Obs}^q_M \mid_U \cong \mathrm{Obs}^q_U$$

of prefactorization algebras on U, where we consider open subsets in U with compact closure. We need to extend this to an isomorphism of factorization algebras. To do this, we use the following property: for any open subset $W \subset U$,

$$\mathrm{Obs}^q_U(W) = \operatorname*{colim}_{V \subset W} \mathrm{Obs}^q_U(V)$$

where the colimit is over all open subsets with compact closure. (The colimit is taken, of course, in the category of filtered differentiable cochain complexes, and is simply the naive and not homotopy colimit). The same holds if we replace Obs^q_U by Obs^q_M. Thus we have constructed an isomorphism

$$\mathrm{Obs}^q_U(W) \cong \mathrm{Obs}^q_M(W)$$

for all open subsets W. The associativity axioms of prefactorization algebras, combined with the fact that $\mathrm{Obs}^q(W)$ is a colimit of $\mathrm{Obs}^q(V)$ for V with compact closure and the fact that the isomorphisms we have constructed respect the product structure for such open subsets V, implies that we have constructed an isomorphism of factorization algebras on U. $\square$

CHAPTER 10

Further aspects of quantum observables

10.1. Translation-invariant factorization algebras from translation-invariant quantum field theories

In this section, we will show that a translation-invariant quantum field theory on $\mathbb{R}^n$ gives rise to a smoothly translation-invariant factorization algebra on $\mathbb{R}^n$ (see section **??**). We will also show that a holomorphically translation-invariant field theory on $\mathbb{C}^n$ gives rise to a holomorphically translation-invariant factorization algebra.

10.1.1. First, we need to define what it means for a field theory to be translation-invariant. Let us consider a classical field theory on $\mathbb{R}^n$. Recall that this is given by

(1) A graded vector bundle E whose sections are $\mathscr{E}$;
(2) An antisymmetric pairing $E \otimes E \to \mathrm{Dens}_{\mathbb{R}^n}$;
(3) A differential operator $Q : \mathscr{E} \to \mathscr{E}$ making $\mathscr{E}$ into an elliptic complex, which is skew-self adjoint;
(4) A local action functional $I \in \mathscr{O}_{loc}(\mathscr{E})$ satisfying the classical master equation.

A classical field theory is translation-invariant if

(1) The graded bundle E is translation-invariant, so that we are given an isomorphism between E and the trivial bundle with fibre E_0.
(2) The pairing, differential Q, and local functional I are all translation-invariant.

It takes a little more work to say what it means for a quantum field theory to be translation-invariant. Suppose we have a translation-invariant classical field theory, equipped with a translation-invariant gauge fixing operator Q^{GF}. As before, a quantization of such a field theory is given by a family of interactions $I[\Phi] \in \mathscr{O}_{sm,P}(\mathscr{E})$, one for each parametrix Φ.

10.1.1.1 Definition. *A translation-invariant quantization of a translation-invariant classical field theory is a quantization with the property that, for all translation-invariant parametrices Φ, $I[\Phi]$ is translation-invariant.*

Remark: In general, in order to give a quantum field theory on a manifold M, we do not need to give an effective interaction $I[\Phi]$ for all parametrices. We only need to specify $I[\Phi]$ for a collection of parametrices such that the intersection of the supports of Φ is the small diagonal $M \subset M^2$. The functional $I[\Psi]$ for all other parametrices Ψ is defined by the renormalization group flow. It is easy to construct a collection of translation-invariant parametrices satisfying this condition. ♢

10.1.1.2 Proposition. *The factorization algebra associated to a translation-invariant quantum field theory is smoothly translation-invariant (see section* **??** *in Chapter* **??** *for the definition).*

PROOF. Let Obs^q denote the factorization algebra of quantum observables for our translation-invariant theory. An observable supported on $U \subset \mathbb{R}^n$ is defined by a family $O[\Phi] \in \mathscr{O}(\mathscr{E})[[\hbar]]$, one for each translation-invariant parametrix, which satisfiese the RG flow and (in the sense we explained in section 9.4) is supported on U for sufficiently small parametrices. The renormalization group flow

$$W_\Psi^\Phi : \mathscr{O}(\mathscr{E})[[\hbar]] \to \mathscr{O}(\mathscr{E})[[\hbar]]$$

for translation-invariant parametrices Ψ, Φ commutes with the action of $\mathbb{R}^n$ on $\mathscr{O}(\mathscr{E})$ by translation, and therefore acts on $\mathrm{Obs}^q(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$ and $U \subset \mathbb{R}^n$, let $T_x U$ denote the x-translate of U. It is immediate that the action of $x \in \mathbb{R}^n$ on $\mathrm{Obs}^q(\mathbb{R}^n)$ takes $\mathrm{Obs}^q(U) \subset \mathrm{Obs}^q(\mathbb{R}^n)$ to $\mathrm{Obs}^q(T_x U)$. It is not difficult to verify that the resulting map

$$\mathrm{Obs}^q(U) \to \mathrm{Obs}^q(T_x U)$$

is an isomorphism of differentiable pro-cochain complexes and that it is compatible with the structure of a factorization algebra.

We need to verify the smoothness hypothesis of a smoothly translation-invariant factorization algebra. This is the following. Suppose that $U_1, \dots, U_k$ are disjoint open subsets of $\mathbb{R}^n$, all contained in an open subset V. Let $A' \subset \mathbb{R}^{nk}$ be the subset consisting of those $x_1, \dots, x_k$ such that the closures of $T_{x_i} U_i$ remain disjoint and in V. Let A be the connected component of 0 in A'. We need only examine the case where A is non-empty.

We need to show that the composed map

$$m_{x_1,\dots,x_k} : \mathrm{Obs}^q(U_1) \times \cdots \times \mathrm{Obs}^q(U_k) \to \\ \mathrm{Obs}^q(T_{x_1} U_1) \times \cdots \times \mathrm{Obs}^q(T_{x_k} U_k) \to \mathrm{Obs}^q(V)$$

varies smoothly with $(x_1, \dots, x_k) \in A$. In this diagram, the first map is the product of the translation isomorphisms $\mathrm{Obs}^q(U_i) \to \mathrm{Obs}^q(T_{x_i} U_i)$, and the second map is the product map of the factorization algebra.

The smoothness property we need to check says that the map $m_{x_1,\dots,x_k}$ lifts to a multilinear map of differentiable pro-cochain complexes

$$\mathrm{Obs}^q(U_1) \times \cdots \times \mathrm{Obs}^q(U_k) \to C^\infty(A, \mathrm{Obs}^q(V)),$$

where on the right hand side the notation $C^\infty(A, \mathrm{Obs}^q(V))$ refers to the smooth maps from A to $\mathrm{Obs}^q(V)$.

This property is local on A, so we can replace A by a smaller open subset if necessary.

Let us assume (replacing A by a smaller subset if necessary) that there exist open subsets U_i' containing U_i, which are disjoint and contained in V and which have the property that for each $(x_1, \dots, x_k) \in A$, $T_{x_i} U_i \subset U_i'$.

Then, we can factor the map $m_{x_1,\dots,x_k}$ as a composition

$$(\dagger) \quad \mathrm{Obs}^q(U_1) \times \cdots \times \mathrm{Obs}^q(U_k) \xrightarrow{i_{x_1} \times \cdots \times i_{x_k}} \mathrm{Obs}^q(U_1') \times \cdots \times \mathrm{Obs}^q(U_k') \to \mathrm{Obs}^q(V).$$

Here, the map $i_{x_i} : \mathrm{Obs}^q(U_i) \to \mathrm{Obs}^q(U_i')$ is the composition

$$\mathrm{Obs}^q(U_i) \to \mathrm{Obs}^q(T_{x_i} U_i) \to \mathrm{Obs}^q(U_i')$$

of the translation isomorphism with the natural inclusion map $\mathrm{Obs}^q(T_{x_i} U_i) \to \mathrm{Obs}^q(U_i')$. The second map in equation (†) is the product map associated to the disjoint subsets $U_1', \dots, U_k' \subset V$.

By possibly replacing A by a smaller open subset, let us assume that $A = A_1 \times \cdots \times A_k$, where the A_i are open subsets of $\mathbb{R}^n$ containing the origin. It remains to show that the map

$$i_{x_i} : \mathrm{Obs}^q(U_i) \to \mathrm{Obs}^q(U_i')$$

is smooth in x_i, that is, extends to a smooth map

$$\mathrm{Obs}^q(U_i) \to C^\infty(A_i, \mathrm{Obs}^q(U_i')).$$

Indeed, the fact that the product map

$$m : \mathrm{Obs}^q(U_1') \times \cdots \times \mathrm{Obs}^q(U_k') \to \mathrm{Obs}^q(V)$$

is a smooth multilinear map implies that, for every collection of smooth maps $\alpha_i : Y_i \to \mathrm{Obs}^q(U_i')$ from smooth manifolds Y_i, the resulting map

$$\begin{aligned} Y_1 \times \cdots \times Y_k &\to \mathrm{Obs}^q(V) \\ (y_1, \dots y_k) &\mapsto m(\alpha_1(y), \dots, \alpha_k(y)) \end{aligned}$$

is smooth.

Thus, we have reduced the result to the following statement: for all open subsets $A \subset \mathbb{R}^n$ and for all $U \subset U'$ such that $T_x U \subset U'$ for all $x \in A$, the map $i_x : \mathrm{Obs}^q(U) \to \mathrm{Obs}^q(U')$ is smooth in $x \in A$.

But this statement is tractable. Let

$$O \in \mathrm{Obs}^q(U) \subset \mathrm{Obs}^q(U') \subset \mathrm{Obs}^q(\mathbb{R}^n)$$

be an observable. It is obvious that the family of observables $T_x O$, when viewed as elements of $\mathrm{Obs}^q(\mathbb{R}^n)$, depends smoothly on x. We need to verify that it depends smoothly on x when viewed as an element of $\mathrm{Obs}^q(U')$.

This amounts to showing that the support conditions which ensure an observable is in $\mathrm{Obs}^q(U')$ hold uniformly for x in compact sets in A.

The fact that O is in $\mathrm{Obs}^q(U)$ means the following. For each (i,k), there exists a compact subset $K \subset U$ and $\varepsilon > 0$ such that for all translation-invariant parametrices Φ supported within ε of the diagonal and for all $(r,s) \leq (i,k)$ in the lexicographical ordering, the Taylor coefficient $O_{r,s}[\Phi]$ is supported on K^s.

We need to enlarge K to a subset $L \subset U' \times A$, mapping properly to A, such that $T_x O$ is supported on L in this sense (again, for $(r,s) \leq (i,k)$). Taking $L = K \times A$, embedded in $U' \times A$ by

$$(k,x) \mapsto (T_x k, x)$$

suffices. $\square$

Remark: Essentially the same proof will give us the somewhat stronger result that for any manifold M with a smooth action of a Lie group G, the factorization algebra corresponding to a G-equivariant field theory on M is smoothly G-equivariant. $\lozenge$

10.2. Holomorphically translation-invariant theories and their factorization algebras

Similarly, we can talk about holomorphically translation-invariant classical and quantum field theories on $\mathbb{C}^n$. In this context, we will take our space of fields to be $\Omega^{0,*}(\mathbb{C}^n, V)$, where V is some translation-invariant holomorphic vector bundle on $\mathbb{C}^n$. The pairing must arise from a translation-invariant map of holomorphic vector bundles

$$V \otimes V \to K_{\mathbb{C}^n}$$

of cohomological degree $n-1$, where $K_{\mathbb{C}^n}$ denotes the canonical bundle. This means that the composed map

$$\Omega_c^{0,*}(\mathbb{C}^n, V)^{\otimes 2} \to \Omega_c^{0,*}(\mathbb{C}^n, K_{\mathbb{C}^n}) \xrightarrow{\int} \mathbb{C}$$

is of cohomological degree -1.

Let

$$\eta_i = \frac{\partial}{\partial \overline{z}_i} \vee - : \Omega^{0,k}(\mathbb{C}^n, V) \to \Omega^{0,k-1}(\mathbb{C}^n, V)$$

be the contraction operator. The cohomological differential operator Q on $\Omega^{0,*}(V)$ must be of the form

$$Q = \overline{\partial} + Q_0$$

where Q_0 is translation-invariant and satisfies the following conditions:

(1)
(2) Q_0 (and hence Q) must be skew self-adjoint with respect to the pairing on $\Omega_c^{0,*}(\mathbb{C}^n, V)$.
(3) We assume that Q_0 is a purely holomorphic differential operator, so that we can write Q_0 as a finite sum
$$Q_0 = \sum \frac{\partial}{\partial z^I} \mu_I$$
where $\mu_I : V \to V$ are linear maps of cohomological degree 1. (Here we are using multi-index notation). Note that this implies that
$$[Q_0, \eta_i] = 0,$$
for $i = 1, \dots, n$. In terms of the μ_I, the adjointness condition says that μ_I is skew-symmetric if $|I|$ is even and symmetric if $|I|$ is odd.

The other piece of data of a classical field theory is the local action functional $I \in \mathscr{O}_{loc}(\Omega^{0,*}(\mathbb{C}^n, V))$. We assume that I is translation-invariant, of course, but also that
$$\eta_i I = 0$$
for $i = 1 \dots n$, where the linear map η_i on $\Omega^{0,*}(\mathbb{C}^n, V)$ is extended in the natural way to a derivation of the algebra $\mathscr{O}(\Omega_c^{0,*}(\mathbb{C}^n, V))$ preserving the subspace of local functionals.

Any local functional I on $\Omega^{0,*}(\mathbb{C}^n, V)$ can be written as a sum of functionals of the form
$$\phi \mapsto \int_{\mathbb{C}^n} \mathrm{d}z_1 \dots \mathrm{d}z_n A(D_1\phi \dots D_k\phi)$$
where $A : V^{\otimes k} \to \mathbb{C}$ is a linear map, and each D_i is in the space
$$\mathbb{C}\left[\mathrm{d}\bar{z}_i, \eta_i, \frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial z_i}\right].$$
(Recall that η_i indicates $\frac{\partial}{\partial \mathrm{d}\bar{z}_i}$). The condition that $\eta_i I = 0$ for each i means that we only consider those D_i which are in the subspace
$$\mathbb{C}\left[\eta_i, \frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial z_i}\right].$$
In other words, as a differential operator on the graded algebra $\Omega^{0,*}(\mathbb{C}^n)$, each D_i has constant coefficients.

It turns out that, under some mild hypothesis, any such action functional I is equivalent (in the sense of the BV formalism) to one which has only z_i derivatives, and no $\bar{z}_i$ or $\mathrm{d}\bar{z}_i$ derivatives.

10.2.0.1 Lemma. *Suppose that $Q = \bar{\partial}$, so that $Q_0 = 0$. Then, any interaction I satisfying the classical master equation and the condition that $\eta_i I = 0$ for $i = 1, \dots, n$ is equivalent to one only involving derivatives in the z_i.*

PROOF. Let $\mathscr{E} = \Omega^{0,*}(\mathbb{C}^n, V)$ denote the space of fields of our theory, and let $\mathscr{O}_{loc}(\mathscr{E})$ denote the space of local functionals on $\mathscr{E}$. Let $\mathscr{O}_{loc}(\mathscr{E})^{hol}$ denote those functions which are translation-invariant and are in the kernel of the operators η_i, and let $\mathscr{O}_{loc}(\mathscr{E})^{hol'}$ denote those which in addition have only z_i derivatives. We will show that the inclusion map

$$\mathscr{O}_{loc}(\mathscr{E})^{hol'} \to \mathscr{O}_{loc}(\mathscr{E})^{hol}$$

is a quasi-isomorphism, where both are equipped with just the $\overline{\partial}$ differential. Both sides are graded by polynomial degree of the local functional, so it suffices to show this for local functionals of a fixed degree.

Note that the space V is filtered, by saying that F^i consists of those elements of degrees $\geq i$. This induces a filtration on $\mathscr{E}$ by the subspaces $\Omega^{0,*}(\mathbb{C}^n, F^i V)$. After passing to the associated graded, the operator Q becomes $\overline{\partial}$. By considering a spectral sequence with respect to this filtration, we see that it suffices to show we have a quasi-isomorphism in the case $Q = \overline{\partial}$.

But this follows immediately from the fact that the inclusion

$$\mathbb{C}\left[\frac{\partial}{\partial z_i}\right] \hookrightarrow \mathbb{C}\left[\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \overline{z}_i}, \eta_i\right]$$

is a quasi-isomorphism, where the right hand side is equipped with the differential $[\overline{\partial}, -]$. To see that this map is a quasi-isomorphism, note that the $\overline{\partial}$ operator sends η_i to $\frac{\partial}{\partial \overline{z}_i}$. $\square$

Recall that the action functional I induces the structure of L_∞ algebra on $\Omega^{0,*}(\mathbb{C}^n, V)[-1]$ whose differential is Q, and whose L_∞ structure maps are encoded by the Taylor components of I. Under the hypothesis of the previous lemma, this L_∞ algebra is L_∞ equivalent to one which is the Dolbeault complex with coefficients in a translation-invariant local L_∞ algebra whose structure maps only involve z_i derivatives.

There are many natural examples of holomorphically translation-invariant classical field theories. Geometrically, they arise from holomorphic moduli problems. For instance, one could take the cotangent theory to the derived moduli of holomorphic G bundles on $\mathbb{C}^n$, or the cotangent theory to the derived moduli space of such bundles equipped with holomorphic sections of some associated bundles, or the cotangent theory to the moduli of holomorphic maps from C^n to some complex manifold.

As is explained in great detail in [], holomorphically translation-invariant field theories arise very naturally in physics as holomorphic (or minimal) twists of supersymmetric field theories in even dimensions.

10.2.1. A holomorphically translation invariant classical theory on $\mathbb{C}^n$ has a natural gauge fixing operator, namely

$$\overline{\partial}^* = -\sum \eta_i \frac{\partial}{\partial \overline{z}_i}.$$

Since $[\eta_i, Q_0] = 0$, we see that $[Q, \overline{\partial}^*] = [\overline{\partial}, \overline{\partial}^*]$ is the Laplacian. (More generally, we can consider a family of gauge fixing operators coming from the $\overline{\partial}^*$ operator for a family of flat Hermitian metrics on $\mathbb{C}^n$. Since the space of such metrics is $GL(n,\mathbb{C})/U(n)$ and thus contractible, we see that everything is independent up to homotopy of the choice of gauge fixing operator.)

We say a translation-invariant parametrix

$$\Phi \in \overline{\Omega}^{0,*}(\mathbb{C}^n, V)^{\otimes 2}$$

is *holomorphically translation-invariant* if

$$(\eta_i \otimes 1 + 1 \otimes \eta_i)\Phi = 0$$

for $i = 1, \ldots, n$. For example, if Φ_0 is a parametrix for the scalar Laplacian

$$\triangle = -\sum \frac{\partial}{\partial z_i} \frac{\partial}{\partial \overline{z}_i}$$

then

$$\Phi_0 \prod_{i=1}^{n} \mathrm{d}(\overline{z}_i - \overline{w}_i) c$$

defines such a parametrix. Here z_i and w_i indicate the coordinates on the two copies of $\mathbb{C}^n$, and $c \in V \otimes V$ is the inverse of the pairing on v. Clearly, we can find holomorphically translation-invariant parametrices which are supported arbitrarily close to the diagonal. This means that we can define a field theory by only considering $I[\Phi]$ for holomorphically translation-invariant parametrices Φ.

10.2.1.1 Definition. *A holomorphically translation-invariant quantization of a holomorphically translation-invariant classical field theory as above is a translation-invariant quantization such that for each holomorphically translation-invariant parametrix Φ, the effective interaction $I[\Phi]$ satisfies*

$$\eta_i I[\Phi] = 0$$

for $i = 1, \ldots, n$. Here η_i abusively denotes the natural extension of the contraction η_i to a derivation on $\mathscr{O}(\Omega_c^{0,*}(\mathbb{C}^n, V))$.

The usual obstruction theory arguments hold for constructing holomorphically-translation invariant quantizations. At each order in $\hbar$, the obstruction-deformation complex is the subcomplex of the complex $\mathscr{O}_{loc}(\mathscr{E})^{\mathbb{C}^n}$ of translation-invariant local functionals which are also in the kernel of the operators η_i.

10.2.1.2 Proposition. *A holomorphically translation-invariant quantum field theory on $\mathbb{C}^n$ leads to a holomorphically translation-invariant factorization algebra.*

PROOF. This follows immediately from proposition 10.1.1.2. Indeed, quantum observables form a smoothly translation-invariant factorization algebra. Such an observable O on U is specified by a family $O[\Phi] \in \mathscr{O}(\Omega^{0,*}(\mathbb{C}^n, V))$ of functionals defined for each holomorphically translation-invariant parametrix Φ, which are supported on U for Φ sufficiently small. The operators $\frac{\partial}{\partial z_i}$, $\frac{\partial}{\partial \overline{z}_i}$, η_i act in a natural way on $\mathscr{O}(\Omega^{0,*}(\mathbb{C}^n, V))$ by derivations, and each commutes with the renormalization group flow W_Ψ^Φ for holomorphically translation-invariant parametrices Ψ, Φ. Thus, $\frac{\partial}{\partial z_i}$, $\frac{\partial}{\partial \overline{z}_i}$ and η_i define derivations of the factorization algebra Obs^q. Explicitly, if $O \in \mathrm{Obs}^q(U)$ is an observable, then for each holomorphically translation-invariant parametrix Φ,

$$\left(\frac{\partial}{\partial z_i} O\right)[\Phi] = \frac{\partial}{\partial z_i}(O[\Phi]),$$

and similarly for $\frac{\partial}{\partial \overline{z}_i}$ and η_i.

By definition (Definition **??**), a holomorphically translation-invariant factorization algebra is a translation-invariant factorization algebra where the derivation operator $\frac{\partial}{\partial \overline{z}_i}$ on observables is homotopically trivialized.

Note that, for a holomorphically translation-invariant parametrix Φ, $[\eta_i, \triangle_\Phi] = 0$ and η_i is a derivation for the Poisson bracket $\{-,-\}_\Phi$. It follows that

$$[Q + \{I[\Phi], -\}_\Phi + \hbar\triangle_\Phi, \eta_i] = [Q, \eta_i]$$

as operators on $\mathscr{O}(\Omega^{0,*}(\mathbb{C}^n, V))$. Since we wrote $Q = \overline{\partial} + Q_0$ and required that $[Q_0, \eta_i] = 0$, we have

$$[Q, \eta_i] = [\overline{\partial}, \eta_i] = \frac{\partial}{\partial \overline{z}_i}.$$

Since the differential on $\mathrm{Obs}^q(U)$ is defined by

$$(\widehat{Q}O)[\Phi] = QO[\Phi] + \{I[\Phi], O[\Phi]\}_\Phi + \hbar\triangle_\Phi O[\Phi],$$

we see that $[\widehat{Q}, \eta_i] = \frac{\partial}{\partial \overline{z}_i}$, as desired. □

As we showed in Chapter **??**, a holomorphically translation invariant factorization algebra in one complex dimension, with some mild additional conditions, gives rise to a vertex algebra. Let us verify that these conditions hold in the examples of interest. We first need a definition.

10.2.1.3 Definition. *A holomorphically translation-invariant field theory on $\mathbb{C}$ is S^1-invariant if the following holds. First, we have an S^1 action on the vector space V, inducing an action of S^1 on the space $\mathscr{E} = \Omega^{0,*}(\mathbb{C}, V)$ of fields, by combining the S^1 action on V with the natural one* *on $\Omega^{0,*}(\mathbb{C})$ coming from rotation on $\mathbb{C}$. . We suppose that all the structures of the field theory are S^1-invariant. More precisely, the symplectic pairing on $\mathscr{E}$ and the differential Q on $\mathscr{E}$ must be S^1-invariant. Further, for every S^1-invariant parametrix Φ, the effective interaction $I[\Phi]$ is S^1-invariant.*

10.2.1.4 Lemma. *Suppose we have a holomorphically translation invariant field theory on $\mathbb{C}$ which is also S^1-invariant. Then, the corresponding factorization algebra satisfies the conditions stated in theorem* **??** *of Chapter* **??** *allowing us to construct a vertex algebra structure on the cohomology.*

PROOF. Let $\mathcal{F}$ denote the factorization algebra of observables of our theory. Note that if $U \subset \mathbb{C}$ is an S^1-invariant subset, then S^1 acts on $\mathcal{F}(U)$.

Recall that $\mathcal{F}$ is equipped with a complete decreasing filtration, and is viewed as a factorization algebra valued in pro-differentiable cochain complexes. Recall that we need to check the following properties.

(1) The S^1 action on $\mathcal{F}(D(0,r))$ extends to a smooth action of the algebra $\mathcal{D}(S^1)$ of distributions on S^1.
(2) Let $\operatorname{Gr}^m \mathcal{F}(D(0,r))$ denote the associated graded with respect to the filtration on $\mathcal{F}(D(0,r))$. Let $\operatorname{Gr}^m_k \mathcal{F}(D(0,r))$ refer to the kth S^1-eigenspace in $\operatorname{Gr}^m \mathcal{F}(D(0,r))$. Then, we require that the map
$$\operatorname{Gr}^m_k \mathcal{F}(D(0,r)) \to \operatorname{Gr}^m_k \mathcal{F}(D(0,r'))$$
is a quasi-isomorphism of differentiable vector spaces.
(3) The differentiable vector space $H^*(\operatorname{Gr}^m_k \mathcal{F}(D(0,r)))$ is finite-dimensional for all k and is zero for $k \gg 0$.

Let us first check that the S^1 action extends to a $\mathcal{D}(S^1)$-action. If $\lambda \in S^1$ let ρ_λ^* denote this action. We need to check that for any observable $\{O[\Phi]\}$ and for every distribution $D(\lambda)$ on $\mathcal{S}^1$ the expression
$$\int_{\lambda \in S^1} D(\lambda) \rho_\lambda^* O[\Phi]$$
makes sense and defines another observable. Further, this construction must be smooth in both $D(\lambda)$ and the observable $O[\Phi]$, meaning that it must work families.

For fixed Φ, each $O_{i,k}[\Phi]$ is simply a distribution on $\mathbb{C}^k$ with some coefficients. For any distribution α on $\mathbb{C}^k$, the expression $\int_\lambda D(\lambda)\rho_\lambda^* \alpha$ makes sense and is continuous in both α and the distribution$D$. Indeed, $\int_\lambda D(\lambda)\rho_\lambda^*\alpha$ is simply the push-forward map in distributions applied to the action map $S^1 \times \mathbb{C}^k \to \mathbb{C}^k$.

It follows that, for each distribution D on S^1, we can define

$$D * O_{i,k}[\Phi] := \int_{\lambda \in S^1} D(\lambda) \rho_\lambda^* O_{i,k}[\Phi].$$

As a function of D and $O_{i,k}[\Phi]$, this construction is smooth. Further, sending an observable $O[\Phi]$ to $D * O[\Phi]$ commutes with the renormalization group flow (between S^1-equivariant parametrices). It follows that we can define a new observable $D * O$ by

$$(D * O)_{i,k}[\Phi] = D * (O_{i,k}[\Phi]).$$

Now, a family of observables O^x (parametrized by $x \in M$, a smooth manifold) is smooth if and only if the family of functionals $O^x_{i,k}[\Phi]$ are smooth for all i, k and all Φ. In fact one need not check this for all Φ, but for any collection of parametrices which includes arbitrarily small parametrices. If follows that the map sending D and O to $D * O$ is smooth, that is, takes smooth families to smooth families.

Let us now check the remaining assumptions of theorem **??**. Let $\mathcal{F}$ denote the factorization algebra of quantum observables of the theory and let $\mathcal{F}_k$ denote the kth eigenspace of the S^1 action. We first need to check that the inclusion

$$\operatorname{Gr}_k^m \mathcal{F}(D(0,r)) \to \operatorname{Gr}_k^m \mathcal{F}(D(0,r'))$$

is a quasi-isomorphism for $r < r'$. We need it to be a quasi-isomorphism of completed filtered differentiable vector spaces. The space $\operatorname{Gr}^m \mathcal{F}(D(0,r))$ is a finite direct sum of spaces of the form

$$\overline{\Omega}_c^{0,*}(D(0,r)^l, V^{\boxtimes l})_{S_l}.$$

It thus suffices to check that for the map

$$\overline{\Omega}_c^{0,*}(D(0,r)^m) \to \overline{\Omega}_c^{0,*}(D(0,r')^m)$$

is a quasi-isomorphism on each S^1-eigenspace. This is immediate.

The same holds to check that the cohomology of $\operatorname{Gr}_k^m \mathcal{F}(D(0,r))$ is zero for $k \gg 0$ and that it is finite-dimensional as a differentiable vector space. $\square$

We have seen that any S^1-equivariant and holmorphically translation-invariant factorization algebra on $\mathbb{C}$ gives rise to a vertex algebra. We have also seen that the obstruction-theory method applies in this situation to construct holomorphically translation invariant factorization algebras from appropriate Lagrangians. In this way, we have a very general method for constructing vertex algebras.

10.3. Renormalizability and factorization algebras

A central concept in field theory is that of *renormalizability*. This is discussed in detail in [**Cos11b**]. The basic idea is the following.

The group $\mathbb{R}_{>0}$ acts on the collection of field theories on $\mathbb{R}^n$, where the action is induced from the scaling action of $\mathbb{R}_{>0}$ on $\mathbb{R}^n$. This action is implemented differently in different models for field theories. In the language if factorization algebras it is very simple, because any factorization algebra on $\mathbb{R}^n$ can be pushed forward under any diffeomorphism of $\mathbb{R}^n$ to yield a new factorization algebra on $\mathbb{R}^n$. Push-forward of factorization algebras under the map $x \mapsto \lambda^{-1}x$ (for $\lambda \in \mathbb{R}_{>0}$) defines the renormalization group flow on factorization algebra.

We will discuss how to implement this rescaling in the definition of field theory given in [**Cos11b**] shortly. The main result of this section is the statement that the map which assigns to a field theory the corresponding factorization algebra of observables intertwines the action of $\mathbb{R}_{>0}$.

Acting by elements $\lambda \in \mathbb{R}_{>0}$ on a fixed quantum field theory produces a one-parameter family of theories, depending on λ. Let F denote a fixed theory, either in the language of factorization algebras, the language of [**Cos11b**], or any other approach to quantum field theory. We will call this family of theories $\rho_\lambda(F)$. We will view the theory $\rho_\lambda(F)$ as being obtained from F by "zooming in" on $\mathbb{R}^n$ by an amount dicated by λ, if $\lambda < 1$, or by zooming out if $\lambda > 1$.

We should imagine the theory F as having some number of continuous parameters, called coupling constants. Classically, the coupling constants are simply constants appearing next to various terms in the Lagrangian. At the quantum level, we could think of the structure constants of the factorization algebra as being functions of the coupling constants (we will discuss this more precisely below).

Roughly speaking, a theory is *renormalizable* if, as $\lambda \to 0$, the family of theories $\rho_\lambda(F)$ converges to a limit. While this definition is a good one non-pertubatively, in perturbation theory it is not ideal. The reason is that often the coupling constants depend on the scale through quantities like $\lambda^\hbar$. If $\hbar$ was an actual real number, we could analyze the behaviour of $\lambda^\hbar$ for λ small. In perturbation theory, however, $\hbar$ is a formal parameter, and we must expand $\lambda^\hbar$ in a series of the form $1 + \hbar \log \lambda + \dots$. The coefficients of this series always grow as $\lambda \to 0$.

In other words, from a perturbative point of view, one can't tell the difference between a theory that has a limit as $\lambda \to 0$ and a theory whose coupling constants have logarithmic growth in λ.

This motivates us to define a theory to be *perturbatively renormalizable* if it has logarithmic growth as $\lambda \to 0$. We will introduce a formal definition of perturbative renormalizability shortly. Let us first indicate why this definition is important.

It is commonly stated (especially in older books) that perturbative renormalizability is a necessary condition for a theory to exist (in perturbation theory) at the quantum level.

This is *not* the case. Instead, renormalizability is a criterion which allows one to select a finite-dimensional space of well-behaved quantizations of a given classical field theory, from a possibly infinite dimensional space of all possible quantizations.

There are other criteria which one wants to impose on a quantum theory and which also help select a small space of quantizations: for instance, symmetry criteria. (In addition, one also requires that the quantum master equation holds, which is a strong constraint. This, however, is part of the definition of a field theory that we use). There are examples of non-renormalizable field theories for which nevertheless a unique quantization can be selected by other criteria. (An example of this nature is BCOV theory).

10.3.1. The renormalization group action on factorization algebras. Let us now discuss the concept of renormalizability more formally. We will define the action of the group $\mathbb{R}_{>0}$ on the set of theories in the definition used in [Cos11b], and on the set of factorization algebras on $\mathbb{R}^n$. We will see that the map which assigns a factorization algebra to a theory is $\mathbb{R}_{>0}$-equivariant.

Let us first define the action of $\mathbb{R}_{>0}$ on the set of factorization algebras on $\mathbb{R}^n$.

10.3.1.1 Definition. *If $\mathcal{F}$ is a factorization algebra on $\mathbb{R}^n$, and $\lambda \in \mathbb{R}_{>0}$, let $\rho_\lambda(\mathcal{F})$ denote the factorization algebra on $\mathbb{R}^n$ which is the push-forward of $\mathcal{F}$ under the diffeomorphism $\lambda^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ given by multiplying by λ^{-1}. Thus,*

$$\rho_\lambda(\mathcal{F})(U) = \mathcal{F}(\lambda(U))$$

and the product maps in $\rho_\lambda(\mathcal{F})$ arise from those in $\mathcal{F}$. We will call this action of $\mathbb{R}_{>0}$ on the collection of factorization algebras on $\mathbb{R}^n$ the local renormalization group *action.*

Thus, the action of $\mathbb{R}_{>0}$ on factorization algebras on $\mathbb{R}^n$ is simply the obvious action of diffeomorphisms on $\mathbb{R}^n$ on factorization algebras on $\mathbb{R}^n$.

10.3.2. The renormalization group flow on classical theories. The action on field theories as defined in [Cos11b] is more subtle. Let us start by describing the action of $\mathbb{R}_{>0}$ on classical field theories. Suppose we have a translation-invariant classical field on $\mathbb{R}^n$, with space of fields $\mathscr{E}$. The space $\mathscr{E}$ is the space of sections of a trivial vector bundle on $\mathbb{R}^n$ with fibre E_0. The vector space E_0 is equipped with a degree -1 symplectic pairing valued in the line ω_0, the fibre of the bundle of top forms on $\mathbb{R}^n$ at 0. We also, of course, have a translation-invariant local functional $I \in \mathscr{O}_{loc}(\mathscr{E})$ satisfying the classical master equation.

Let us choose an action ρ^0_λ of the group $\mathbb{R}_{>0}$ on the vector space E_0 with the property that the symplectic pairing on E_0 is $\mathbb{R}_{>0}$-equivariant, where the action of $\mathbb{R}_{>0}$ acts on the line ω_0 with weight $-n$. Let us further assume that this action is diagonalizable, and that the eigenvalues of ρ^0_λ are rational integer powers of λ. (In practise, only integer or half-integer powers appear).

The choice of such an action, together with the action of $\mathbb{R}_{>0}$ on $\mathbb{R}^n$ by rescaling, induces an action of $\mathbb{R}_{>0}$ on

$$\mathscr{E} = C^\infty(\mathbb{R}^n) \otimes E_0$$

which sends

$$\phi \otimes e_0 \mapsto \phi(\lambda^{-1}x)\rho^0_\lambda(e_0),$$

where $\phi \in C^\infty(\mathbb{R}^n)$ and $e_0 \in E_0$. The convention that $x \mapsto \lambda^{-1}x$ means that for small λ, we are looking at small scales (for instance, as $\lambda \to 0$ the metric becomes large).

This action therefore induces an action on spaces associated to $\mathscr{E}$, such as the spaces $\mathscr{O}(\mathscr{E})$ of functionals and $\mathscr{O}_{loc}(\mathscr{E})$ of local functionals. The compatibility between the action of $\mathbb{R}_{>0}$ and the symplectic pairing on E_0 implies that the Poisson bracket on the space $\mathscr{O}_{loc}(\mathscr{E})$ of local functionals on $\mathscr{E}$ is preserved by the $\mathbb{R}_{>0}$ action. Let us denote the action of $\mathbb{R}_{>0}$ on $\mathscr{O}_{loc}(\mathscr{E})$ by ρ_λ.

10.3.2.1 Definition. *The local renormalization group flow on the space of translation-invariant classical field theories sends a classical action functional $I \in \mathscr{O}_{loc}(\mathscr{E})$ to $\rho_\lambda(I)$.*

This definition makes sense, because ρ_λ preserves the Poisson bracket on $\mathscr{O}_{loc}(\mathscr{E})$. Note also that, if the action of ρ^0_λ on E_0 has eigenvalues in $\frac{1}{n}\mathbb{Z}$, then the action of ρ_λ on the space $\mathscr{O}_{loc}(\mathscr{E})$ is diagonal and has eigenvalues again in $\frac{1}{n}\mathbb{Z}$.

The action of $\mathbb{R}_{>0}$ on the space of classical field theories up to isomorphism is independent of the choice of action of $\mathbb{R}_{>0}$ on E_0. If we choose a different action, inducing a different action ρ'_λ of $\mathbb{R}_{>0}$ on everything, then $\rho_\lambda I$ and $\rho'_\lambda I$ are related by a linear and symplectic change of coordinates on the space of fields which covers the identity on $\mathbb{R}^n$. Field theories related by such a change of coordinates are equivalent.

It is often convenient to choose the action of $\mathbb{R}_{>0}$ on the space E_0 so that the quadratic part of the action is invariant. When we can do this, the local renormalization group flow acts only on the interactions (and on any small deformations of the quadratic part that one considers). Let us give some examples of the local renormalization group flow on classical field theories. Many more details are given in [**Cos11b**].

Consider the free massless scalar field theory on $\mathbb{R}^n$. The complex of fields is

$$C^\infty(\mathbb{R}^n) \xrightarrow{\mathrm{D}} C^\infty(\mathbb{R}^n).$$

We would like to choose an action of $\mathbb{R}_{>0}$ so that both the symplectic pairing and the action functional $\int \phi \,\mathrm{D}\, \phi$ are invariant. This action must, of course, cover the action of $\mathbb{R}_{>0}$ on $\mathbb{R}^n$ by rescaling. If ϕ, ψ denote fields in the copies of $C^\infty(\mathbb{R}^n)$ in degrees 0 and 1 respectively, the desired action sends

$$\rho_\lambda(\phi(x)) = \lambda^{\frac{2-n}{2}} \phi(\lambda^{-1}x)$$
$$\rho_\lambda(\psi(x)) = \lambda^{\frac{-n-2}{2}} \psi(\lambda^{-1}x).$$

Let us then consider how ρ_λ acts on possible interactions. We find, for example, that if

$$I_k(\phi) = \int \phi^k$$

then

$$\rho_\lambda(I_k) = \lambda^{n-\frac{k(n-2)}{2}} I_k.$$

10.3.2.2 Definition. *A classical theory is* renormalizable *if, as $\lambda \to 0$, it flows to a fixed point under the local renormalization group flow.*

For instance, we see that in dimension 4, the most general renormalizable classical action for a scalar field theory which is invariant under the symmetry $\phi \mapsto -\phi$ is

$$\int \phi \, \mathrm{D} \phi + m^2\phi^2 + c\phi^4.$$

Indeed, the ϕ^4 term is fixed by the local renormalization group flow, whereas the ϕ^2 term is sent to zero as $\lambda \to 0$.

10.3.2.3 Definition. *A classical theory is* strictly renormalizable *if it is a fixed point under the local renormalization group flow.*

A theory which is renormalizable has good small-scale behaviour, in that the coupling constants (classically) become small at small scales. (At the quantum level there may also be logaritmic terms which we will discuss shortly). A renormalizable theory may, however, have bad large-scale behaviour: for instance, in four dimensions, a mass term $\int \phi^2$ becomes large at large scales. A strictly renormalizable theory is one which is classically scale invariant. At the quantum level, we will define a strictly renormalizable theory to be one which is scale invariant up to logarithmic corrections.

Again in four dimensions, the only strictly renormalizable interaction for the scalar field theory which is invariant under $\phi \mapsto -\phi$ is the ϕ^4 interaction. In six dimensions, the ϕ^3 interaction is strictly renormalizable, and in three dimensions the ϕ^6 interaction (together with finitely many other interactions involving derivatives) are strictly renormalizable.

As another example, recall that the graded vector space of fields of pure Yang-Mills theory (in the first order formalism) is

$$\left(\Omega^0[1] \oplus \Omega^1 \oplus \Omega^2_+ \oplus \Omega^2_+[-1] \oplus \Omega^3[-1] \oplus \Omega^4[-2]\right) \otimes \mathfrak{g}.$$

(Here Ω^i indicates forms on $\mathbb{R}^4$). The action of $\mathbb{R}_{>0}$ is the natural one, coming from pull-back of forms under the map $x \mapsto \lambda^{-1}x$. The Yang-Mills action functional

$$S(A,B) = \int F(A) \wedge B + B \wedge B$$

is obviously invariant under the action of $\mathbb{R}_{>0}$, since it only involves wedge product and integration, as well as projection to Ω^2_+. (Here $A \in \Omega^1 \otimes \mathfrak{g}$) and $B \in \Omega^2_+ \otimes \mathfrak{g}$). The other terms in the full BV action functional are also invariant, because the symplectic pairing on the space of fields and the action of the gauge group are both scale-invariant.

Something similar holds for Chern-Simons theory on $\mathbb{R}^3$, where the space of fields is $\Omega^*(\mathbb{R}^3) \otimes \mathfrak{g}[1]$. The action of $\mathbb{R}> 0$ is by pull-back by the map $x \mapsto \lambda^{-1}x$, and the Chern-Simons functional is obviously invariant.

10.3.2.4 Lemma. *The map which assigns to a translation-invariant classical field theory on $\mathbb{R}^n$ the associated P_0 factorization algebra commutes with the action of the local renormalization group flow.*

PROOF. The action of $\mathbb{R}_{>0}$ on the space of fields of the theory induces an action on the space $\mathrm{Obs}^{cl}(\mathbb{R}^n)$ of classical observables on $\mathbb{R}^n$, by sending an observable O (which is a functon on the space $\mathscr{E}(\mathbb{R}^n)$ of fields) to the observable

$$\rho_\lambda O : \phi \mapsto O(\rho_\lambda(\phi)).$$

This preserves the Poisson bracket on the subspace $\widetilde{\mathrm{Obs}}^{cl}(\mathbb{R}^n)$ of functionals with smooth first derivative, because by assumption the symplectic pairing on the space of fields is scale invariant. Further, it is immediate from the definition of the local renormalization group flow on classical field theories that

$$\rho_\lambda\{S,O\} = \{\rho_\lambda(I), \rho_\lambda(O)\}$$

where $S \in \mathscr{O}_{loc}(\mathscr{E})$ is a translation-invariant solution of the classical master equation (whose quadratic part is elliptic).

Let $\mathrm{Obs}^{cl}_\lambda$ denote the factorization algebra on $\mathbb{R}^n$ coming from the theory $\rho_\lambda(S)$ (where S is some fixed classical action). Then, we see that we have an isomorphism of cochain complexes

$$\rho_\lambda : \mathrm{Obs}^{cl}(\mathbb{R}^n) \cong \mathrm{Obs}^{cl}_\lambda(\mathbb{R}^n).$$

We next need to check what this isomorphism does to the support conditions. Let $U \subset \mathbb{R}^n$ and let $O \in \mathrm{Obs}^{cl}(U)$ be an observable supported on U. Then, one can check easily that $\rho_\lambda(O)$ is supported on $\lambda^{-1}(U)$. Thus, ρ_λ gives an isomorphism

$$\mathrm{Obs}^{cl}(U) \cong \mathrm{Obs}^{cl}_\lambda(\lambda^{-1}(U)).$$

and so,

$$\mathrm{Obs}^{cl}(\lambda U) \cong \mathrm{Obs}^{cl}_\lambda(U).$$

The factorization algebra
$$\rho_\lambda \operatorname{Obs}^{cl} = \lambda_* \operatorname{Obs}^{cl}$$
assigns to an open set $U \subset \mathbb{R}^n$ the value of Obs^{cl} on $\lambda(U)$. Thus, we have constructed an isomorphism of precosheaves on $\mathbb{R}^n$,
$$\rho_\lambda \operatorname{Obs}^{cl} \cong \operatorname{Obs}^{cl}_\lambda .$$
This isomorphism compatible with the commutative product and the (homotopy) Poisson bracket on both side, as well as the factorization product maps. □

10.3.3. The renormalization group flow on quantum field theories. The most interesting version of the renormalization group flow is, of course, that on quantum field theories. Let us fix a classical field theory on $\mathbb{R}^n$, with space of fields as above $\mathscr{E} = C^\infty(\mathbb{R}^n) \otimes E_0$ where E_0 is a graded vector space. In this section we will define an action of the group $\mathbb{R}_{>0}$ on the simplicial set of quantum field theories with space of fields $\mathscr{E}$, quantizing the action on classical field theories that we constructed above. We will show that the map which assigns to a quantum field theory the corresponding factorization algebra commutes with this action.

Let us assume, for simplicity, that we have chosen the linear action of $\mathbb{R}_{>0}$ on E_0 so that it leaves invariant a quadratic action functional on $\mathscr{E}$ defining a free theory. Let $Q : \mathscr{E} \to \mathscr{E}$ be the corresponding cohomological differential, which, by assumption, is invariant under the $\mathbb{R}_{>0}$ action. (This step is not necessary, but will make the exposition simpler).

Let us also assume (again for simplicity) that there exists a gauge fixing operator $Q^{GF} : \mathscr{E} \to \mathscr{E}$ with the property that
$$\rho_\lambda Q^{GF} \rho_{-\lambda} = \lambda^k Q^{GF}$$
for some $k \in \mathbb{Q}$. For example, for a massless scalar field theory on $\mathbb{R}^n$, we have seen that the action of $\mathbb{R}_{>0}$ on the space $C^\infty(\mathbb{R}^n) \oplus C^\infty(\mathbb{R}^n)[-1]$ of fields sends ϕ to $\lambda^{\frac{2-n}{2}}\phi(\lambda^{-1}x)$ and ψ to $\lambda^{\frac{-2-n}{2}}\psi(\lambda^{-1}x)$ (where ϕ is the field of cohomological degree 0 and ψ is the field of cohomological degree 1). The gauge fixing operator is the identity operator from $C^\infty(\mathbb{R}^n)[-1]$ to $C^\infty(\mathbb{R}^n)[0]$. In this case, we have $\rho_\lambda Q^{GF} \rho_{-\lambda} = \lambda^2 Q^{GF}$.

As another example, consider pure Yang-Mills theory on $\mathbb{R}^4$. The fields, as we have described above, are built from forms on $\mathbb{R}^4$, equipped with the natural action of $\mathbb{R}_{>0}$. The gauge fixing operator is d^*. It is easy to see that $\rho_\lambda \mathrm{d}^* \rho_{-\lambda} = \lambda^2 \mathrm{d}^*$. The same holds for Chern-Simons theory, which also has a gauge fixing operator defined by d^* on forms.

A translation-invariant quantum field theory is defined by a family
$$\{I[\Phi] \in \mathscr{O}^+_{P,sm}(\mathscr{E})^{\mathbb{R}^n}[[\hbar]] \mid \Phi \text{ a translation-invariant parametrix}\}$$
which satisfies the renormalization group equation, quantum master equation, and the locality condition. We need to explain how scaling of $\mathbb{R}^n$ by $\mathbb{R}_{>0}$ acts on the (simplicial)

set of quantum field theories. To do this, we first need to explain how this scaling action acts on the set of parametrices.

10.3.3.1 Lemma. *If Φ is a translation-invariant parametrix, then $\lambda^k \rho_\lambda(\Phi)$ is also a parametrix, where as above k measures the failure of Q^{GF} to commute with ρ_λ.*

PROOF. All of the axioms characterizing a parametrix are scale invariant, except the statement that

$$([Q, Q^{GF}] \otimes 1)\Phi = K_{id} - \text{ something smooth.}$$

We need to check that $\lambda^k \rho_\lambda \Phi$ also satisfies this. Note that

$$\rho_\lambda([Q, Q^{GF}] \otimes 1)\Phi = \lambda^k([Q, Q^{GF}] \otimes 1)\rho_\lambda(\Phi)$$

since ρ_λ commutes with Q but not with Q^{GF}. Also, ρ_λ preserves K_{id} and smooth kernels, so the desired identity holds. □

This lemma suggests a way to define the action of the group $\mathbb{R}_{>0}$ on the set of quantum field theories.

10.3.3.2 Lemma. *If $\{I[\Phi]\}$ is a theory, define $I_\lambda[\Phi]$ by*

$$I_\lambda[\Phi] = \rho_\lambda(I[\lambda^{-k}\rho_{-\lambda}(\Phi)]).$$

Then, the collection of functionals $\{I_\lambda[\Phi]\}$ define a new theory.

On the right hand side of the equation in the lemma, we are using the natural action of ρ_λ on all spaces associated to $\mathscr{E}$, such as the space $\mathscr{E}\widehat{\otimes}_\pi\mathscr{E}$ (to define $\rho_{-\lambda}(\Phi)$) and the space of functions on $\mathscr{E}$ (to define how ρ_λ acts on the function $I[\lambda^{-k}\rho_{-\lambda}(\Phi)]$).

Note that this lemma, as well as most things we discuss about renormalizability of field theories which do not involve factorization algebras, is discussed in more detail in [Cos11b], except that there the language of heat kernels is used. We will prove the lemma here anyway, because the proof is quite simple.

PROOF. We need to check that $I_\lambda[\Phi]$ satisfies the renormalization group equation, locality action, and quantum master equation. Let us first check the renormalization group flow. As a shorthand notation, let us write Φ_λ for the parametrix $\lambda^k \rho_\lambda(\Phi)$. Then, note that the propagator $P(\Phi_\lambda)$ is

$$P(\Phi_\lambda) = \rho_\lambda P(\Phi).$$

Indeed,

$$\begin{aligned}\rho_\lambda \tfrac{1}{2}(Q^{GF} \otimes 1 + 1 \otimes Q^{GF})\Phi &= \lambda^k \tfrac{1}{2}(Q^{GF} \otimes 1 + 1 \otimes Q^{GF})\rho_\lambda(\Phi) \\ &= P(\Phi_\lambda).\end{aligned}$$

It follows from this that, for all functionals $I \in \mathscr{O}_P^+(\mathscr{E})[[\hbar]]$,

$$\rho_\lambda(W(P(\Phi) - P(\Psi), I) = W(P(\Phi_\lambda) - P(\Psi_\lambda), \rho_\lambda(I).)$$

We need to verify the renormalization group equation, which states that

$$W(P(\Phi) - P(\Psi), I_\lambda[\Psi]) = I_\lambda[\Phi].$$

Because $I_\lambda[\Phi] = \rho_\lambda I[\Phi_{-\lambda}]$, this is equivalent to

$$\rho_{-\lambda} W(P(\Phi) - P(\Psi), \rho_\lambda(I[\Psi_{-\lambda}]) = I[\Phi_{-\lambda}].$$

Bringing $\rho_{-\lambda}$ inside the W reduces us to proving the identity

$$W(P(\Phi_{-\lambda} - P(\Psi_{-\lambda}, I[\Psi_{-\lambda}]) = I[\Phi_{-\lambda}]$$

which is the renormalization group identity for the functionals $I[\Phi]$.

The fact that $I_\lambda[\Phi]$ satisfies the quantum master equation is proved in a similar way, using the fact that

$$\rho_\lambda(\triangle_\Phi I) = \triangle_{\Phi_\lambda} \rho_\lambda(I)$$

where $\triangle_\Phi$ denotes the BV Laplacian associated to Φ and I is any functional.

Finally, the locality axiom is an immediate consequence of that for the original functionals $I[\Phi]$. □

10.3.3.3 Definition. *Define the* local renormalization group flow *to be the action of* $\mathbb{R}_{>0}$ *on the set of theories which sends, as in the previous lemma, a theory* $\{I[\Phi]\}$ *to the theory*

$$\{I_\lambda[\Phi]\} = \rho_\lambda(I[\lambda^{-k}\rho_{-\lambda}\Phi]).$$

Note that this works in families, and so defines an action of $\mathbb{R}_{>0}$ *on the simplicial set of theories.*

Note that this definition simply means that we act by $\mathbb{R}_{>0}$ on everything involved in the definition of a theory, including the parametrices.

Let us now quote some results from [**Cos11b**], concerning the behaviour of this action. Let us recall that to begin with, we chose an action of $\mathbb{R}_{>0}$ on the space $\mathscr{E} = C^\infty(\mathbb{R}^4) \otimes E_0$ of fields, which arose from the natural rescaling action on $C^\infty(\mathbb{R}^4)$ and an action on the finite-dimensional vector space E_0. We assumed that the action on E_0 is diagonalizable, where on each eigenspace ρ_λ acts by λ^a for some $a \in \mathbb{Q}$. Let $m \in \mathbb{Z}$ be such that the exponents of each eigenvalue are in $\frac{1}{m}\mathbb{Z}$.

10.3.3.4 Theorem. *For any theory* $\{I[\Phi]\}$ *and any parametrix* Φ, *the family of functionals* $I_\lambda[\Phi]$ *depending on* λ *live in*

$$\mathscr{O}^+_{sm,P}(\mathscr{E})\left[\log\lambda, \lambda^{\frac{1}{m}}, \lambda^{-\frac{1}{m}}\right][[\hbar]].$$

In other words, the functionals $I_\lambda[\Phi]$ depend on λ only through polynomials in $\log \lambda$ and $\lambda^{\pm \frac{1}{m}}$. (More precisely, each functional $I_{\lambda,i,k}[\Phi]$ in the Taylor expansion of $I_\lambda[\Phi]$ has such polynomial dependence, but as we quantify over all i and k the degree of the polynomials may be arbitrarily large).

In [**Cos11b**], this result is only stated under the hypothesis that $m = 2$, which is the case that arises in most examples, but the proof in [**Cos11b**] works in general.

10.3.3.5 Lemma. *The action of $\mathbb{R}_{>0}$ on quantum field theories lifts that on classical field theories described earlier.*

This basic point is also discussed in [**Cos11b**]; it follows from the fact that at the classical level, the limit of $I[\Phi]$ as $\Phi \to 0$ exists and is the original classical interaction.

10.3.3.6 Definition. *A quantum theory is* renormalizable *if the functionals $I_\lambda[\Phi]$ depend on λ only by polynomials in $\log \lambda$ and $\lambda^{\frac{1}{m}}$ (where we assume that $m > 0$). A quantum theory is* strictly renormalizable *if it only depends on λ through polynomials in $\log \lambda$.*

Note that at the classical level, a strictly renormalizable theory must be scale-invariant, because logarithmic contributions to the dependence on λ only arise at the quantum level.

10.3.4. Quantization of renormalizable and strictly renormalizable theories.

Let us decompose $\mathscr{O}_{loc}(\mathscr{E})^{\mathbb{R}^4}$, the space of translation-invariant local functionals on $\mathscr{E}$, into eigenspaces for the action of $\mathbb{R}_{>0}$. For $k \in \frac{1}{m}\mathbb{Z}$, we let $\mathscr{O}_{loc}^{(k)}(\mathscr{E})^{\mathbb{R}^4}$ be the subspace on which ρ_λ acts by λ^k. Let $\mathscr{O}_{loc}^{(\geq 0)}(\mathscr{E})^{\mathbb{R}^4}$ denote the direct sum of all the non-negative eigenspaces.

Let us suppose that we are interested in quantizing a classical theory, given by an interaction I, which is either strictly renormalizable or renormalizable. In the first case, I is in $\mathscr{O}_{loc}^{(0)}(\mathscr{E})^{\mathbb{R}^4}$, and in the second, it is in $\mathscr{O}_{loc}^{(\geq 0)}(\mathscr{E})^{\mathbb{R}^4}$.

By our initial assumptions, the Lie bracket on $\mathscr{O}_{loc}(\mathscr{E})^{\mathbb{R}^4}$ commutes with the action of $\mathbb{R}_{>0}$. Thus, if we have a strictly renormalizable classical theory, then $\mathscr{O}_{loc}^{(0)}(\mathscr{E})^{\mathbb{R}^4}$ is a cochain complex with differential $Q + \{I-,\}$. This is the cochain complex controlling first-order deformations of our classical theory as a strictly renormalizable theory. In physics terminology, this is the cochain complex of marginal deformations.

If we start with a classical theory which is simply renormalizable, then the space $\mathscr{O}_{loc}^{(\geq 0)}(\mathscr{E})^{\mathbb{R}^4}$ is a cochain complex under the differential $Q + \{I-,\}$. This is the cochain complex of renormalizable deformations.

Typically, the cochain complexes of marginal and renormalizable deformations are finite-dimensional. (This happens, for instance, for scalar field theories in dimensions greater than 2).

Here is the quantization theorem for renormalizable and strictly renormalizalble quantizations.

10.3.4.1 Theorem. *Fix a classical theory on $\mathbb{R}^n$ which is renormalizable with classical interaction I. Let $\mathcal{R}^{(n)}$ denote the set of renormalizable quantizations defined modulo $\hbar^{n+1}$. Then, given any element in $\mathcal{R}^{(n)}$, there is an obstruction to quantizing to the next order, which is an element*

$$O_{n+1} \in H^1\left(\mathscr{O}_{loc}^{(\geq 0)}(\mathscr{E})^{\mathbb{R}^4}, Q + \{I, -\}\right).$$

If this obstruction vanishes, then the set of quantizations to the next order is a torsor for $H^0\left(\mathscr{O}_{loc}^{(\geq 0)}(\mathscr{E})^{\mathbb{R}^4}\right)$.

This statement holds in the simplicial sense too: if $\mathcal{R}_{\triangle}^{(n)}$ denotes the simplicial set of renormalizable theories defined modulo $\hbar^{n+1}$ and quantizing a given classical theory, then there is a homotopy fibre diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{R}_{\triangle}^{(n+1)} \arrow[r] \arrow[d] & \mathcal{R}_{\triangle}^{(n)} \arrow[d] \\
0 \arrow[r] & \mathrm{DK}\left(\mathscr{O}_{loc}^{(\geq 0)}(\mathscr{E})^{\mathbb{R}^4}[1], Q + \{I, -\}\right)
\end{tikzcd}$$

On the bottom right DK *indicates the Dold-Kan functor from cochain complexes to simplicial sets.*

All of these statements hold for the (simplicial) sets of strictly renormalizable theories quantizing a given strictly renormalizable classical theory, except that we should replace $\mathscr{O}_{loc}^{(\geq 0)}$ by $\mathscr{O}_{loc}^{(0)}$ everywhere. Further, all these results hold in families iwth evident modifications.

Finally, if $\mathscr{G}\mathscr{F}$ denotes the simplicial set of translation-invariant gauge fixing conditions for our fixed classical theory (where we only consider gauge-fixing conditions which scale well with respect to ρ_λ as discussed earlier), then the simplicial sets of (strictly) renormalizable theories with a fixed gauge fixing condition are fibres of a simplicial set fibred over $\mathcal{GF}$. As before, this means that the simplicial set of theories is independent up to homotopy of the choice of gauge fixing condition.

This theorem is proved in [**Cos11b**], and is the analog of the quantization theorem for theories without the renormalizability criterion.

Let us give some examples of how this theorem allows us to construct small-dimensional families of quantizations of theories where without the renormalizability criterion there would be an infinite dimensional space of quantizations.

Consider, as above, the massless ϕ^4 theory on $\mathbb{R}^4$, with interaction $\int \phi \mathrm{D} \phi + \phi^4$. At the classical level this theory is scale-invariant, and so strictly renormalizable. We have the following.

10.3.4.2 Lemma. *The space of strictly-renormalizable quantizations of the massless ϕ^4 theory in 4 dimensions which are also invariant under the $\mathbb{Z}/2$ action $\phi \mapsto -\phi$ is isomorphic to $\hbar\mathbb{R}[[\hbar]]$. That is, there is a single $\hbar$-dependent coupling constant.*

PROOF. We need to check that the obstruction group for this problem is zero, and the deformation group is one-dimensional. The obstruction group is zero for degree reasons, because for a theory without gauge symmetry the complex of local functionals is concentrated in degrees ≤ 0. To compute the deformation group, note that the space of local functionals which are scale invariant and invariant under $\phi \mapsto -\phi$ is two-dimensional, spanned by $\int \phi^4$ and $\int \phi \mathrm{D} \phi$. The quotient of this space by the image of the differential $Q + \{I, -\}$ is one dimensional, because we can eliminate one of the two possible terms by a change of coordinates in ϕ. $\square$

Let us give another, and more difficult, example.

10.3.4.3 Theorem. *The space of renormalizable (or strictly renormalizable) quantizations of pure Yang-Mills theory on $\mathbb{R}^4$ with simple gauge Lie algebra $\mathfrak{g}$ is isomorphic to $\hbar\mathbb{R}[[\hbar]]$. That is, there is a single $\hbar$-dependent coupling constant.*

PROOF. The relevant cohomology groups were computed in [**Cos11b**], where it was shown that the deformation group is one dimensional and that the obstruction group is $H^5(\mathfrak{g})$. The obstruction group is zero unless $\mathfrak{g} = \mathrm{sl}_n$ and $n \geq 3$. By considering the outer automorphisms of sl_n, it was argued in [**Cos11b**] that the obstruction must always vanish. $\square$

This theorem then tells us that we have an essentially canonical quantization of pure Yang-Mills theory on $\mathbb{R}^4$, and hence a corresonding factorization algebra.

The following is the main new result of this section.

10.3.4.4 Theorem. *The map from translation-invariant quantum theories on $\mathbb{R}^n$ to factorization algebras on $\mathbb{R}^n$ commutes with the local renormalization group flow.*

PROOF. Suppose we have a translation-invariant quantum theory on $\mathbb{R}^n$ with space of fields $\mathscr{E}$ and family of effective actions $\{I[\Phi]\}$. Recall that the RG flow on theories sends this theory to the theory defined by

$$I_\lambda[\Phi] = \rho_\lambda(I[\lambda^{-k}\rho_{-\lambda}(\Phi)]).$$

We let $\Phi_\lambda = \lambda^k \rho_\lambda \Phi$. As we have seen in the proof of lemma 10.3.3.2, we have

$$P(\Phi_\lambda) = \rho_\lambda(P(\Phi))$$
$$\triangle_{\Phi_\lambda} = \rho_\lambda(\triangle_\Phi).$$

Suppose that $\{O[\Phi]\}$ is an observable for the theory $\{I[\Phi]\}$. First, we need to show that

$$O_\lambda[\Phi] = \rho_\lambda(O[\Phi_{-\lambda}])$$

is an observable for the theory $O_\lambda[\Phi]$. The fact that $O_\lambda[\Phi]$ satisfies the renormalization group flow equation is proved along the same lines as the proof that $I_\lambda[\Phi]$ satisfies the renormalization group flow equation in lemma 10.3.3.2.

If Obs^q_λ denotes the factorization algebra for the theory I_λ, then we have constructed a map

$$\mathrm{Obs}^q(\mathbb{R}^n) \to \mathrm{Obs}^q_\lambda(\mathbb{R}^n)$$
$$\{O[\Phi]\} \mapsto \{O_\lambda[\Phi]\}.$$

The fact that $\triangle_{\Phi_\lambda} = \rho_\lambda(\triangle_\Phi)$ implies that this is a cochain map. Further, it is clear that this is a smooth map, and so a map of differentiable cochain complexes.

Next we need to check is the support condition. We need to show that if $\{O[\Phi]\}$ is in $\mathrm{Obs}^q(U)$, where $U \subset \mathbb{R}^n$ is open, then $\{O_\lambda[\Phi]\}$ is in $\mathrm{Obs}^q(\lambda^{-1}(U))$. Recall that the support condition states that, for all i, k, there is some parametrix Φ_0 and a compact set $K \subset U$ such that $O_{i,k}[\Phi]$ is supported in K for all $\Phi \leq \Phi_0$.

By making Φ_0 smaller if necessary, we can assume that $O_{i,k}[\Phi_\lambda]$ is supported on K for $\Phi \leq \Phi_0$. (If Φ is supported within ε of the diagonal, then Φ_λ is supported within $\lambda^{-1}\varepsilon$.) Then, $\rho_\lambda O_{i,k}[\Phi_\lambda]$ will be supported on $\lambda^{-1}K$ for all $\Phi \leq \Phi_0$. This says that O_λ is supported on $\lambda^{-1}K$ as desired.

Thus, we have constructed an isomorphism

$$\mathrm{Obs}^q(U) \cong \mathrm{Obs}^q_\lambda(\lambda^{-1}(U)).$$

This isomorphism is compatible with inclusion maps and with the factorization product. Therefore, we have an isomorphism of factorization algebras

$$(\lambda^{-1})_* \mathrm{Obs}^q \cong \mathrm{Obs}^q_\lambda$$

where $(\lambda^{-1})_*$ indicates pushforward under the map given by multiplication by λ^{-1}. Since the action of the local renormalization group flow on factorization algebras on $\mathbb{R}^n$ sends $\mathcal{F}$ to $(\lambda^{-1})_*\mathcal{F}$, this proves the result. $\square$

The advantage of the factorization algebra formulation of the local renormalization group flow is that it is very easy to define; it captures precisely the intuition that the renormalization group flow arises the action of $\mathbb{R}_{>0}$ on $\mathbb{R}^n$. This theorem shows that the less-obvious definition of the renormalization group flow on theories, as defined in [Cos11b], coincides with the very clear definition in the language of factorization algebras. The advantage of the definition presented in [Cos11b] is that it is possible to compute with this definition, and that the relationship between this definition and how physicists define the β-function is more or less clear. For example, the one-loop β-function (one-loop contribution to the renormalization group flow) is calculated explicitly for the ϕ^4 theory in [Cos11b].

10.4. Cotangent theories and volume forms

In this section we will examine the case of a cotangent theory, in which our definition of a quantization of a classical field theory acquires a particularly nice interpretation. Suppose that $\mathcal{L}$ is an elliptic L_∞ algebra on a manifold M describing an elliptic moduli problem, which we denote by $B\mathcal{L}$. As we explained in Chapter **??**, section 5.6, we can construct a classical field theory from $\mathcal{L}$, whose space of fields is $\mathscr{E} = \mathcal{L}[1] \oplus \mathcal{L}^![-2]$. The main observation of this section is that a quantization of this classical field theory can be interpreted as a kind of "volume form" on the elliptic moduli problem $B\mathcal{L}$. This point of view was developed in [Cos13], and used in [Cos11a] to provide a geometric interpretation of the Witten genus.

The relationship between quantization of field theories and volume forms was discussed already at the very beginning of this book, in Chapter **??**. There, we explained how to interpret (heuristically) the BV operator for a free field theory as the divergence operator for a volume form.

While this heuristic interpretation holds for many field theories, cotangent theories are a class of theories where this relationship becomes very clean. If we have a cotangent theory to an elliptic moduli problem $\mathcal{L}$ on a compact manifold, then the L_∞ algebra $\mathcal{L}(M)$ has finite-dimensional cohomology. Therefore, the formal moduli problem $B\mathcal{L}(M)$ is an honest finite-dimensional formal derived stack. We will find that a quantization of a cotangent theory leads to a volume form on $B\mathcal{L}(M)$ which is of a "local" nature.

Morally speaking, the partition function of a cotangent theory should be the volume of $B\mathcal{L}(M)$ with respect to this volume form. If, as we've been doing, we work in perturbation theory, then the integral giving this volume often does not converge. One has to replace $B\mathcal{L}(M)$ by a global derived moduli space of solutions to the equations of motion to have a chance at defining the volume. The volume form on a global moduli space is obtained by doing perturbation theory near every point and then gluing together the formal volume forms so obtained near each point.

This program has been successfully carried out in a number of examples, such as [**?**, GG11, **?**]. For example, in [Cos11a], the cotangent theory to the space of holomorphic maps from an elliptic curve to a complex manifold was studied, and it was shown that the partition function (defined in the way we sketched above) is the Witten elliptic genus.

10.4.1. A finite dimensional model. We first need to explain an algebraic interpretation of a volume form in finite dimensions. Let X be a manifold (or complex manifold or smooth algebraic variety; nothing we will say will depend on which geometric category we work in). Let $\mathscr{O}(X)$ denote the smooth functions on X, and let $\operatorname{Vect}(X)$ denote the vector fields on X.

If ω is a volume form on X, then it gives a divergence map

$$\operatorname{Div}_\omega : \operatorname{Vect}(X) \to \mathscr{O}(X)$$

defined via the Lie derivative:

$$\operatorname{Div}_\omega(V)\omega = \mathcal{L}_V \omega$$

for $V \in \operatorname{Vect}(X)$. Note that the divergence operator $\operatorname{Div}_\omega$ satisfies the equations

$$(\dagger) \qquad \begin{aligned} \operatorname{Div}_\omega(fV) &= f \operatorname{Div}_\omega V + V(f). \\ \operatorname{Div}_\omega([V,W]) &= V \operatorname{Div}_\omega W - W \operatorname{Div}_\omega V. \end{aligned}$$

The volume form ω is determined up to a constant by the divergence operator $\operatorname{Div}_\omega$.

Conversely, to give an operator $\operatorname{Div} : \operatorname{Vect}(X) \to \mathscr{O}(X)$ satisfying equations ($\dagger$) is the same as to give a flat connection on the canonical bundle K_X of X, or, equivalently, to give a right D-module structure on the structure sheaf $\mathscr{O}(X)$.

10.4.1.1 Definition. *A* projective volume form *on a space X is an operator* $\operatorname{Div} : \operatorname{Vect}(X) \to \mathscr{O}(X)$ *satisfying equations ($\dagger$).*

The advantage of this definition is that it makes sense in many contexts where more standard definitions of a volume form are hard to define. For example, if A is a quasi-free differential graded commutative algebra, then we can define a projective volume form on the dg scheme $\operatorname{Spec} A$ to be a cochain map $\operatorname{Der}(A) \to A$ satisfying equations ($\dagger$). Similarly, if $\mathfrak{g}$ is a dg Lie or L_∞ algebra, then a projective volume form on the formal moduli problem $B\mathfrak{g}$ is a cochain map $C^*(\mathfrak{g}, \mathfrak{g}[1]) \to C^*(\mathfrak{g})$ satisfying equations ($\dagger$).

10.4.2. There is a generalization of this notion that we will use where, instead of vector fields, we take any Lie algebroid.

10.4.2.1 Definition. *Let A be a commutative differential graded algebra over a base ring k. A Lie algebroid L over A is a dg A-module with the following extra data.*

(1) *A Lie bracket on L making it into a dg Lie algebra over k. This Lie bracket will be typically not A-linear.*
(2) *A homomorphism of dg Lie algebras $\alpha : L \to \operatorname{Der}^*(A)$, called the anchor map.*

(3) These structures are related by the Leibniz rule

$$[l_1, f l_2] = (\alpha(l_1)(f))\, l_2 + (-1)^{|l_1||f|} f [l_1, l_2]$$

for $f \in A$, $l_i \in L$.

In general, we should think of L as providing the derived version of a foliation. In ordinary as opposed to derived algebraic geometry, a foliation on a smooth affine scheme with algebra of functions A consists of a Lie algebroid L on A which is projective as an A-module and whose anchor map is fibrewise injective.

10.4.2.2 Definition. *If A is a commutative dg algebra and L is a Lie algebroid over A, then an L-projective volume form on A is a cochain map*

$$\mathrm{Div} : L \to A$$

satisfying

$$\begin{aligned} \mathrm{Div}(al) &= a \,\mathrm{Div}\, l + (-1)^{|l||a|} \alpha(l) a. \\ \mathrm{Div}([l_1, l_2]) &= l_1 \,\mathrm{Div}\, l_2 - (-1)^{|l_1||l_2| l_2} \,\mathrm{Div}\, l_1. \end{aligned}$$

Of course, if the anchor map is an isomorphism, then this structure is the same as a projective volume form on A. In the more general case, we should think of an L-projective volume form as giving a projective volume form on the leaves of the derived foliation.

10.4.3. Let us explain how this definition relates to the notion of quantization of P_0 algebras.

10.4.3.1 Definition. *Give the operad P_0 a $\mathbb{C}^\times$ action where the product has weight 0 and the Poisson bracket has weight 1. A graded P_0 algebra is a $\mathbb{C}^\times$-equivariant differential graded algebra over this dg operad.*

Note that, if X is a manifold, $\mathscr{O}(T^*[-1]X)$ has the structure of graded P_0 algebra, where the $\mathbb{C}^\times$ action on $\mathscr{O}(T^*[-1]X)$ is given by rescaling the cotangent fibers.

Similarly, if L is a Lie algebroid over a commutative dg algebra A, then $\mathrm{Sym}^*_A L[1]$ is a $\mathbb{C}^\times$-equivariant P_0 algebra. The P_0 bracket is defined by the bracket on L and the L-action on A; the $\mathbb{C}^\times$ action gives $\mathrm{Sym}^k L[1]$ weight $-k$.

10.4.3.2 Definition. *Give the operad BD over $\mathbb{C}[[\hbar]]$ a $\mathbb{C}^\times$ action, covering the $\mathbb{C}^\times$ action on $\mathbb{C}[[\hbar]]$, where $\hbar$ has weight -1, the product has weight 0, and the Poisson bracket has weight 1.*

Note that this $\mathbb{C}^\times$ action respects the differential on the operad BD, which is defined on generators by

$$d(- * -) = \hbar\{-,-\}.$$

Note also that by describing the operad BD as a $\mathbb{C}^\times$-equivariant family of operads over $\mathbb{A}^1$, we have presented BD as a filtered operad whose associated graded operad is P_0.

10.4.3.3 Definition. *A filtered BD algebra is a BD algebra A with a $\mathbb{C}^\times$ action compatible with the $\mathbb{C}^\times$ action on the ground ring $\mathbb{C}[[\hbar]]$, where $\hbar$ has weight -1, and compatible with the $\mathbb{C}^\times$ action on BD.*

10.4.3.4 Lemma. *If L is Lie algebroid over a dg commutative algebra A, then every L-projective volume form yields a filtered BD algebra structure on $\mathrm{Sym}^*_A(L[1])[[\hbar]]$, quantizing the graded P_0 algebra $\mathrm{Sym}^*_A(L[1])$.*

PROOF. If $\mathrm{Div} : L \to A$ is an L-projective volume form, then it extends uniquely to an order two differential operator $\triangle$ on $\mathrm{Sym}^*_A(L[1])$ which maps

$$\mathrm{Sym}^i_A(L[1]) \to \mathrm{Sym}^{i-1}_A(L[1]).$$

Then $\mathrm{Sym}^*_A L[1][[\hbar]]$, with differential $d + \hbar\triangle$, gives the desired filtered BD algebra.

□

10.4.4. Let $B\mathcal{L}$ be an elliptic moduli problem on a compact manifold M. The main result of this section is that there exists a special kind of quantization of the cotangent field theory for $B\mathcal{L}$ that gives a projective volume on this formal moduli problem $B\mathcal{L}$. Projective volume forms arising in this way have a special "locality" property, reflecting the locality appearing in our definition of a field theory.

Thus, let $\mathcal{L}$ be an elliptic L_∞ algebra on M. This gives rise to a classical field theory whose space of fields is $\mathscr{E} = \mathcal{L}[1] \oplus \mathcal{L}^![-2]$, as described in Chapter **??**, section 5.6. Let us give the space $\mathscr{E}$ a $\mathbb{C}^\times$-action where $\mathcal{L}[1]$ has weight 0 and $\mathcal{L}^![-1]$ has weight 1. This induces a $\mathbb{C}^\times$ action on all associated spaces, such as $\mathscr{O}(\mathscr{E})$ and $\mathscr{O}_{loc}(\mathscr{E})$.

This $\mathbb{C}^\times$ action preserves the differential $Q + \{I,-\}$ on $\mathscr{O}(\mathscr{E})$, as well as the commutative product. Recall (Chapter **??**, section 6.2) that the subspace

$$\widetilde{\mathrm{Obs}}^{cl}(M) = \mathscr{O}_{sm}(\mathscr{E}) \subset \mathscr{O}(\mathscr{E})$$

of functionals with smooth first derivative has a Poisson bracket of cohomological degree 1, making it into a P_0 algebra. This Poisson bracket is of weight 1 with respect to the $\mathbb{C}^\times$ action on $\widetilde{\mathrm{Obs}}^{cl}(M)$, so $\widetilde{\mathrm{Obs}}^{cl}(M)$ is a graded P_0 algebra.

We are interested in quantizations of our field theory where the BD algebra $\mathrm{Obs}^q_\Phi(M)$ of (global) quantum observables (defined using a parametrix Φ) is a filtered BD algebra.

10.4.4.1 Definition. *A cotangent quantization of a cotangent theory is a quantization, given by effective interaction functionals $I[\Phi] \in \mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]]$ for each parametrix Φ, such that $I[\Phi]$ is of weight -1 under the $\mathbb{C}^\times$ action on the space $\mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]]$ of functionals.*

This $\mathbb{C}^\times$ action gives $\hbar$ weight -1. Thus, this condition means that if we expand

$$I[\Phi] = \sum \hbar^i I_i[\Phi],$$

then the functionals $I_i[\Phi]$ are of weight $i-1$.

Since the fields $\mathscr{E} = \mathcal{L}[1] \oplus \mathcal{L}^![-2]$ decompose into spaces of weights 0 and 1 under the $\mathbb{C}^\times$ action, we see that $I_0[\Phi]$ is linear as a function of $\mathcal{L}^![-2]$, that $I_1[\Phi]$ is a function only of $\mathcal{L}[1]$, and that $I_i[\Phi] = 0$ for $i > 1$.

Remark:
(1) The quantization $\{I[\Phi]\}$ is a cotangent quantization if and only if the differential $Q + \{I[\Phi], -\}_\Phi + \hbar \triangle_\Phi$ preserves the $\mathbb{C}^\times$ action on the space $\mathscr{O}(\mathscr{E})[[\hbar]]$ of functionals. Thus, $\{I[\Phi]\}$ is a cotangent quantization if and only if the BD algebra $\mathrm{Obs}^q_\Phi(M)$ is a filtered BD algebra for each parametrix Φ.
(2) The condition that $I_0[\Phi]$ is of weight -1 is automatic.
(3) It is easy to see that the renormalization group flow

$$W\left(P(\Phi) - P(\Psi), -\right)$$

commutes with the $\mathbb{C}^\times$ action on the space $\mathscr{O}^+_{sm,P}(\mathscr{E})[[\hbar]]$.

$\Diamond$

10.4.5. Let us now explain the volume-form interpretation of cotangent quantization. Let $\mathcal{L}$ be an elliptic L_∞ algebra on M, and let $\mathscr{O}(B\mathcal{L}) = C^*(\mathcal{L})$ be the Chevalley-Eilenberg cochain complex of M. The cochain complexes $\mathscr{O}(B\mathcal{L}(U))$ for open subsets $U \subset M$ define a commutative factorization algebra on M.

As we have seen in Chapter **??**, section **??**, we should interpret modules for an L_∞ algebra $\mathfrak{g}$ as sheaves on the formal moduli problem $B\mathfrak{g}$. The $\mathfrak{g}$-module $\mathfrak{g}[1]$ corresponds to the tangent bundle of $B\mathfrak{g}$, and so vector fields on $\mathfrak{g}$ correspond to the $\mathscr{O}(B\mathfrak{g})$-module $C^*(\mathfrak{g}, \mathfrak{g}[1])$.

Thus, we use the notation

$$\mathrm{Vect}(B\mathcal{L}) = C^*(\mathcal{L}, \mathcal{L}[1]);$$

this is a dg Lie algebra and acts on $C^*(\mathcal{L})$ by derivations (see Appendix **??**, section B.2, for details).

For any open subset $U \subset M$, the $\mathcal{L}(U)$-module $\mathcal{L}(U)[1]$ has a sub-module $\mathcal{L}_c(U)[1]$ given by compactly supported elements of $\mathcal{L}(U)[1]$. Thus, we have a sub-$\mathscr{O}(B\mathcal{L}(U))$-module

$$\mathrm{Vect}_c(B\mathcal{L}(U)) = C^*(\mathcal{L}(U), \mathcal{L}_c(U)[1]) \subset \mathrm{Vect}(B\mathcal{L}(U)).$$

This is in fact a sub-dg Lie algebra, and hence a Lie algebroid over the dg commutative algebra $\mathscr{O}(B\mathcal{L}(U))$. Thus, we should view the subspace $\mathcal{L}_c(U)[1] \subset \mathcal{L}_c(U)[1]$ as providing a foliation of the formal moduli problem $B\mathcal{L}(U)$, where two points of $B\mathcal{L}(U)$ are in the same leaf if they coincide outside a compact subset of U.

If $U \subset V$ are open subsets of M, there is a restriction map of L_∞ algebras $\mathcal{L}(V) \to \mathcal{L}(U)$. The natural extension map $\mathcal{L}_c(U)[1] \to \mathcal{L}_c(V)[1]$ is a map of $\mathcal{L}(V)$-modules. Thus, by taking cochains, we find a map

$$\mathrm{Vect}_c(B\mathcal{L}(U)) \to \mathrm{Vect}_c(B\mathcal{L}(V)).$$

Geometrically, we should think of this map as follows. If we have an R-point α of $B\mathcal{L}(V)$ for some dg Artinian ring R, then any compactly-supported deformation of the restriction $\alpha \mid_U$ of α to U extends to a compactly supported deformation of α.

We want to say that a cotangent quantization of $\mathcal{L}$ leads to a "local" projective volume form on the formal moduli problem $B\mathcal{L}(M)$ if M is compact. If M is compact, then $\mathrm{Vect}_c(B\mathcal{L}(M))$ coincides with $\mathrm{Vect}(B\mathcal{L}(M))$. A local projective volume form on $B\mathcal{L}(M)$ should be something like a divergence operator

$$\mathrm{Div} : \mathrm{Vect}(B\mathcal{L}(M)) \to \mathscr{O}(B\mathcal{L}(M))$$

satisfying the equations (†), with the locality property that Div maps the subspace

$$\mathrm{Vect}_c(B\mathcal{L}(U)) \subset \mathrm{Vect}(B\mathcal{L}(M))$$

to the subspace $\mathscr{O}(B\mathcal{L}(U)) \subset \mathscr{O}(B\mathcal{L}(M))$.

Note that a projective volume form for the Lie algebroid $\mathrm{Vect}_c(B\mathcal{L}(U))$ over $\mathscr{O}(B\mathcal{L}(U))$ is a projective volume form on the leaves of the foliation of $B\mathcal{L}(U)$ given by $\mathrm{Vect}_c(B\mathcal{L}(U))$. The leaf space for this foliation is described by the L_∞ algebra

$$\mathcal{L}_\infty(U) = \mathcal{L}(U)/\mathcal{L}_c(U) = \operatorname*{colim}_{K \subset U} \mathcal{L}(U \setminus K).$$

(Here the colimit is taken over all compact subsets of U.) Consider the one-point compactification U_∞ of U. Then the formal moduli problem $\mathcal{L}_\infty(U)$ describes the germs at ∞ on U_∞ of sections of the sheaf on U of formal moduli problems given by $\mathcal{L}$.

Thus, the structure we're looking for is a projective volume form on the fibers of the maps $B\mathcal{L}(U) \to B\mathcal{L}_\infty(U)$ for every open subset $U \subset M$, where the divergence operators describing these projective volume forms are all compatible in the sense described above.

What we actually find is something a little weaker. To state the result, recall (section 9.2) that we use the notation $\mathscr{P}$ for the contractible simplicial set of parametrices, and $\mathscr{C}\mathscr{P}$ for the cone on $\mathscr{P}$. The vertex of the cone $\mathscr{C}\mathscr{P}$ will denoted $\overline{0}$.

10.4.5.1 Theorem. *A cotangent quantization of the cotangent theory associated to the elliptic L_∞ algebra $\mathcal{L}$ leads to the following data.*

(1) *A commutative dg algebra $\mathscr{O}_{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ over $\Omega^*(\mathscr{C}\mathscr{P})$. The underlying graded algebra of this commutative dg algebra is $\mathscr{O}(B\mathcal{L}) \otimes \Omega^*(\mathscr{C}\mathscr{P})$. The restriction of this commutative dg algebra to the vertex $\overline{0}$ of $\mathscr{C}\mathscr{P}$ is the commutative dg algebra $\mathscr{O}(B\mathcal{L})$.*
(2) *A dg Lie algebroid $\mathrm{Vect}_c^{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ over $\mathscr{O}_{\mathscr{C}\mathscr{P}}(B\mathcal{L})$, whose underlying graded $\mathscr{O}_{\mathscr{C}\mathscr{P}}(B\mathcal{L})$-module is $\mathrm{Vect}_c(B\mathcal{L}) \otimes \Omega^*(\mathscr{C}\mathscr{P})$. At the vertex $\overline{0}$ of $\mathscr{C}\mathscr{P}$, the dg Lie algebroid $\mathrm{Vect}_c^{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ coincides with the dg Lie algebroid $\mathrm{Vect}_c(B\mathcal{L})$.*
(3) *We let $\mathscr{O}_{\mathscr{P}}(B\mathcal{L})$ and $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$ be the restrictions of $\mathscr{O}_{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ and $\mathrm{Vect}_c^{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ to $\mathscr{P} \subset \mathscr{C}\mathscr{P}$. Then we have a divergence operator*
$$\mathrm{Div}_{\mathscr{P}} : \mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L}) \to \mathscr{O}_{\mathscr{P}}(B\mathcal{L})$$
defining the structure of a $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$ projective volume form on $\mathscr{O}_{\mathscr{P}}(B\mathcal{L})$ and $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$.

Further, when restricted to the sub-simplicial set $\mathscr{P}_U \subset \mathscr{P}$ of parametrices with support in a small neighborhood of the diagonal $U \subset M \times M$, all structures increase support by an arbitrarily small amount (more precisely, by an amount linear in U, in the sense explained in section 9.2).

PROOF. This follows almost immediately from theorem 9.2.2.1. Indeed, because we have a cotangent theory, we have a filtered BD algebra
$$\mathrm{Obs}^q_{\mathscr{P}}(M) = \left(\mathscr{O}(\mathscr{E})[[\hbar]] \otimes \Omega^*(\mathscr{P}), \widehat{Q}_{\mathscr{P}}, \{-,-\}_{\mathscr{P}}\right).$$
Let us consider the sub-BD algebra $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)$, which, as a graded vector space, is $\mathscr{O}_{sm}(\mathscr{E})[[\hbar]] \otimes \Omega^*(\mathscr{P})$ (as usual, $\mathscr{O}_{sm}(\mathscr{E})$ indicates the space of functionals with smooth first derivative).

Because we have a filtered BD algebra, there is a $\mathbb{C}^\times$-action on this complex $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)$. We let
$$\mathscr{O}_{\mathscr{P}}(B\mathcal{L}) = \widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^0$$
be the weight 0 subspace. This is a commutative differential graded algebra over $\Omega^*(\mathscr{P})$, whose underlying graded algebra is $\mathscr{O}(B\mathcal{L})$; further, it extends (using again the results of 9.2.2.1) to a commutative dg algebra $\mathscr{O}_{\mathscr{C}\mathscr{P}}(B\mathcal{L})$ over $\Omega^*(\mathscr{C}\mathscr{P})$, which when restricted to the vertex is $\mathscr{O}(B\mathcal{L})$.

Next, consider the weight -1 subspace. As a graded vector space, this is
$$\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1} = \mathrm{Vect}_c(B\mathcal{L}) \otimes \Omega^*(\mathscr{P}) \oplus \hbar\mathscr{O}_{\mathscr{P}}(B\mathcal{L}).$$

We thus let
$$\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L}) = \widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1}/\hbar\mathscr{O}_{\mathscr{P}}(B\mathcal{L}).$$
The Poisson bracket on $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)$ is of weight 1, and it makes the space $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1}$ into a sub Lie algebra.

We have a natural decomposition of graded vector spaces
$$\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1} = \mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L}) \oplus \hbar\mathscr{O}_{\mathscr{P}}(B\mathcal{L}).$$
The dg Lie algebra structure on $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1}$ gives us

(1) The structure of a dg Lie algebra on $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$ (as the quotient of $\widetilde{\mathrm{Obs}}^q_{\mathscr{P}}(M)^{-1}$ by the differential Lie algebra ideal $\hbar\mathscr{O}_{\mathscr{P}}(B\mathcal{L})$).
(2) An action of $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$ on $\mathscr{O}_{\mathscr{P}}(B\mathcal{L})$ by derivations; this defines the anchor map for the Lie algebroid structure on $\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L})$.
(3) A cochain map
$$\mathrm{Vect}_c^{\mathscr{P}}(B\mathcal{L}) \to \hbar\mathscr{O}_{\mathscr{P}}(B\mathcal{L}).$$
This defines the divergence operator.

It is easy to verify from the construction of theorem 9.2.2.1 that all the desired properties hold. □

10.4.6. The general results about quantization of [**Cos11b**] thus apply to this situation, to show that the following.

10.4.6.1 Theorem. *Consider the cotangent theory $\mathscr{E} = \mathcal{L}[1] \oplus \mathcal{L}^![-2]$ to an elliptic moduli problem described by an elliptic L_∞ algebra $\mathcal{L}$ on a manifold M.*

The obstruction to constructing a cotangent quantization is an element in
$$H^1(\mathscr{O}_{loc}(\mathscr{E})^{\mathbb{C}^\times}) = H^1(\mathscr{O}_{loc}(B\mathcal{L})).$$
If this obstruction vanishes, then the simplicial set of cotangent quantizations is a torsor for the simplicial Abelian group arising from the cochain complex $\mathscr{O}_{loc}(B\mathcal{L})$ by the Dold-Kan correspondence.

As in Chapter **??**, section 4.5, we are using the notation $\mathscr{O}_{loc}(B\mathcal{L})$ to refer to a "local" Chevalley-Eilenberg cochain for the elliptic L_∞ algebra $\mathcal{L}$. If L is the vector bundle whose sections are $\mathcal{L}$, then as we explained in [**Cos11b**], the jet bundle $J(L)$ is a D_M L_∞ algebra and
$$\mathscr{O}_{loc}(B\mathcal{L}) = \mathrm{Dens}_M \otimes_{D_M} C^*_{red}(J(L)).$$
There is a de Rham differential (see section 5.3) mapping $\mathscr{O}_{loc}(B\mathcal{L})$ to the complex of local 1-forms,
$$\Omega^1_{loc}(B\mathcal{L}) = C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1]).$$

The de Rham differential maps $\mathscr{O}_{loc}(B\mathcal{L})$ isomorphically to the subcomplex of $\Omega^1_{loc}B\mathcal{L})$ of closed local one-forms. Thus, the obstruction is a local closed 1-form on $B\mathcal{L}$ of cohomology degree 1: it is in

$$H^1(\Omega^1_{loc}(B\mathcal{L}).$$

Since the obstruction to quantizing the theory is the obstruction to finding a locally-defined volume form on $B\mathcal{L}$, we should view this obstruction as being the local first Chern class of $B\mathcal{L}$.

10.5. Correlation functions

So far in this chapter, we have proved the quantization theorem showing that from a field theory we can construct a factorization algebra. We like to think that this factorization algebra encodes most things one would want to with a quantum field theory in perturbation theory. To illustrate this, in this section, we will explain how to construct correlation frunctions form the factorization algebra, under certain additional hypothesis.

Suppose we have a field theory on a compact manifold M, with space of fields $\mathscr{E}$ and linearized differential Q on the space of fields. Let us suppose that

$$H^*(\mathscr{E}(M), Q) = 0.$$

This means the following: the complex $(\mathscr{E}(M), Q)$ is tangent complex to the formal moduli space of solutions to the equation of motion to our field theory, at the base point around which we are doing perturbation theory. The statement that this tangent complex has no cohomology means that there the trivial solution of the equation of motion has no deformations (up to whatever gauge symmetry we have). In other words, we are working with an isolated point in the moduli of solutions to the equations of motion.

As an example, consider a massive interacting scalar field theory on a compact manifold M, with action functional for example

$$\int_M \phi(\mathrm{D} + m^2)\phi + \phi^4$$

where $\phi \in C^\infty(M)$ and $m > 0$. Then, the complex $\mathscr{E}(M)$ of fields is the complex

$$C^\infty(M) \xrightarrow{\mathrm{D} + m^2} C^\infty(M).$$

Hodge theory tells us that this complex has no cohomology.

Let Obs^q denote the factorization algebra of quantum observables of a quantum field theory which satisfies this (classical) condition.

10.5.0.1 Lemma. *In this situation, there is a canonical isomorphism*

$$H^*(\mathrm{Obs}^q(M)) = \mathbb{C}[[\hbar]].$$

(Note that we usually work, for simplicity, with complex vector spaces; this result holds where everything is real too, in which case we find $\mathbb{R}[[\hbar]]$ on the right hand side).

PROOF. There's a spectral sequence

$$H^*(\mathrm{Obs}^{cl}(M))[[\hbar]] \twoheadrightarrow H^*(\mathrm{Obs}^q(M)).$$

Further, $\mathrm{Obs}^{cl}(M)$ has a complete decreasing filtration whose associated graded is the complex

$$\mathrm{Gr}\,\mathrm{Obs}^{cl}(M) = \prod_n \mathrm{Sym}^n(\mathscr{E}(M)^\vee)$$

with differential arising from the linear differential Q on $\mathscr{E}(M)$. The condition that $H^*(\mathscr{E}(M), Q) = 0$ implies that the cohomology of $\mathrm{Sym}^n(\mathscr{E}(M)^\vee)$ is also zero, so that $H^*(\mathrm{Obs}^{cl}(M)) = \mathbb{C}$. This shows that there is an isomorphism of $\mathbb{C}[[\hbar]]$-modules from $H^*(\mathrm{Obs}^q(M))$ to $\mathbb{C}[[\hbar]]$. To make this isomorphism canonical, we declare that the vacuum observable $|0\rangle \in H^0(\mathrm{Obs}^q(M))$ (that is, the unit in the factorization algebra) gets sent to $1 \in \mathbb{C}[[\hbar]]$. □

10.5.0.2 Definition. *As above, let* Obs^q *denote the factorization algebra of obsevables of a QFT on M which satisfies $H^*(\mathscr{E}(M), Q) = 0$.*

Let $U_1, \dots, U_n \subset M$ be disjoint open sets, and let $O_i \in \mathrm{Obs}^q(U_i)$. Define the expectation value *(or correlation function) of the obsevables O_i, denoted by*

$$\langle O_1, \dots, O_n \rangle \in \mathbb{C}[[\hbar]],$$

to be the image of the product observable

$$O_1 * \cdots * O_n \in H^*(\mathrm{Obs}^q(M))$$

under the canonical isomorphism between $H^*(\mathrm{Obs}^q(M))$ and $\mathbb{C}[[\hbar]]$.

We have already encountered this definition when we discussed free theories (see definition **??** in Chapter **??**). There we saw that this definition reproduced usual physics definitions of correlation functions for free field theories.

Part 3

A factorization enhancement of Noether's theorem

CHAPTER 11

Noether's theorem in classical field theory

Noether's theorem is a central result in field theory, which states that there is a bijection between symmetries of a field theory and conserved currents. In this chapter we will develop a very general version of Noether's theorem for classical field theories in the language of factorization algebras. In the following chapter, we will develop the analogous theorem for quantum field theories.

The statement for classical field theories is the following. Suppose we have a classical field theory on a manifold M, and let $\widetilde{\mathrm{Obs}}^{cl}$ denote the P_0 factorization algebra of observables of the theory. Suppose that $\mathcal{L}$ is a local L_∞ algebra on M which acts on our classical field theory (we will define precisely what we mean by an action shortly). Let $\mathcal{L}_c$ denote the precosheaf of L_∞ algebras on M given by compactly supported section of $\mathcal{L}$. Note that the P_0 structure on $\widetilde{\mathrm{Obs}}^{cl}$ means that $\widetilde{\mathrm{Obs}}^{cl}[-1]$ is a precosheaf of dg Lie algebras.

The formulation of Noether's theorem we will prove involves shifted central extensions of hte cosheaf $\mathcal{L}_c$ of L_∞ algebras on M. Such central extensions were discussed insection **??**; we are interested in -1-shifted central extensions, which fit into short exact sequences

$$0 \to \underline{\mathbb{C}}[-1] \to \widetilde{\mathcal{L}}_c \to \mathcal{L}_c \to 0,$$

where $\underline{\mathbb{C}}$ is the constant precosheaf.

The theorem is the following.

Theorem. *Suppose that a local L_∞ algebra $\mathcal{L}$ acts on a classical field theory with observables Obs^{cl}. Then, there is a -1-shifted central extension $\widetilde{\mathcal{L}}_c$ of the precosheaf $\mathcal{L}_c$ of L_∞ algebras on M, and a map of precosheaves of L_∞ algebras*

$$\widetilde{\mathcal{L}}_c \to \widetilde{\mathrm{Obs}}^{cl}[-1]$$

which, for every open subset U, sends the central element of $\widetilde{\mathcal{L}}_c$ to the observable $1 \in \mathrm{Obs}^{cl}(U)[-1]$.

This map is not arbitrary. Rather, it is compatible with the action of the cosheaf $\mathcal{L}_c$ on Obs^{cl} arising from the action $\mathcal{L}_c$ on the field theory. Let us explain the form this compatibility takes.

Note that the dg Lie algebra $\widetilde{\mathrm{Obs}}^{cl}(U)[-1]$ acts on $\mathrm{Obs}^{cl}(U)$ by the Poisson bracket, in such a way that the subspace spanned by the observable 1 acts by zero. The L_∞ map we just discussed therefore gives an action of $\widetilde{\mathcal{L}}_c(U)$ on $\mathrm{Obs}^{cl}(U)$, which descends to an action of $\mathcal{L}_c(U)$ because the central element acts by zero.

Theorem. *In this situation, the action of $\mathcal{L}_c(U)$ coming from the L_∞-map $\widetilde{\mathcal{L}}_c \to \widetilde{\mathrm{Obs}}^{cl}$ and the action coming from the action of $\mathcal{L}$ on the classical field theory coincide up to a homotopy.*

. Let us relate this formulation of Noether's theorem to familiar statements in classical field theory. Suppose we have a symplectic manifold X with an action of a Lie algebra $\mathfrak{g}$ by symplectic vector fields. Let us work locally on X, so that we can assume $H^1(X) = 0$. Then, there is a short exact sequence of Lie algebras

$$0 \to \mathbb{R} \to C^\infty(X) \to \mathrm{SympVect}(X) \to 0$$

where $\mathrm{SympVect}(X)$ is the Lie algebra of symplectic vector fields on X, and $C^\infty(X)$ is a Lie algebra under the Poisson bracket.

We can pull back this central extension under the Lie algebra homomorphism $\mathfrak{g} \to \mathrm{SympVect}(X)$ to obtain a central extension $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$. This is the analog of the central extension $\widetilde{\mathcal{L}}_c$ that appeared in our formulation of Noether's theorem.

The Poisson algebra $C^\infty(X)$ is observables of the classical field theory. The map $\widetilde{\mathfrak{g}} \to C^\infty(X)$ sends the central element of $\widetilde{\mathfrak{g}}$ to $1 \in C^\infty(X)$. Further, the action of $\mathfrak{g}$ on $C^\infty(X)$ arising from the homomorphism $\mathfrak{g} \to C^\infty(X)$ coincides with the one arising from the original homomorphism $\mathfrak{g} \to \mathrm{SympVect}(X)$.

Thus, the map $\mathfrak{g} \to C^\infty(X)$ is entirely analogous to the map that appears in our formulation of classical Noether's theorem. Indeed, we defined a field theory to be a sheaf of formal moduli problems with a -1-shifted symplectic form. The P_0 Poisson bracket on the observables of a classical field theory is analogous to the Poisson bracket on observables in classical mechanics. Our formulation of Noether's theorem can be rephrased as saying that, after passing to a central extension, an action of a sheaf of Lie algebras by symplectic symmetries on a sheaf of formal moduli probblems is Hamiltonian.

This similarity is more than just an analogy. After some non-trivial work, one can show that our formulation of Noether's theorem, when applied to classical mechanics, yields the statement discussed above about actions of a Lie algebra on a symplectic manifold. The key result one needs in order to translate is a result of Nick Rozenblyum []. Observables of classical mechanics form a locally-constant P_0 factorization algebra on the real line, and so (by a theorem of Lurie discussed in section **??**) an E_1 algebra in P_0 algebras. Rozenblyum shows E_1 algebras in P_0 algebras are the same as P_1, that is ordinary Poisson, algebras. This allows us to translate the shifted Poisson bracket on the factorization algebra on $\mathbb{R}$ of observable of classical mechanics into the ordinary unshifted Poisson bracket that is more familiar in classical mechanics, and to translate our formulation of Noether's theorem into the statement about Lie algebra actions on symplectic manifolds discussed above.

11.0.1.

11.0.1.1 Theorem. *Suppose we have a quantum field theory on a manifold M, which is acted on by a local dg Lie (or L_∞) algebra on M. Then, there is a map of factorization algebras on M from the twisted factorization envelope (**??**) of $\mathcal{L}$ to observables of the field theory.*

Of course, this is not an arbitrary map; rather, the action of $\mathcal{L}$ on observables can be recovered from this map together with the factorization product.

This theorem may seem quite different from Noether's theorem as it is usually stated. We explain the link between this result and the standard formulation in section **??**.

For us, the power of this result is that it gives us a very general method for understanding quantum observables. The factorization envelope of a local L_∞ algebra is a very explicit and easily-understood object. By contrast, the factorization algebra of quantum observables of an interacting field theory is a complicated object which resists explicit description. Our formulation of Noether's theorem shows us that, if we have a field theory which has many symmetries, we can understand explicitly a large part of the factorization algebra of quantum observables.

11.1. Symmetries of a classical field theory

We will start our discussion of Noether's theorem by examining what it means for a homotopy Lie algebra to act on a field theory. We are particularly interested in what it means for a *local L_∞ algebra* to act on a classical field theory. Recall **??** that a local L_∞ algebra $\mathcal{L}$ is a sheaf of L_∞ algebras which is the sheaf of sections of a graded vector bundle L, and where the L_∞-structure maps are poly-differential operators.

We know from chapter **??** that a perturbative classical field theory is described by an elliptic moduli problem on M with a degree -1 symplectic form. Equivalently, it is described by a local L_∞ algebra $\mathcal{M}$ on M equipped with an invariant pairing of degree -3. Therefore, an action of $\mathcal{L}$ on $\mathcal{M}$ should be an L_∞ action of $\mathcal{L}$ on $\mathcal{M}$. Thus, the first thing we need to understand is what it means for one L_∞ algebra to act on the other.

11.1.1. Actions of L_∞ algebras. If $\mathfrak{g}, \mathfrak{h}$ are ordinary Lie algebras, then it is straightforward to say what it means for $\mathfrak{g}$ to act on $\mathfrak{h}$. If $\mathfrak{g}$ does act on $\mathfrak{h}$, then we can define the semi-direct product $\mathfrak{g} \ltimes \mathfrak{h}$. This semi-direct product lives in a short exact sequence of Lie algebras

$$0 \to \mathfrak{h} \to \mathfrak{g} \ltimes \mathfrak{h} \to \mathfrak{g} \to 0.$$

Further, we can recover the action of $\mathfrak{g}$ on $\mathfrak{h}$ from the data of a short exact sequence of Lie algebras like this.

We will take this as a model for the action of one L_∞ algebra $\mathfrak{g}$ on another L_∞ algebra $\mathfrak{h}$.

11.1.1.1 Definition. *An action of an L_∞ algebra $\mathfrak{g}$ on an L_∞ algebra $\mathfrak{h}$ is, by definition, an L_∞-algebra structure on $\mathfrak{g} \oplus \mathfrak{h}$ with the property that the (linear) maps in the exact sequence*

$$0 \to \mathfrak{h} \to \mathfrak{g} \oplus \mathfrak{h} \to \mathfrak{g} \to 0$$

are maps of L_∞ algebras.

Remark:

(1) The set of actions of $\mathfrak{g}$ on $\mathfrak{h}$ enriches to a simplicial set, whose n-simplices are families of actions over the dg algebra $\Omega^*(\triangle^n)$.
(2) There are other possible notions of action of $\mathfrak{g}$ on $\mathfrak{h}$ which might seem more natural to some readers. For instance, an abstract notion is to say that an action of $\mathfrak{g}$ on $\mathfrak{h}$ is an L_∞ algebra $\widetilde{\mathfrak{h}}$ with a map $\phi : \widetilde{\mathfrak{h}} \to \mathfrak{g}$ and an isomorphism of L_∞ algebras between the homotopy fibre $\phi^{-1}(0)$ and $\mathfrak{h}$. One can show that this more fancy definition is equivalent to the concrete one proposed above, in the sense that the two ∞-groupoids of possible actions are equivalent.

If $\mathfrak{h}$ is finite dimensional, then we can identify the dg Lie algebra of derivations of $C^*(\mathfrak{h})$ with $C^*(\mathfrak{h}, \mathfrak{h}[1])$ with a certain dg Lie bracket. We can thus view $C^*(\mathfrak{h}, \mathfrak{h}[1])$ as the dg Lie algebra of vector fields on the formal moduli problem $B\mathfrak{h}$.

11.1.1.2 Lemma. *Actions of $\mathfrak{g}$ on $\mathfrak{h}$ are the same as L_∞-algebra maps $\mathfrak{g} \to C^*(\mathfrak{h}, \mathfrak{h}[1])$.*

PROOF. This is straightforward. □

This lemma shows that an action of $\mathfrak{g}$ on $\mathfrak{h}$ is the same as an action of $\mathfrak{g}$ on the formal moduli problem $B\mathfrak{h}$ which *may not* preserve the base-point of $B\mathfrak{h}$.

11.1.2. Actions of local L_∞ algebras.

Now let us return to the setting of local L_∞ algebras, and define what it means for one local L_∞ algebra to act on another.

11.1.2.1 Definition. *Let $\mathcal{L}$, $\mathcal{M}$ be local L_∞ algebras on M. Then an $\mathcal{L}$ action on $\mathcal{M}$ is given by a local L_∞ structure on $\mathcal{L} \oplus \mathcal{M}$, such that the exact sequence*

$$0 \to \mathcal{M} \to \mathcal{L} \oplus \mathcal{M} \to \mathcal{L} \to 0$$

is a sequence of L_∞ algebras.

More explicitly, this says that $\mathcal{M}$ (with its original L_∞ structure) is a sub-L_∞-algebra of $\mathcal{M} \oplus \mathcal{L}$, and is also an L_∞-ideal: all operations which take as input at least one element of $\mathcal{M}$ land in $\mathcal{M}$. We will refer to the L_∞ algebra $\mathcal{L} \oplus \mathcal{M}$ with the L_∞ structure defining the action as $\mathcal{L} \ltimes \mathcal{M}$.

11.1.2.2 Definition. *Suppose that $\mathcal{M}$ has an invariant pairing. An action of $\mathcal{L}$ on $\mathcal{M}$ preserves the pairing if, for local compactly supported sections α_i, β_j of $\mathcal{L}$ and $\mathcal{M}$ the tensor*

$$\langle l_{r+s}(\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s), \beta_{s+1} \rangle$$

is totally symmetric if $s+1$ is even (or antisymmetric if $s+1$ is odd) under permutation of the β_i.

11.1.2.3 Definition. *An action of a local L_∞ algebra $\mathcal{L}$ on a classical field theory defined by a local L_∞ algebra $\mathcal{M}$ with an invariant pairing of degree -3 is, as above, an L_∞ action of $\mathcal{L}$ on $\mathcal{M}$ which preserves the pairing.*

As an example, we have the following.

11.1.2.4 Lemma. *Suppose that $\mathcal{L}$ acts on an elliptic L_∞ algebra $\mathcal{M}$. Then $\mathcal{L}$ acts on the cotangent theory for $\mathcal{M}$.*

PROOF. This is immediate by naturality, but we can also write down explicitly the semi-direct product L_∞ algebra describing the action. Note that $\mathcal{L} \ltimes \mathcal{M}$ acts linearly on

$$(\mathcal{L} \ltimes \mathcal{M})^! [-3] = \mathcal{L}^! [-3] \oplus \mathcal{M}^! [-3].$$

Further, $\mathcal{L}^![-3]$ is a submodule for this action, so that we can form the quotient $\mathcal{M}^![-3]$. Then,

$$(\mathcal{L} \ltimes \mathcal{M}) \ltimes \mathcal{M}^! [-3]$$

is the desired semi-direct product. $\square$

Remark: Note that this construction is simply giving the -1-shifted relative cotangent bundle to the map

$$B(\mathcal{L} \ltimes \mathcal{M}) \to B\mathcal{L}.$$

The definition we gave above of an action of a local L_∞ algebra on a classical field theory is a little abstract. We can make it more concrete as follows.

Recall that the space of fields of the classical field theory associated to $\mathcal{M}$ is $\mathcal{M}[1]$, and that the L_∞ structure on $\mathcal{M}$ is entirely encoded in the action functional

$$S \in \mathscr{O}_{loc}(\mathcal{M}[1])$$

which satisfies the classical master equation $\{S, S\} = 0$. (The notation $\mathscr{O}_{loc}$ always indicates local functionals modulo constants).

An action of a local L_∞ algebra $\mathcal{L}$ on $\mathcal{M}$ can also be encoded in a certain local functional, which depends on $\mathcal{L}$. We need to describe the precise space of functionals that arise in this interpretation.

If X denotes the space-time manifold on which $\mathcal{L}$ and $\mathcal{M}$ are sheaves, then $\mathcal{L}(X)$ is an L_∞ algebra. Thus, we can form the Chevalley-Eilenberg cochain complex

$$C^*(\mathcal{L}(X)) = (\mathscr{O}(\mathcal{L}(X)[1]), \mathrm{d}_{\mathcal{L}})$$

as well as it's reduced version $C^*_{red}(\mathcal{L}(X))$.

We can form the completed tensor product of this dg algebra with the shifted Lie algebra $\mathscr{O}_{loc}(\mathcal{M}[1])$, to form a new shifted dg Lie algebra $C^*_{red}(\mathcal{L}(X)) \otimes \mathscr{O}_{loc}(\mathcal{M}[1])$.

Inside this is the subspace

$$\mathscr{O}_{loc}(\mathcal{L}[1] \oplus \mathcal{M}[1]) / \left(\mathscr{O}_{loc}(\mathcal{L}[1]) \oplus \mathscr{O}_{loc}(\mathcal{M}[1])\right) \subset C^*_{red}(\mathcal{L}(X)) \otimes \mathscr{O}_{loc}(\mathcal{M}[1])$$

of functionals which are local as a function of $\mathcal{L}[1]$. Note that we are working with functionals which must depend on *both* $\mathcal{L}[1]$ and $\mathcal{M}[1]$: we discard those functionals which depend only on one or the other.

One can check that this graded subspace is preserved both by the Lie bracket $\{-,-\}$, the differential $\mathrm{d}_{\mathcal{L}}$ and the differential $\mathrm{d}_{\mathcal{M}}$ (coming from the L_∞ structure on $\mathcal{L}$ and $\mathcal{M}$). This space thus becomes a shifted dg Lie algebra, with the differential $\mathrm{d}_{\mathcal{L}} \oplus \mathrm{d}_{\mathcal{M}}$ and with the degree $+1$ bracket $\{-,-\}$.

11.1.2.5 Lemma. *To give an action of a local L_∞ algebra $\mathcal{L}$ on a classical field theory corresponding to a local L_∞ algebra $\mathcal{M}$ with invariant pairing, is the same as to give an action functional*

$$S^{\mathcal{L}} \in \mathscr{O}_{loc}(\mathcal{L}[1] \oplus \mathcal{M}[1]) / \left(\mathscr{O}_{loc}(\mathcal{L}[1]) \oplus \mathscr{O}_{loc}(\mathcal{M}[1])\right)$$

which is of cohomological degree 0*, and satisfied the Maurer-Cartan equation*

$$(\mathrm{d}_{\mathcal{L}} + \mathrm{d}_{\mathcal{M}}) S^{\mathcal{L}} + \tfrac{1}{2}\{S^{\mathcal{L}}, S^{\mathcal{L}}\} = 0.$$

PROOF. Given such an $S^{\mathcal{L}}$, then

$$\mathrm{d}_{\mathcal{L}} + \mathrm{d}_{\mathcal{M}} + \{S^{\mathcal{L}}, -\}$$

defines a differential on $\mathscr{O}(\mathcal{L}(X)[1] \oplus \mathcal{M}(X)[1])$. The classical master equation implies that this differential is of square zero, so that it defines an L_∞ structure on $\mathcal{L}(X) \oplus \mathcal{M}(X)$. The locality condition on $S^{\mathcal{L}}$ guarantees that this is a local L_∞ algebra structure. A simple analysis shows that this L_∞ structure respects the exact sequence

$$0 \to \mathcal{M} \to \mathcal{L} \oplus \mathcal{M} \to \mathcal{L} \to 0$$

and the invariant pairing on $\mathcal{M}$. $\square$

This lemma suggests that we should look at a classical field theory with an action of $\mathcal{L}$ as a family of classical field theories over the sheaf of formal moduli problems $B\mathcal{L}$. Further justification for this idea will be offered in proposition **??**.

11.1.3. Let $\mathfrak{g}$ be an ordinary L_∞ algebra (not a sheaf of such), which we assume to be finite-dimensional for simplicity. Let $C^*(\mathfrak{g})$ be it's Chevalley-Eilenberg cochain algebra, viewed as a pro-nilpotent commutative dga. Suppose we have a classical field theory, represented as an elliptic L_∞ algebra $\mathcal{M}$ with an invariant pairing. Then we can define the notion of a $\mathfrak{g}$-action on $\mathcal{M}$ as follows.

11.1.3.1 Definition. *A $\mathfrak{g}$-action on $\mathcal{M}$ is any of the following equivalent data.*

(1) An L_∞ structure on $\mathfrak{g} \oplus \mathcal{M}(X)$ such that the exact sequence

$$0 \to \mathcal{M}(X) \to \mathfrak{g} \oplus \mathcal{M}(X) \to \mathfrak{g}$$

is a sequence of L_∞-algebras, and such that the structure maps

$$\mathfrak{g}^{\otimes n} \otimes \mathcal{M}(X)^{\otimes m} \to \mathcal{M}(X)$$

are poly-differential operators in the $\mathcal{M}$-variables.

(2) An L_∞-homomorphism

$$\mathfrak{g} \to \mathscr{O}_{loc}(B\mathcal{M})[-1]$$

(the shift is so that $\mathscr{O}_{loc}(B\mathcal{M})[-1]$ is an ordinary, and not shifted, dg Lie algebra).

(3) An element

$$S^{\mathfrak{g}} \in C^*_{red}(\mathfrak{g}) \otimes \mathscr{O}_{loc}(B\mathcal{M})$$

which satisfies the Maurer-Cartan

$$\mathrm{d}_{\mathfrak{g}} S^{\mathfrak{g}} \mathrm{d}_{\mathcal{M}} S^{\mathfrak{g}} + \tfrac{1}{2}\{S^{\mathfrak{g}}, S^{\mathfrak{g}}\} = 0.$$

It is straightforward to verify that these three notions are identical. The third version of the definition can be viewed as saying that a $\mathfrak{g}$-action on a classical field theory is a family of classical field theories over the dg ring $C^*(\mathfrak{g})$ which reduces to the original classical field theory modulo the maximal ideal $C^{>0}(\mathfrak{g})$. This version of the definition generalizes to the quantum level.

Our formulation of Noether's theorem will be phrased in terms of the action of a local L_∞ algebra on a field theory. However, we are often presented with the action of an ordinary, finite-dimensional L_∞-algebra on a theory, and we would like to apply Noether's theorem to this situation. Thus, we need to be able to formulate this kind of action as an action of a local L_∞ algebra.

The following lemma shows that we can do this.

11.1.3.2 Lemma. *Let $\mathfrak{g}$ be an L_∞-algebra. Then, the simplicial sets describing the following are canonically homotopy equivalent:*

(1) *Actions of* $\mathfrak{g}$ *on a fixed classical field theory on a space-time manifold* X.
(2) *Actions of the local* L_∞ *algebra* $\Omega^*_X \otimes \mathfrak{g}$ *on the same classical field theory.*

Note that the sheaf $\Omega^*_X \otimes \mathfrak{g}$ is a fine resolution of the constant sheaf of L_∞ algebras with value $\mathfrak{g}$. The lemma can be generalized to show that, given any locally-constant sheaf of L_∞ algebras $\mathfrak{g}$, an action of $\mathfrak{g}$ on a theory is the same thing as an action of a fine resolution of $\mathfrak{g}$.

PROOF. Suppose that $\mathcal{M}$ is a classical field theory, and suppose that we have an action of the local L_∞ algebra $\Omega^*_X \otimes \mathfrak{g}$ on $\mathcal{M}$.

Actions of $\mathfrak{g}$ on $\mathcal{M}$ are Maurer-Cartan elements of the pro-nilpotent dg Lie algebra

$$\mathrm{Act}(\mathfrak{g}, \mathcal{M}) \overset{\text{def}}{=} C^*_{red}(\mathfrak{g}) \otimes \mathscr{O}_{loc}(\mathcal{M}[1])$$

, with dg Lie structure described above. Actions of $\Omega^*_X \otimes \mathfrak{g}$ are Maurer-Cartan elements of the pro-nilpotent dg Lie algebra

$$\mathrm{Act}(\Omega^*_X \otimes \mathfrak{g}, \mathcal{M}) \overset{\text{def}}{=} \mathscr{O}_{loc}(\Omega^*_X \otimes \mathfrak{g}[1] \oplus \mathcal{M}[1]) / \left(\mathscr{O}_{loc}(\Omega^*_X \otimes \mathfrak{g}[1]) \oplus \mathscr{O}_{loc}(\mathcal{M}[1])\right),$$

again with the dg Lie algebra structure defined above.

Recall that there is an inclusion of dg Lie algebras

$$\mathscr{O}_{loc}(\Omega^*_X \otimes \mathfrak{g}[1] \oplus \mathcal{M}[1]) / \mathscr{O}_{loc}(\Omega^*_X \otimes \mathfrak{g}[1]) \subset C^*(\Omega^*(X) \otimes \mathfrak{g}) \otimes \mathscr{O}_{loc}(\mathcal{M}[1]).$$

Further, there is an inclusion of L_∞ algebras

$$\mathfrak{g} \hookrightarrow \Omega^*(X) \otimes \mathfrak{g}$$

(by tensoring with the constant functions). Composing these maps gives a map of dg Lie algebras

$$\mathrm{Act}(\Omega^*_X \otimes \mathfrak{g}, \mathcal{M}) \to \mathrm{Act}(\mathfrak{g}, \mathcal{M}).$$

It suffices to show that this map is an equivalence.

We will do this by using the D_X-module interpretation of the left hand side. Let $J(\mathcal{M})$ and $J(\Omega^*_X)$ refer to the D_X-modules of jets of sections of $\mathcal{M}$ and of the de Rham complex, respectively. Note that the natural map of D_X-modules

$$C^\infty_X \to J(\Omega^*_X)$$

is a quasi-isomorphism (this is the Poincaré lemma.).

Recall that

$$\mathscr{O}_{loc}(\mathcal{M}[1]) = \omega_X \otimes_{D_X} C^*_{red}(J(\mathcal{M})).$$

The cochain complex underlying the dg Lie algebra $\mathrm{Act}(\Omega^*_X \otimes \mathfrak{g}, \mathcal{M})$ has the following interpretation in the language of D_X-modules:

$$\mathrm{Act}(\Omega^*_X \otimes \mathfrak{g}, \mathcal{M}) = \omega_X \otimes_{D_X} \left(C^*_{red}(J(\Omega^*_X) \otimes_{\mathbb{C}} \mathfrak{g}) \otimes_{C^\infty_X} C^*_{red}(J(\mathcal{M})) \right).$$

Under the other hand, the complex $\mathrm{Act}(\mathfrak{g}, \mathcal{M})$ has the D_X-module interpretation

$$\mathrm{Act}(\mathfrak{g}, \mathcal{M}) = \omega_X \otimes_{D_X} \left(C^*_{red}(\mathfrak{g} \otimes C^\infty_X) \otimes_{C^\infty_X} C^*_{red}(J(\mathcal{M})) \right).$$

Because the map $C^\infty_X \to J(\Omega^*_X)$ is a quasi-isomorphism of D_X-modules, the natural map

$$C^*_{red}(J(\Omega^*_X) \otimes_{\mathbb{C}} \mathfrak{g}) \otimes_{C^\infty_X} C^*_{red}(J(\mathcal{M})) \to C^*_{red}(\mathfrak{g} \otimes C^\infty_X) \otimes_{C^\infty_X} C^*_{red}(J(\mathcal{M}))$$

is a quasi-isomorphism of D_X-modules. Now, both sides of this equation are flat as left D_X-modules; this follows from the fact that $C^*_{red}(J(\mathcal{M}))$ is a flat D_X-module. If follows that this map is still a quasi-isomorphism after tensoring over D_X with ω_X. □

11.2. Examples of classical field theories with an action of a local L_∞ algebra

One is often interested in particular classes of field theories: for example, conformal field theories, holomorphic field theories, or field theories defined on Riemannian manifolds. It turns out that these ideas can be formalized by saying that a theory is acted on by a particular local L_∞ algebra, corresponding to holomorphic, Riemannian, or conformal geometry. This generalizes to any geometric structure on a manifold which can be described by a combination of differential equations and symmetries.

In this section, we will describe the local L_∞ algebras corresponding to holomorphic, conformal, and Riemannian geometry, and give examples of classical field theories acted on by these L_∞ algebras.

We will first discuss the holomorphic case. Let X be a complex manifold, and define a local dg Lie algebra algebra $\mathcal{L}^{hol}$ by setting

$$\mathcal{L}^{hol}(X) = \Omega^{0,*}(X, TX),$$

equipped with the Dolbeault differential and the Lie bracket of vector fields. A holomorphic classical field theory will be acted on by $\mathcal{L}^{hol}(X)$.

Remark: A stronger notion of holomorphicity might require the field theory to be acted on by the group of holomorphic symmetries of X, and that the derivative of this action extends to an action of the local dg Lie algebra $\mathcal{L}^{hol}$.

Let us now give some examples of field theories acted on by $\mathcal{L}^{hol}$.

Example: Let X be a complex manifold of complex dimension d, and let $\mathfrak{g}$ be a finite-dimensional Lie algebra with Lie group G. Then, $\Omega^{0,*}(X, \mathfrak{g})$ describes the formal moduli space of principle G-bundles on X. We can form the cotangent theory to this, which is a classical field theory, by letting

$$\mathcal{M} = \Omega^{0,*}(X, \mathfrak{g}) \oplus \Omega^{d,*}(X, \mathfrak{g}^\vee)[d-3].$$

As discussed in [Cos13], this example is important in physics. If $d = 2$ it describes a holomorphic twist of $N = 1$ supersymmetric gauge theory. In addition, one can use the formalism of L_∞ spaces [Cos11a, Cos13] to write twisted supersymmetric σ-models in these terms (when $d = 1$ and $\mathfrak{g}$ is a certain sheaf of L_∞ algebras on the target space).

The dg Lie algebra $\mathcal{L}^{hol}(X)$ acts by Lie derivative on $\Omega^{k,*}(X)$ for any k. One can make this action explicit as follows: the contraction map

$$\begin{aligned} \Omega^{0,*}(X, TX) \times \Omega^{k,*}(X) &\to \Omega^{k-1,*}(X) \\ (V, \omega) &\mapsto \iota_V \omega \end{aligned}$$

is $\Omega^{0,*}(X)$-linear and defined on $\Omega^{0,0}(X, TX)$ in the standard way. The Lie derivative is defined by the Cartan homotopy formula

$$\mathcal{L}_V \omega = [\iota_V, \partial]\omega.$$

In this way, $\mathcal{L}$ acts on $\mathcal{M}$. This action preserves the invariant pairing.

We can write this in terms of an $\mathcal{L}$-dependent action functional, as follows. If $\alpha \in \Omega^{0,*}(X, \mathfrak{g})[1]$, $\beta \in \Omega^{d,*}(X, \mathfrak{g}^\vee)[d-2]$ and $V \in \Omega^{0,*}(X, TX)[1]$, we define

$$S^{\mathcal{L}}(\alpha, \beta, V) = \int \left\langle \beta, (\overline{\partial} + \mathcal{L}_V)\alpha \right\rangle + \tfrac{1}{2} \left\langle \beta, [\alpha, \alpha] \right\rangle.$$

(The fields α, β, V can be of mixed degree).

Note that if $V \in \Omega^{0,*}(X, TX)$ is of cohomological degree 1, it defines a deformation of complex structure of X, and the $\overline{\partial}$-operator for this deformed complex structure is $\overline{\partial} + \mathcal{L}_V$. The action functional $S^{\mathcal{L}}$ therefore describes the variation of the original action functinal S as we vary the complex structure on X. Other terms in $S^{\mathcal{L}}$ encode the fact that S is invariant under holomorphic symmetries of X.

We will return to this example throughout our discussion of Noether's theorem. We will see that, in dimension $d = 1$ and with $\mathfrak{g}$ Abelian, it leads to a version of the Segal-Sugawara construction: a map from the Virasoro vertex algebra to the vertex algebra associated to a free $\beta - \gamma$ system.

Example: Next, let us discuss the situation of field theories defined on a complex manifold X together with a holomorphic principal G-bundle. In the case that X is a Riemann surface, field theories of this form play an important role in the mathematics of chiral conformal field theory.

For this example, we define a local dg Lie algebra $\mathcal{L}$ on a complex manifold X by

$$\mathcal{L}(X) = \Omega^{0,*}(X, TX) \ltimes \Omega^{0,*}(X, \mathfrak{g})$$

so that $\mathcal{L}(X)$ is the semi-direct product of the Dolbeault resolution of holomorphic vector fields with the Dolbeault complex with coefficients in $\mathfrak{g}$. Thus, $\mathcal{L}(X)$ is the dg Lie algebra controlling deformations of X as a complex manifold equipped with a holomorphic G-bundle, near the trivial bundle. (The dg Lie algebra controlling deformations of the pair (X, P) where P is a non-trivial principal G-bundle on X is $\Omega^{0,*}(X, \mathrm{At}_P)$ where At_P is the Atiyah algebroid of P, and everything that follows works in the more general case when P is non-trivial and we use $\Omega^{0,*}(X, \mathrm{At}_P)$ in place of $\mathcal{L}$).

Let V be a representation of G. We can form the cotangent theory to the elliptic moduli problem of sections of V, defined by the Abelian elliptic L_∞ algebra

$$\mathcal{M}(X) = \Omega^{0,*}(X, V)[-1] \oplus \Omega^{0,*}(X, V^\vee)[d-2].$$

This is acted on by the local L_∞ algebra $\mathcal{L}$ we described above.

More generally, we could replace V by a complex manifold M with a G-action and consider the cotangent theory to the moduli of holomorphic maps to M.

Example: In this example we will introduce the local dg Lie algebra $\mathcal{L}^{Riem}$ on a Riemannian manifold X which controls deformations of X as a Riemannian manifold. This local dg Lie algebra acts on field theories which are defined on Riemannian manifolds; we will show this explicitly in the case of scalar field theories.

Let (X, g_0) be a Riemannian manifold of dimension d, which for simplicity we assume to be oriented.

Consider the local dg Lie algebra

$$\mathcal{L}^{Riem}(X) = \mathrm{Vect}(X) \oplus \Gamma(X, \mathrm{Sym}^2 TX)[-1].$$

The differential is $\mathrm{d}V = \mathscr{L}_V g_0$ where $\mathscr{L}_V$ indicates Lie derivative. The Lie bracket is defined by saying that the bracket of a vector field V with anything is given by Lie derivative.

Note that $\mathcal{L}^{Riem}(X)$ is the dg Lie algebra describing the formal neighbourhood of X in the moduli space of Riemannian manifolds.

Consider the free scalar field theory on X, defined by the abelian elliptic dg Lie algebra

$$\mathcal{M}(X) = C^\infty(X)[-1] \xrightarrow{\triangle_{g_0}} \Omega^d(X)[-2]$$

where the superscript indicates cohomological degree, and

$$\triangle_{g_0} = \mathrm{d} * \mathrm{d}$$

is the Laplacian for the metric g_0, landing in top-forms.

We define the action of $\mathcal{L}^{Riem}(X)$ on $\mathcal{M}(X)$ by defining an action functional $S^{\mathcal{L}}$ which couples the fields in $\mathcal{L}^{Riem}(X)$ to those in $\mathcal{M}(X)$. If $\phi, \psi \in \mathcal{M}(X)[1]$ are fields of cohomological degree 0 and 1, and $V \in \mathrm{Vect}(X)$, $\alpha \in \Gamma(X, \mathrm{Sym}^2 TX)$, then we define $S^{\mathcal{L}}$ by

$$S^{\mathcal{L}}(\phi, \psi, V, \alpha) = \int \phi(\triangle_{g_0+\alpha} - \triangle_{g_0})\phi + \int (V\phi)\psi.$$

On the right hand side we interpret $\triangle_{g_0+\alpha}$ as a formal power series in the field α. The fact that this satisfies the master equation follows from the fact that the Laplacian $\triangle_{g_0+\alpha}$ is covariant under infinitesimal diffeomorphisms:

$$\triangle_{g_0+\alpha} + \varepsilon[V, \triangle_{g_0+\alpha}] = \triangle_{g_0+\alpha+\varepsilon\mathcal{L}_V g_0+\varepsilon\mathscr{L}_V\alpha}.$$

One can rewrite this in the language of L_∞ algebras by Taylor expanding $\triangle_{g_0+\alpha}$ in powers of α. The resulting semi-direct product L_∞-algebra $\mathcal{L}^{Riem}(X) \ltimes \mathcal{M}(X)$ is the describes the formal moduli space of Riemannian manifolds together with a harmonic function ϕ.

Example: Let us modify the previous example by considering a scalar field theory with a polynomial interaction, so that the action functional is of the form

$$\int \phi\triangle_{g_0}\phi + \sum_{n\geq 2} \lambda_n \tfrac{1}{n!}\phi^n \mathrm{d}Vol_{g_0}.$$

In this case, $\mathcal{M}$ is deformed into a non-abelian L_∞ algebra, with maps l_n defined by

$$\begin{aligned} l_n &: C^\infty(X)^{\otimes n} \to \Omega^d(X) \\ l_n(\phi_1, \dots, \phi_n) &= \lambda_n \phi_1 \cdots \phi_n \mathrm{d}Vol_{g_0}. \end{aligned}$$

The action of $\mathcal{L}^{Riem}$ on $\mathcal{M}$ is defined, as above, by declaring that the action functional $S^{\mathcal{L}}$ coupling the two theories is

$$S^{\mathcal{L}}(\phi, \psi, V, \alpha) + S(\phi, \psi) = \int \phi\triangle_{g_0+\alpha}\phi + \sum_{n\geq 2} \lambda_n \tfrac{1}{n!}\phi^n \mathrm{d}Vol_{g_0+\alpha} + \int (V\phi)\psi.$$

Example: Next let us discuss the classical conformal field theories.

As above, let (X, g_0) be a Riemannian manifold. Define a local dg Lie algebra $\mathcal{L}^{conf}$ on X by setting

$$\mathcal{L}^{conf}(X) = \mathrm{Vect}(X) \oplus C^\infty(X) \oplus \Gamma(X, \mathrm{Sym}^2 TX)[-1].$$

The copy of $C^\infty(X)$ corresponds to Weyl rescalings.

The differential on $\mathcal{L}^{conf}(X)$ is

$$\mathrm{d}(V, f) = \mathscr{L}_V g_0 + f g_0$$

where $V \in \mathrm{Vect}(X)$ and $f \in C^\infty(X)$. The Lie bracket is defined by saying that $\mathrm{Vect}(X)$ acts on everything by Lie derivative, and that if $f \in C^\infty(X)$ and $\alpha \in \Gamma(X, \mathrm{Sym}^2 TX)$, $[f, \alpha] = f\alpha$.

It is easy to verify that $H^0(\mathcal{L}^{conf}(X))$ is the Lie algebra of conformal symmetries of X, and that $H^1(\mathcal{L}^{conf}(X))$ is the space of first-order conformal deformations of X. The local dg Lie algebra $\mathcal{L}^{conf}$ will act on any classical conformal field theory.

We will see this explicitly in the case of the free scalar field theory in dimension 2. Let $\mathcal{M}$ be the elliptic dg Lie algebra corresponding to the free scalar field theory on a Riemannian 2-manifold X, as described in the previous example.

The action of $\mathcal{L}^{conf}$ on $\mathcal{M}$ is defined such that the sub-algebra $\mathcal{L}^{Riem}$ acts in the same way as before, and that $C^\infty(X)$ acts by zero.

This does not define an action for the two-dimensional theory with polynomial interaction, because the polynomial interaction is not conformally invariant.

There are many other, more complicated, examples of this nature. If X is a conformal 4-manifold, then Yang-Mills theory on X is conformally invariant at the classical level. The same goes for self-dual Yang-Mills theory. One can explicitly write an action of $\mathcal{L}^{conf}$ on the elliptic L_∞-algebra on X describing either self-dual or full Yang-Mills theory.

Example: In this example, we will see how we can describe sources for local operators in the language of local dg Lie algebras.

Let X be a Riemannian manifold, and consider a scalar field theory on X with a ϕ^3 interaction, whose associated elliptic L_∞ algebra $\mathcal{M}$ has been described above. Recall that $\mathcal{M}(X)$ consists of $C^\infty(X)$ in degree 1 and of $\Omega^d(X)$ in degree 2, where $d = \dim X$.

The action functional encoding the L_∞ structure on $\mathcal{M}$ is the functional on $\mathcal{M}[1]$ defined by

$$S(\phi, \psi) = \int \phi \triangle \phi + \int \phi^3.$$

where ϕ is a degree 0 element of $\mathcal{M}(X)[1]$, so that ϕ is a smooth function.

Let us view the sheaf $C^\infty_X[-1]$ as an Abelian local L_∞-algebra on X, situated in degree 1 with zero differential and bracket.

Let us define an action of $\mathcal{L}$ on $\mathcal{M}$ by giving an action functional

$$S^{\mathcal{L}}(\alpha, \phi, \psi) = \int \phi \triangle \phi + \int \phi \alpha.$$

Here,

$$\alpha \in \mathcal{L}[1] = C^\infty(M)$$

and ϕ, ψ are elements of degrees 0 and 1 of $\mathcal{M}(X)[1]$. This gives a semi-direct product L_∞ algebra $\mathcal{L} \ltimes \mathcal{M}$, whose underlying cochain complex is

$$\begin{tikzcd} C^\infty(X)^1 \arrow[dr, "\mathrm{Id}"] & \\ C^\infty(X)^1 \arrow[r, "\triangle"] & C^\infty(X)^2 \end{tikzcd}$$

where the first row is $\mathcal{L}$ and the second row is $\mathcal{M}$. The only non-trivial Lie bracket is the original Lie bracket on $\mathcal{M}$.

11.3. The factorization algebra of equivariant observable

11.3.0.1 Proposition. *Suppose that $\mathcal{M}$ is a classical field theory with an action of $\mathcal{L}$. Then, there is a P_0 factorization algebra $\mathrm{Obs}^{cl}_{\mathcal{L}}$ of equivariant observables, which is a factorization algebra in modules for the factorization algebra in commutative dg algebras $C^*(\mathcal{L})$, which assigns to an open subset U the commutative dga $C^*(\mathcal{L}(U))$.*

PROOF. Since $\mathcal{L}$ acts on $\mathcal{M}$, we can construct the semi-direct product local L_∞ algebra $\mathcal{L} \ltimes \mathcal{M}$. We define the equivariant classical observables

$$\mathrm{Obs}^{cl}_{\mathcal{L}} = C^*(\mathcal{L} \ltimes \mathcal{M})$$

to be the Chevalley-Eilenberg cochain factorization algebra associated to this semi-direct product.

As in section 6.4, we will construct a sub-factorization algebra on which the Poisson bracket is defined and which is quasi-isomorphic. We simply let

$$\widetilde{\mathrm{Obs}}^{cl}_{\mathcal{L}}(U) \subset \mathrm{Obs}^{cl}_{\mathcal{L}}(U)$$

be the subcomplex consisting of those functionals which have smooth first derivative but only in the $\mathcal{M}$-directions. As in section 6.4, there is a P_0 structure on this subcomplex, which on generators is defined by the dual of the non-degenerate invariant pairing on $\mathcal{M}$. Those functionals which lie in $C^*(\mathcal{L}(U))$ are central for this Poisson bracket.

It is clear that this constructs a P_0-factorization algebra over the factorization algebra $C^*(\mathcal{L})$. □

11.4. Inner actions

A stronger notion of action of a local L_∞ algebra on a classical field theory will be important for Noeother's theorem. We will call this stronger notion an *inner* action of a local L_∞ algebra on a classical field theory. For classical field theories, every action can be lifted canonically to an inner action, but at the quantum level this is no longer the case.

We defined an action of a local L_∞ algebra $\mathcal{L}$ on a field theory $\mathcal{M}$ (both on a manifold X) to be a Maurer-Cartan element in the differential graded Lie algebra $\mathrm{Act}(\mathcal{L}, \mathcal{M})$ whose underlying cochain complex is

$$\mathscr{O}_{loc}(\mathcal{L}[1] \oplus \mathcal{M}[1]) / \left(\mathscr{O}_{loc}(\mathcal{L}[1]) \oplus \mathscr{O}_{loc}(\mathcal{M}[1])\right)$$

with the Chevalley-Eilenberg differential for the direct sum L_∞ algebra $\mathcal{L} \oplus \mathcal{M}$.

An inner action will be defined as a Maurer-Cartan element in a larger dg Lie algebra which is a central extension of $\mathrm{Act}(\mathcal{L}, \mathcal{M})$ by $\mathscr{O}_{loc}(\mathcal{L}[1])$. Note that

$$\mathscr{O}_{loc}(\mathcal{L}[1] \oplus \mathcal{M}[1]) \subset C^*_{red}(\mathcal{L}(X) \oplus \mathcal{M})$$

has the structure of dg Lie algebra, where the differential is the Chevalley-Eilenberg differential for the direct sum dg Lie algebra, and the bracket arises, as usual, from the invariant pairing on $\mathcal{M}$.

Further, there's a natural map of dg Lie algebras from this to $\mathscr{O}_{loc}(\mathcal{M}[1])$, which arises by applying the functor of Lie algebra cochains to the inclusion $\mathcal{M} \hookrightarrow \mathcal{M} \oplus \mathcal{L}$ of L_∞ algebras.

We let

$$\mathrm{InnerAct}(\mathcal{L}, \mathcal{M}) \subset \mathscr{O}_{loc}(\mathcal{L}[1] \oplus \mathcal{M}[1])$$

be the kernel of this map. Thus, as a cochain complex,

$$\mathrm{InnerAct}(\mathcal{L}, \mathcal{M}) = \mathscr{O}_{loc}(\mathcal{L}[1] \oplus \mathcal{M}[1]) / \mathscr{O}_{loc}(\mathcal{M}[1])$$

with differential the Chevalley-Eilenberg differential for the direct sum L_∞ algebra $\mathcal{L} \oplus \mathcal{M}$. Note that the Lie bracket on $\mathrm{InnerAct}(\mathcal{L}, \mathcal{M})$ is of cohomological degree $+1$.

11.4.0.1 Definition. *An* inner *action of $\mathcal{L}$ on $\mathcal{M}$ is a Maurer-Cartan element*

$$S^{\mathcal{L}} \in \mathrm{InnerAct}(\mathcal{L}, \mathcal{M}).$$

Thus, $S^{\mathcal{L}}$ is of cohomological degree 0, and satisfies the master equation

$$\mathrm{d}S^{\mathcal{L}} + \tfrac{1}{2}\{S^{\mathcal{L}}, S^{\mathcal{L}}\}.$$

11.4.0.2 Lemma. *Suppose we have an action of $\mathcal{L}$ on a field theory $\mathcal{M}$. Then there is an obstruction class in $H^1(\mathscr{O}_{loc}(\mathcal{L}[1]))$ such that the action extends to an inner action if and only if this class vanishes.*

PROOF. There is a short exact sequence of dg Lie algebras

$$0 \to \mathscr{O}_{loc}(\mathcal{L}[1]) \to \mathrm{InnerAct}(\mathcal{L}, \mathcal{M}) \to \mathrm{Act}(\mathcal{L}, \mathcal{M}) \to 0$$

and $\mathscr{O}_{loc}(\mathcal{L}[1])$ is central. The result follows from general facts about Maurer-Cartan simplicial sets.

More explicitly, the obstruction is calculated as follows. Suppose we have an action functional

$$S^{\mathcal{L}} \in \mathrm{Act}(\mathcal{L}, \mathcal{M}) \subset C^*_{red}(\mathcal{L}(X)) \otimes C^*_{red}(\mathcal{M}(X)).$$

Then, let us view $S^{\mathcal{L}}$ as a functional in

$$\widetilde{S}^{\mathcal{L}} \, \mathrm{InnerAct}(\mathcal{L}, \mathcal{M}) \subset C^*_{red}(\mathcal{L}(X)) \otimes C^*(\mathcal{M}(X))$$

using the natural inclusion $C^*_{red}(\mathcal{M}(X)) \hookrightarrow C^*(\mathcal{M}(X))$. The obstruction is simply the failure of $\widetilde{S}^{\mathcal{L}}$ to satisfy the Maurer-Cartan equation in $\mathrm{InnerAct}(\mathcal{L}, \mathcal{M})$. □

Let us now briefly remark on some refinements of this lemma, which give some more control over obstruction class.

Recall that we sometimes use the notation $C^*_{red,loc}(\mathcal{L})$ for the complex $\mathscr{O}_{loc}(\mathcal{L}[1])$. Thus, $C^*_{red,loc}(\mathcal{L})$ is a subcomplex of $C^*_{red}(\mathcal{L}(X))$. We let $C^{\geq 2}_{,loc}(\mathcal{L})$ be the kernel of the natural map

$$C^*_{red,loc}(\mathcal{L}) \to \mathcal{L}^!(X)[-1].$$

Thus, $C^{\geq 2}_{red,loc}(\mathcal{L})$ is a subcomplex of $C^{\geq 2}_{red}(\mathcal{L}(X))$.

11.4.0.3 Lemma. *If a local L_∞ algebra $\mathcal{L}$ acts on a classical field theory $\mathcal{M}$, then the obstruction to extending $\mathcal{L}$ to an inner action lifts naturally to an element of the subcomplex*

$$C^{\geq 2}_{red,loc}(\mathcal{L}) \subset C^*_{red,loc}(\mathcal{L}).$$

PROOF. Suppose that the action of $\mathcal{L}$ on $\mathcal{M}$ is encoded by an action functional $S^{\mathcal{L}}$, as before. The obstruction is

$$\left(\mathrm{d}_{\mathcal{L}} S^{\mathcal{L}} + \mathrm{d}_{\mathcal{M}} S^{\mathcal{L}} + \tfrac{1}{2}\{S^{\mathcal{L}}, S^{\mathcal{L}}\} \right) |_{\mathscr{O}_{loc}(\mathcal{L}[1])} \in \mathscr{O}_{loc}(\mathcal{L}[1]).$$

Here, $\mathrm{d}_{\mathcal{L}}$ and $\mathrm{d}_{\mathcal{M}}$ are the Chevalley-Eilenberg differentials for the two L_∞ algebras.

We need to verify that no terms in this expression can be linear in $\mathcal{L}$. Recall that the functional $S^{\mathcal{L}}$ has no linear terms. Further, the differentials $\mathrm{d}_{\mathcal{L}}$ and $\mathrm{d}_{\mathcal{M}}$ respect the filtration on everything by polynomial degree, so that they can not produce a functional with a linear term from a functional which does not have a linear term. □

Remark: There is a somewhat more general situation when this lemma is false. When one works with families of classical field theories over some dg ring R with a nilpotent ideal I, one allows the L_∞ algebra $\mathcal{M}$ describing the field theory to be curved, as long as the curving vanishes modulo I. This situation is encountered in the study of σ-models: see [Cos11a]. When $\mathcal{M}$ is curved, the differential $\mathrm{d}_{\mathcal{M}}$ does not preserve the filtration by polynomial degree, so that this argument fails.

Let us briefly discuss a special case when the obstruction vanishes.

11.4.0.4 Lemma. *Suppose that the action of $\mathcal{L}$ on $\mathcal{M}$, when viewed as an action of $\mathcal{L}$ on the sheaf of formal moduli problems $B\mathcal{M}$, preserves the base point of $B\mathcal{M}$. In the language of L_∞ algebras, this means that the L_∞ structure on $\mathcal{L} \oplus \mathcal{M}$ defining the action has no terms mapping*

$$\mathcal{L}^{\otimes n} \to \mathcal{M}$$

for some $n > 0$.

Then, the action of $\mathcal{L}$ extends canonically to an inner action.

PROOF. We need to verify that the obstruction

$$\left(\mathrm{d}_{\mathcal{L}} S^{\mathcal{L}} + \mathrm{d}_{\mathcal{M}} S^{\mathcal{L}} + \tfrac{1}{2}\{S^{\mathcal{L}}, S^{\mathcal{L}}\}\right) |_{\mathscr{O}_{loc}(\mathcal{L}[1])} \in \mathscr{O}_{loc}(\mathcal{L}[1]).$$

is identically zero. Our assumptions on $S^{\mathcal{L}}$ mean that it is at least quadratic as a function on $\mathcal{M}[1]$. It follows that the obstruction is also at least quadratic as a function of $\mathcal{M}[1]$, so that it is zero when restricted to being a function of just $\mathcal{L}[1]$. $\square$

11.5. Classical Noether's theorem

As we showed in lemma **??**, there is a bijection between classes in $H^1(C^*_{red,loc}(\mathcal{L}))$ and local central extensions of $\mathcal{L}_c$ shifted by -1.

11.5.0.1 Theorem. *Let $\mathcal{M}$ be a classical field theory with an action of a local L_∞ algebra $\mathcal{L}$. Let $\widetilde{\mathcal{L}}_c$ be the central extension corresponding to the obstruction class $\alpha \in H^1(C^*_{red,loc}(\mathcal{L}))$ for lifting $\mathcal{L}$ to an inner action. Let $\widetilde{\mathrm{Obs}}^{cl}$ be the classical observables of the field theory $\mathcal{M}$, equipped with its P_0 structure. Then, there is an L_∞-map of precosheaves of L_∞-algebras*

$$\widetilde{\mathcal{L}}_c \to \widetilde{\mathrm{Obs}}^{cl}[-1]$$

which sends the central element c to the unit $1 \in \widetilde{\mathrm{Obs}}^{cl}[-1]$ (note that, after the shift, the unit 1 is in cohomological degree 1, as is the central element c).

Remark: The linear term in the L_∞-morphism is a map of precosheaves of cochain complexes from $\widetilde{\mathcal{L}}_c \to \mathrm{Obs}^{cl}[-1]$. The fact that we have such a map of precosheaves implies that we have a map of commutative dg factorization algebras

$$\widehat{\mathrm{Sym}}^*(\widetilde{\mathcal{L}}_c[1]) \to \widetilde{\mathrm{Obs}}^{cl}.$$

which, as above, sends the central element to 1. This formulation is the one that will quantize: we will find a map from a certain Chevalley-Eilenberg chain complex of $\widetilde{\mathcal{L}}_c[1]$ to quantum observables.

Remark: Lemma 11.4.0.3 implies that the central extension $\widetilde{\mathcal{L}}_c$ is split canonically as a presheaf of cochain complexes:

$$\widetilde{\mathcal{L}}_c(U) = \mathbb{C}[-1] \oplus \mathcal{L}_c(U).$$

Thus, we have a map of precosheaves of cochain complexes

$$\mathcal{L}_c \to \mathrm{Obs}^{cl}.$$

The same proof will show that that this cochain map to a continuous map from the distributional completion $\overline{\mathcal{L}}_c(U)$ to Obs^{cl}.

PROOF. Let us first consider a finite-dimensional version of this statement, in the case when the central extension splits. Let $\mathfrak{g}, \mathfrak{h}$ be L_∞ algebras, and suppose that $\mathfrak{h}$ is equipped with an invariant pairing of degree -3. Then, $C^*(\mathfrak{h})$ is a P_0 algebra. Suppose we are given an element

$$G \in C^*_{red}(\mathfrak{g}) \otimes C^*(\mathfrak{h})$$

of cohomological degree 0, satisfying the Maurer-Cartan

$$\mathrm{d}G + \tfrac{1}{2}\{G, G\} = 0$$

where $\mathrm{d}_{\mathfrak{g}}, \mathrm{d}_{\mathfrak{h}}$ are the Chevalley-Eilenberg differentials for $\mathfrak{g}$ and $\mathfrak{h}$ respectively, and $\{-,-\}$ denotes the Poisson bracket coming from the P_0 structure on $C^*(\mathfrak{h})$.

Then, G is precisely the data of an L_∞ map

$$\mathfrak{g} \to C^*(\mathfrak{h})[-1].$$

Indeed, for any L_∞ algebra $\mathfrak{j}$, to give a Maurer-Cartan element in $C^*_{red}(\mathfrak{g}) \otimes \mathfrak{j}$ is the same as to give an L_∞ map $\mathfrak{g} \to \mathfrak{j}$. Further, the simplicial set of L_∞-maps and homotopies between them is homotopy equivalent to the Maurer-Cartan simplicial set.

Let us now consider the case when we have a central extension. Suppose that we have an element

$$G \in C^*_{red}(\mathfrak{g}) \otimes C^*(\mathfrak{h})$$

of degree 0, and an obstruction element

$$\alpha \in C^*_{red}(\mathfrak{g})$$

of degree 1, such that

$$\mathrm{d}G + \tfrac{1}{2}\{G, G\} = \alpha \otimes 1.$$

Let $\widetilde{\mathfrak{g}}$ be the -1-shifted central extension determined by α, so that there is a short exact sequence

$$0 \to \mathbb{C}[-1] \to \widetilde{\mathfrak{g}} \to \mathfrak{g} \to 0.$$

Then, the data of G and α is the same as a map of L_∞ algebras $\widetilde{\mathfrak{g}} \to C^*(\mathfrak{h})[-1]$ which sends the central element of $\widetilde{\mathfrak{g}}$ to $1 \in C^*(\mathfrak{h})$.

To see this, let us choose a splitting $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C} \cdot c$ where the central element c is of degree 1. Let $c^\vee$ be the linear functional on $\widetilde{\mathfrak{g}}$ which is zero on $\mathfrak{g}$ and sends c to 1.

Then, the image of α under the natural map $C^*(\mathfrak{g}) \to C^*(\widetilde{\mathfrak{g}})$ is made exact by $c^\vee$, viewed as a zero-cochain in $C^*(\widetilde{\mathfrak{g}})$. It follows that

$$G + c^\vee \otimes 1 \in C^*_{red}(\widetilde{\mathfrak{g}}) \otimes C^*(\mathfrak{h})$$

satisfies the Maurer-Cartan equation, and therefore defines (as above) an L_∞-map $\widetilde{\mathfrak{g}} \to C^*(\mathfrak{h})[-1]$. This L_∞-map sends $c \to 1$: this is because G only depends on c by the term $c^\vee \otimes 1$.

Let us apply these remarks to the setting of factorization algebras. First, let us remark a little on the notation: we normally use the notation $\mathscr{O}_{loc}(\mathcal{L}[1])$ to refer to the complex of local functionals on $\mathcal{L}[1]$, with the Chevalley-Eilenberg differential. However, we can also refer to this object as $C^*_{red,loc}(\mathcal{L})$, the reduced, local cochains of $\mathcal{L}$. It is the subcomplex of $C^*_{red}(\mathcal{L}(X))$ of cochains which are local.

Suppose we have an action of a local L_∞-algebra $\mathcal{L}$ on a classical field theory $\mathcal{M}$. Let

$$\alpha \in \mathscr{O}_{loc}(\mathcal{L}[1]) = C^*_{red,loc}(\mathcal{L})$$

be a 1-cocycle representing the obstruction to lifting to an inner action on $\mathcal{M}$. Let

$$\widetilde{\mathcal{L}}_c = \mathcal{L}_c \oplus \underline{\mathbb{C}}[-1]$$

be the corresponding central extension.

By the definition of α, we have a functional

$$S^{\mathcal{L}} \in C^*_{red,loc}(\mathcal{L} \oplus \mathcal{M}) / C^*_{red,loc}(\mathcal{M})$$

of cohomological degree 0 satisfying the Maurer-Cartan equation

$$\mathrm{d}S^{\mathcal{L}} + \tfrac{1}{2}\{S^{\mathcal{L}}, S^{\mathcal{L}}\} = \alpha.$$

For every open subset $U \subset M$, we have an injective cochain map

$$\Phi : C^*_{red,loc}(\mathcal{L} \oplus \mathcal{M}) / C^*_{red,loc}(\mathcal{M}) \to C^*_{red}(\mathcal{L}_c(U)) \widehat{\otimes} \widetilde{C}^*(\mathcal{M}(U)),$$

where $\widehat{\otimes}$ refers to the completed tensor product and $\widetilde{C}^*(\mathcal{M}(U))$ refers to the subcomplex of $C^*(\mathcal{M}(U))$ consisting of functionals with smooth first derivative. The reason we have such a map is simply that a local functional on U is defined when at least if its inputs is compactly supported.

The cochain map Φ is in fact a map of dg Lie algebras, where the Lie bracket arises as usual from the pairing on $\mathcal{M}$. Thus, for every U, we have an element

$$S^{\mathcal{L}}(U) \in C^*_{red}(\mathcal{L}_c(U)) \widehat{\otimes} \widetilde{C}^*(\mathcal{M}(U))$$

satisfying the Maurer-Cartan equation

$$\mathrm{d}S^{\mathcal{L}}(U) + \tfrac{1}{2}\{S^{\mathcal{L}}(U), S^{\mathcal{L}}(U)\} = \alpha(U).$$

It follows, as in the finite-dimensional case discussed above, that $S^{\mathcal{L}}(U)$ gives rise to a map of L_∞ algebras

$$\widetilde{\mathcal{L}}_c(U) \to \widetilde{C}^*(\mathcal{M}(U))[-1] = \widetilde{\mathrm{Obs}}^{cl}(U)[-1]$$

sending the central element c in $\widetilde{\mathcal{L}}_c(U)$ to the unit $1 \in \widetilde{\mathrm{Obs}}^{cl}(U)$. The fact that $S^{\mathcal{L}}$ is local implies immediately that this is a map of precosheaves. □

11.6. Conserved currents

Traditionally, Noether's theorem states that there is a conserved current associated to every symmetry. Let us explain why the version of (classical) Noether's theorem presented above leads to this more traditional statement. Similar remarks will hold for the quantum version of Noether's theorem.

In the usual treatment, a *current* is taken to be an $d-1$-form valued in Lagrangians (if we're dealing with a field theory on a manifold X of dimension d). In our formalism, we make the following definition (which will be valid at the quantum level as well).

11.6.0.1 Definition. *A conserved current in a field theory is a map of precosheaves*

$$J : \overline{\Omega}_c^*[1] \to \mathrm{Obs}^{cl}$$

to the factorization algebra of classical observables.

Dually, J can be viewed as a closed, degree 0 element of

$$J(U)\Omega^*(U)[n-1] \widehat{\otimes} \mathrm{Obs}^{cl}(U)$$

defined for every open subset U, and which is compatible with inclusions of open subsets in the obviious way.

In particular, we can take the component of $J(U)^{n-1,0}$ which is an element

$$J(U)^{n-1,0} \in \Omega^{n-1}(U) \widehat{\otimes} \mathrm{Obs}^{cl}(U)^0.$$

The superscript in $\mathrm{Obs}^{cl}(U)^0$ indicates cohomological degree 0. Thus, $J(U)^{n-1,0}$ is an $n-1$-form valued in observables, which is precisely what is traditionally called a current.

Let us now explain why our definition means that this current is conserved (up to homotopy). We will need to introduce a little notation to explain this point. If $N \subset X$ is a closed subset, we let

$$\mathrm{Obs}^{cl}(N) = \mathrm{holim}_{N \subset U} \mathrm{Obs}^{cl}(U)$$

be the homotopy limit of observables on open neighbourhoods of N. Thus, an element of $\mathrm{Obs}^{cl}(N)$ is an observable defined on every open neighbourhood of N, in a way compatible (up to homotopy) with inclusions of open sets. The fact that we are taking a homotopy limit instead of an ordinary limit is not so important for this discussion, it's to ensure that the answer doesn't depend on arbitrary choices.

For example, if $p \in X$ is a point, then $\mathrm{Obs}^{cl}(p)$ should be thought of as the space of local observables at p.

Suppose we have a conserved current (in the sense of the definition above). Then, for every compact codimension 1 oriented submanifold $N \subset X$, the delta-distribution on N is an element

$$[N] \in \overline{\Omega}^1(U)$$

defined for every open neighbourhood U of N. Applying the map defining the closed current, we get an element

$$J[N] \in \mathrm{Obs}^{cl}(U)$$

for every neighbourhood U of N. This element is compatible with inclusions $U \hookrightarrow U'$, so defines an element of $\mathrm{Obs}^{cl}(N)$.

Let $M \subset X$ be a top-dimensional submanifold with boundary $\partial M = N \amalg N'$. Then,

$$\mathrm{d}J[M] = J[N] - J[N'] \in \mathrm{Obs}^{cl}(M).$$

It follows that the cohomology class $[J[N]]$ of $J[N]$ doesn't change if N is changed by a cobordism.

In particular, let us suppose that our space-time manifold X is a product

$$X = N \times \mathbb{R}.$$

Then, the observable $[J[N_t]]$ associated to the submanifold $N \times \{t\}$ is independent of t.

This is precisely the condition (in the traditional formulation) for a current to be conserved.

Now let us explain why our version of Noether's theorem, as explained above, produces a conserved current from a symmetry.

11.6.0.2 Lemma. *Suppose we have a classical field theory on a manifold X which has an infinitesimal symmetry. To this data, our formulation of Noether's theorem produces a conserved current.*

PROOF. A theory with an infinitesimal symmetry is acted on by the abelian Lie algebra $\mathbb{R}$ (or $\mathbb{C}$). Lemma **??** shows us that such an action is equivalent to the action of the Abelian local dg Lie algebra Ω^*_X. Lemma 11.4.0.3 implies that the central extension $\widetilde{\mathcal{L}}_c$ is split as a cochain complex (*except* in the case that we work in families and the classical field theory $\mathcal{M}$ has curving). We thus get a map

$$\Omega^*_{X,c}[1] \to \mathrm{Obs}^{cl}.$$

The remark following theorem 11.5.0.1 tells us that this map extends to a continuous cochain map

$$\overline{\Omega}^*_{X,c}[1] \to \mathrm{Obs}^{cl}$$

which is our defintion of a conserved current.

□

11.7. Examples of classical Noether's theorem

Let's give some simple examples of this construction. All of the examples we will consider here will satisfy the criterion of lemma **??** which implies that the central extension of the local L_∞ algebra of symmetries is trivial.

Example: Suppose that a field theory on a manifold X of dimension d has an inner action of the Abelian local L_∞ algebra $\Omega^d_X[-1]$. Then, we get a map of presheaves of cochain complexes

$$\overline{\Omega}^d_X \to \mathrm{Obs}^{cl}.$$

Since, for every point $p \in X$, the delta-function δ_p is an element of $\overline{\Omega}^d(X)$, in this way we get a local observable in $\mathrm{Obs}^{cl}(p)$ for every point. This varies smoothly with p.

For example, consider the free scalar field theory on X. We can define an action of $\Omega^d_X[-1]$ on the free scalar field theory as follows. If $\phi \in C^\infty(X)$ and $\psi \in \Omega^d(X)[-1]$ are fields of the free scalar field theory, and $\gamma \in \Omega^d(X)$ is an element of the Abelian local L_∞ algebra we want to act, then the action is described by the action functional describing how $\mathcal{L}$

$$S^{\mathcal{L}}(\phi, \psi, \gamma) = \int \phi\gamma.$$

The corresponding map L_∞ map

$$\Omega^d_c(U) \to \mathrm{Obs}^{cl}(U) \qquad (\dagger)$$

is linear, and sends $\gamma \in \Omega^d_c(U)$ to the observable $\int \phi\gamma$.

Example: Let's consider the example of a scalar field theory on a Riemannian manifold X, described by the action functional

$$\int_X \phi \triangle \phi + \phi^3 \mathrm{d}Vol.$$

This is acted on by the dg Lie algebra $\mathcal{L}^{Riem}$ describing deformations of X as a Riemannian manifold.

If U is an open subset of X, then $\mathcal{L}_c^{Riem}(U)$ consists, in degree 0, of the compactly-supported first-order deformations of the Riemannian metric g_0 on $U \subset X$. If

$$\alpha \in \Gamma_c(U, \mathrm{Sym}^2 TX)$$

is such a deformation, let us Taylor expand the Laplace-Beltrami operator $\phi \triangle_{g_0+\alpha} \phi$ as a sum

$$\triangle_{g_0+\alpha} = \triangle_{g_0} \sum_{n \geq 1} \tfrac{1}{n!} D_n(\alpha, \dots, \alpha)$$

where D_n are poly-differential operators from $\Gamma(X, TX)^{\otimes n}$ to the space of order ≤ 2 differential operators on X. Explicit formula for the operators D_n can be derived from the formula for the Laplace-Beltrami operator in terms of the metric.

Note that if α has compact support in U, then

$$\int \phi D_n(\alpha, \dots, \alpha)\phi$$

defines an observable in $\mathrm{Obs}^{cl}(U)$, and in fact in $\widetilde{\mathrm{Obs}}^{cl}(U)$ (as it has smooth first derivative in ϕ.

The L_∞ map

$$\mathcal{L}_c^{Riem}(U) \to \mathrm{Obs}^{cl}(U)[-1]$$

has Taylor terms

$$\Phi_n : \mathcal{L}_c^{Riem}(U)^{\otimes n} \to \mathrm{Obs}^{cl}(U)$$

defined by the observables

$$\Phi_n((\alpha_1, V_1), \dots, (\alpha_n, V_n))(\phi, \psi) = \begin{cases} \int \phi D_n(\alpha_1, \dots, \alpha_n)\phi & \text{if } n > 1 \\ \int \phi D_1(\alpha)\phi + \int (V\phi)\psi & \text{if } n = 1. \end{cases}$$

One is often just interested in the cochain map

$$\mathcal{L}_c^{Riem}(U) \to \mathrm{Obs}^{cl}(U),$$

corresponding to ϕ_1 above, and not in the higher terms. This cochain map has two terms: one given by the observable describing the first-order variation of the metric, and one given by the observable $\int (V\phi)\psi$ describing the action of vector fields on the fields of the theory.

A similar analysis describes the map from $\mathcal{L}_c^{Riem}$ to the observables of a scalar field theory with polynomial interaction.

Example: Let us consider the $\beta\gamma$ system in one complex dimension, on $\mathbb{C}$. The dg Lie algebra $\mathcal{M}$ describing this theory is

$$\mathcal{M}(\mathbb{C}) = \Omega^{0,*}(\mathbb{C}, V)[-1] \oplus \Omega^{1,*}(\mathbb{C}, V^*)[-1].$$

Let

$$\mathcal{L} = \Omega^{0,*}(\mathbb{C}, T\mathbb{C})$$

be the Dolbeault resolution of holomorphic vector fields on $\mathbb{C}$. $\mathcal{L}$ acts on $\mathcal{M}$ by Lie derivative. We can write down the action functional encoding this action by

$$S^{\mathcal{L}}(\beta, \gamma, V) = \int (\mathscr{L}_V \beta)\gamma.$$

Here, $\beta \in \Omega^{0,*}(\mathbb{C}, V)$, $\gamma \in \Omega^{1,*}(\mathbb{C}, V^*)$ and $V \in \Omega^{0,*}(\mathbb{C}, T\mathbb{C})$.

Lemma **??** implies that in this case there is no central extension. Therefore, we have a map

$$\Phi : \mathcal{L}_c[1] \to \mathrm{Obs}^{cl}$$

of precosheaves of cochain complexes. At the cochain level, this map is very easy to describe: it simply sends a compactly supported vector field $V \in \Omega_c^{0,*}(U, TU)[1]$ to the observable

$$\Phi(V)(\beta, \gamma) = \int_U (\mathscr{L}_V \beta)\, \gamma.$$

We are interested in what this does at the level of cohomology. Let us work on an open annulus $A \subset \mathbb{C}$. We have seen (section **??**??) that the cohomology of $\mathrm{Obs}^{cl}(A)$ can be expressed in terms of the dual of the space of holomorphic functions on A:

$$H^0(\mathrm{Obs}^{cl}(A)) = \widehat{\mathrm{Sym}}^* \left(\mathrm{Hol}(A)^\vee \otimes V^\vee \oplus \Omega^1_{hol}(A)^\vee \otimes V \right).$$

Higher cohomology of $\mathrm{Obs}^{cl}(A)$ vanishes.

Here, $\mathrm{Hol}(A)$ denotes holomorphic functions on A, $\Omega^1_{hol}(A)$ denotes holomorphic 1-forms, and we are taking the continuous linear duals of these spaces. Further, we use, as always, the completed tensor product when defining the symmetric algebra.

In a similar way, we can identify

$$H^*(\Omega_c^{0,*}(A, TA)) = H^*(\Omega^{0,*}(A, K_A^{\otimes 2})^\vee[-1].$$

The residue pairing gives a dense embedding

$$\mathbb{C}[t, t^{-1}]\mathrm{d}t \subset \mathrm{Hol}(A)^\vee.$$

A concrete map

$$\mathbb{C}[t, t^{-1}][-1] \to \Omega_c^{0,*}(A)$$

which realizes this map is defined as follows. Choose a smooth function f on the annulus which takes value 1 near the outer boundary and value 0 near the inner boundary. Then, $\overline{\partial} f$ has compact support. The map sends a polynomial $P(t)$ to $\overline{\partial}(fP)$. One can check, using Stokes' theorem, that this is compatible with the residue pairing: if $Q(t)\mathrm{d}t$ is a holomorphic one-form on the annulus,

$$\oint P(t)Q(t)\mathrm{d}t = \int_A \overline{\partial}(f(t, \overline{t})P(t))Q(t)\mathrm{d}t.$$

In particular, the residue pairing tells us that a dense subspace of $H^1(\mathcal{L}_c(A))$ is

$$\mathbb{C}[t, t^{-1}]\partial_t \subset H^1(\Omega_c^{0,*}(A, TA)).$$

We therefore need to describe a map

$$\Phi : \mathbb{C}[t, t^{-1}]\partial_t \to \widehat{\mathrm{Sym}}^* \left(\mathrm{Hol}(A)^\vee \otimes V^\vee \oplus \Omega^1_{hol}(A)^\vee \otimes V\right).$$

In other words, given an element $P(t)\partial_t \in \mathbb{C}[t, t^{-1}]\mathrm{d}t$, we need to describe a functional $\Phi(P(t)\partial_t)$ on the space of pairs

$$(\beta, \gamma) \in \mathrm{Hol}(A) \otimes V \oplus \Omega^1_{hol}(A) \otimes V^\vee.$$

From what we have explained so far, it is easy to calculate that this functional is

$$\Phi(P(t)\partial_t)(\beta, \gamma) = \oint (P(t)\partial_t \beta(t))\, \gamma(t).$$

The reader familiar with the theory of vertex algebras will see that this is the classical limit of a standard formula for the Virasoro current.

CHAPTER 12

Noether's theorem in quantum field theory

12.1. Quantum Noether's theorem

So far, we have explained the classical version of Noether's theorem, which states that given an action of a local L_∞ algebra $\mathcal{L}$ on a classical field theory, we have a central extension $\widetilde{\mathcal{L}}_c$ of the precosheaf $\mathcal{L}_c$ of L_∞-algebras, and a map of precosheaves of L_∞ algebras

$$\widetilde{\mathcal{L}}_c \to \mathrm{Obs}^{cl}[-1].$$

Our quantum Noether's theorem provides a version of this at the quantum level. Before we explain this theorem, we need to introduce some algebraic ideas about enveloping algebras of homotopy Lie algebras.

Given any dg Lie algebra $\mathfrak{g}$, one can construct its P_0 envelope, which is the universal P_0 algebra containing $\mathfrak{g}$. This functor is the homotopy left adjoint of the forgetful functor from P_0 algebras to dg Lie algebras. Explicitly, the P_0 envelope is

$$U^{P_0}(\mathfrak{g}) = \mathrm{Sym}^* \mathfrak{g}[1]$$

with the obvious product. The Poisson bracket is the unique bi-derivation which on the generators $\mathfrak{g}$ is the given Lie bracket on $\mathfrak{g}$.

Further, if we have a shifted central extension $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$ by $\mathbb{C}[-1]$, determined by a class $\alpha \in H^1(\mathfrak{g})$, we can define the twisted P_0 envelope

$$U^{P_0}_\alpha(\mathfrak{g}) = U^{P_0}(\widetilde{\mathfrak{g}}) \otimes_{\mathbb{C}[c]} \mathbb{C}_{c=1}$$

obtained from the P_0 envelope of $\widetilde{\mathfrak{g}}$ by specializing the central parameter to 1.

We will reformulate the classical Noether theorem using the factorization P_0 envelope of a sheaf of L_∞ algebras on a manifold. The quantum Noether theorem will then be formulated in terms of the factorization BD envelope, which is the quantum version of the factorization P_0 envelope. The factorization BD envelope is a close relative of the factorization envelope of a sheaf of L_∞ algebras that we discussed in Section **??** of Chapter **??**.

For formal reasons, a version of these construction holds in the world of L_∞ algebras. One can show that a commutative dg algebra together with a 1-shifted L_∞ structure with the property that all higher brackets are multi-derivations for the product structure defines a homotopy P_0 algebra. (The point is that the operad describing such gadgets is naturally quasi-isomorphic to the operad P_0).

If $\mathfrak{g}$ is an L_∞ algebra, one can construct a homotopy P_0 algebra which has underlying commutative algebra $\mathrm{Sym}^* \mathfrak{g}[1]$, and which has the unique shifted L_∞ structure where $\mathfrak{g}[1]$ is a sub-L_∞ algebra and all higher brackets are derivations in each variable. This L_∞ structure makes $\mathrm{Sym}^* \mathfrak{g}[1]$ into a homotopy P_0 algebra, and one can show that it is the homotopy P_0 envelope of $\mathfrak{g}$.

We can rephrase the classical version of Noether's theorem as follows.

12.1.0.1 Theorem. *Suppose that a local L_∞ algebra $\mathcal{L}$ acts on a classical field theory, and that the obstruction to lifting this to an inner action is a local cochain α. Then there is a map of homotopy P_0 factorization algebras*

$$U_\alpha^{P_0}(\mathcal{L}_c) \to \mathrm{Obs}^{cl}.$$

Here $U_\alpha^{P_0}(\mathcal{L}_c)$ is the twisted homotopy P_0 factorization envelope, which is defined by saying that on each open subset $U \subset M$ it is $U_\alpha^{P_0}(\mathcal{L}_c(U))$.

The universal property of $U_\alpha^{P_0}(\mathcal{L}_c)$ means that this theorem is a formal consequence of the version of Noether's theorem that we have already proved. At the level of commutative factorization algebras, this map is obtained just by taking the cochain map $\widetilde{\mathcal{L}}_c(U) \to \mathrm{Obs}^{cl}(U)$ and extending it in the unique way to a map of commutative dg algebras

$$\mathrm{Sym}^*(\widetilde{\mathcal{L}}_c(U)) \to \mathrm{Obs}^{cl}(U),$$

before specializing by setting the central parameter to be 1. There are higher homotopies making this into a map of homotopy P_0 algebras, but we will not write them down explicitly (they come from the higher homotopies making the map $\widetilde{\mathcal{L}}_c(U) \to \mathrm{Obs}^{cl}(U)$ into a map of L_∞ algebras).

This formulation of classical Noether's theorem is clearly ripe for quantization. We must simply replace classical observables by quantum observables, and the P_0 envelope by the BD envelope.

Recall that the BD operad is an operad over $\mathbb{C}[[\hbar]]$ which quantizes the P_0 operad. A BD algebra cochain complex with a Poisson bracket of degree 1 and a commutative product, such that the failure of the differential to be a derivation for the commutative product is measured by $\hbar$ times the Poisson bracket.

There is a map of operads over $\mathbb{C}[[\hbar]]$ from the Lie operad to the BD operad. At the level of algebras, this map takes a BD algebra A to the dg Lie algebra $A[-1]$ over $\mathbb{C}[[\hbar]]$, with the Lie bracket given by the Poisson bracket on A. The BD envelope of a dg Lie algebra $\mathfrak{g}$ is defined to be homotopy-universal BD algebra $U^{BD}(\mathfrak{g})$ with a map of dg Lie algebras from $\mathfrak{g}[[\hbar]]$ to $U^{BD}(\mathfrak{g})[-1]$.

One can show that for any dg Lie algebra $\mathfrak{g}$, the homotopy BD envelope of $\mathfrak{g}$ is the Rees module for the Chevalley chain complex $C_*(\mathfrak{g}) = \mathrm{Sym}^*(\mathfrak{g}[-1])$, which is equipped with the increasing filtration defined by the symmetric powers of $\mathfrak{g}$. Concretely,

$$U^{BD}(\mathfrak{g}) = C_*(\mathfrak{g})[[\hbar]] = \mathrm{Sym}^*(\mathfrak{g}[-1])[[\hbar]]$$

with differential $\mathrm{d}_{\mathfrak{g}} + \hbar \mathrm{d}_{CE}$, where $\mathrm{d}_{\mathfrak{g}}$ is the internal differential on $\mathfrak{g}$ and d_{CE} is the Chevalley-Eilenberg differential. The commutative product and Lie bracket are the $\hbar$-linear extensions of those on the P_0 envelope we discussed above. A similar statement holds for L_∞ algebras.

This discussion holds at the level of factorization algebras too: the BD envelope of a local L_∞ algebra $\mathcal{L}$ is defined to be the factorization algebra which assigns to an open subset U the BD envelope of $\mathcal{L}_c(U)$. Thus, it is the Rees factorization algebra associated to the factorization envelope of U. (We will describe this object in more detail in section 12.5 of this chapter).

Now we can state the quantum Noether theorem.

12.1.0.2 Theorem. *Suppose we have a quantum field theory on a manifold M acted on by a local L_∞ algebra $\mathcal{L}$. Let Obs^q be the factorization algebra of quantum observables of this field theory.*

In this situation, there is a $\hbar$-dependent local cocycle

$$\alpha \in H^1(\mathscr{O}_{loc}(\mathcal{L}[1]))[[\hbar]]$$

and a homomorphism of factorization algebras from the twisted BD envelope

$$U^{BD}_{\alpha}(\mathcal{L}_c) \to \mathrm{Obs}^q .$$

The relationship between this formulation of quantum Noether's theorem and the traditional point of view on Noether's theorem was discussed (in the classical case) in section **??**.Let us explain, however, some aspects of this story which are slightly different in the quantum and classical settings.

Suppose that we have an action of an ordinary Lie algebra $\mathfrak{g}$ on a quantum field theory on a manifold M. Then the quantum analogue of the result of lemma **??** (which we will prove below) shows that we have an action of the local dg Lie algebra $\Omega^*_X \otimes \mathfrak{g}$ on the field theory. It follows that we have a central extention of $\Omega^*_X \otimes \mathfrak{g}$, given by a class $\alpha \in H^1(\mathscr{O}_{loc}(\Omega^*_X \otimes \mathfrak{g}[1]))$, and a map from the twisted BD envelope of this central extension to observables of our field theory.

Suppose that $N \subset X$ is an oriented codimension 1 submanifold. (We assume for simplicity that X is also oriented). Let us choose an identification of a tubular neighbourhood of N with $N \times \mathbb{R}$. Let $\pi_N : N \times \mathbb{R} \to \mathbb{R}$ denote the projection map to $\mathbb{R}$. The push forward of the factorization algebra $U^{BD}(\Omega^*_X \otimes \mathfrak{g})$ along the projection $\pi_{\mathbb{R}}$ defines a locally-constant facotrization algebra on $\mathbb{R}$, and so an associative algebra.

Let us assume, for the moment, that the central extension vanishes. Then a variant of lemma **??** shows that there is an isomorphism of associative algebras

$$H^* \left(\pi_N U^{BD}(\Omega^*_X \otimes \mathfrak{g}) \right) \cong \mathrm{Rees}(U(H^*(N) \otimes \mathfrak{g})).$$

The algebra on the right hand side is the Rees algebra for the universal enveloping algebra of $H^*(N) \otimes \mathfrak{g}$. This algebra is a $\mathbb{C}[[\hbar]]$-algebra which specializes at $\hbar = 0$ to the completed symmetric algebra of $H^*(N) \otimes \mathfrak{g}$, but is generically non-commutative.

In this way, we see that Noether's theorem gives us a map of factorization algebras on $\mathbb{R}$

$$\mathrm{Rees}(U(\mathfrak{g})) \to H^0(\pi_N \operatorname{Obs}^q)$$

where on the right hand side we have quantum observables of our theory, projected to $\mathbb{R}$.

Clearly this is closely related to the traditional formulation of Noether's theorem: we are saying that every symmetry (i.e. element of $\mathfrak{g}$) gives rise to an observable on every codimension 1 manifold (that is, a current). The operator product between these observables is the product in the universal enveloping algebra.

Now let us consider the case when the central extension is non-zero. A small calculation shows that the group containing possible central extensions can be identified as

$$H^1(\mathscr{O}_{loc}(\Omega^*_X \otimes \mathfrak{g}[1]))[[\hbar]] = H^{d+1}(X, C^*_{red}(\mathfrak{g})) = \oplus_{i+j=d+1} H^i(X) \otimes H^j_{red}(\mathfrak{g})[[\hbar]],$$

where d is the real dimension of X, and $C^*_{red}(\mathfrak{g})$ is viewed as a constant sheaf of cochain complexes on X.

Let us assume that X is of the form $N \times \mathbb{R}$, where as above N is compact and oriented. Then the cocycle above can be integrated over N to yield an element in $H^2_{red}(\mathfrak{g})$, which can be viewed as an ordinary, unshifted central extension of the Lie algebra $\mathfrak{g}$ (which depends on $\hbar$). We can form the twisted universal enveloping algebra $U_\alpha(\mathfrak{g})$, obtained as usual by taking the universal enveloping algebra of the central extension of $\mathfrak{g}$ and then setting the central parameter to 1. This twisted enveloping algebra admits a filtration, so that we can form the Rees algebra. Our formulation of Noether's theorem then produces a map of factorization algebras on $\mathbb{R}$

$$\mathrm{Rees}(U_\alpha(\mathfrak{g})) \to H^0(\pi^*_N \operatorname{Obs}^q).$$

12.2. Actions of a local L_∞-algebra on a quantum field theory

Let us now turn to the proof of the quantum version of Noether's theorem. As in the discussion of the classical theory, the first thing we need to pin down is what it means for a local L_∞ algebra to act on a quantum field theory.

As in the setting of classical field theories, there are two variants of the definition we need to consider: one defining a field theory with an $\mathcal{L}$ action, and one a field theory with an *inner* $\mathcal{L}$-action. Just as in the classical story, the central extension that appears in our formulation of Noether's theorem appears as the obstruction to lifting a field theory with an action to a field theory with an inner action.

We have used throughout the definition of quantum field theory given in [**Cos11b**]. The concept of field theory with an action of a local L_∞-algebra $\mathcal{L}$ relies on a refined definition of field theory, also given in [**Cos11b**]: the concept of a field theory with background fields. Let us explain this definition.

Let us fix a classical field theory, defined by a local L_∞ algebra $\mathcal{M}$ on X with an invariant pairing of cohomological degree -3. Let us choose a gauge fixing operator Q^{GF} on $\mathcal{M}$, as discussed in section **??**. Then as before, we have an elliptic differential operator $[Q, Q^{GF}]$ (where Q refers to the linear differential on $\mathcal{M}$). As explained in section **??**, this leads to the following data.

(1) A propagator $P(\Phi) \in \overline{\mathcal{M}}[1]^{\otimes 2}$, defined for every parametrix Φ. If Φ, Ψ are parametrices, then $P(\Phi) - P(\Psi)$ is smooth.
(2) A kernel $K_\Phi \in \mathcal{M}[1]^{\otimes 2}$ for every parametrix Φ, satisfying

$$Q(P(\Phi) - P(\Psi)) = K_\Psi - K_\Phi.$$

These kernels lead, in turn, to the definition of the RG flow operator and of the BV Laplacian

$$\begin{aligned} W(P(\Phi) - P(\Psi), -) : \mathscr{O}^+_{P,sm}(\mathcal{M}[1])[[\hbar]] &\to \mathscr{O}^+_{P,sm}(\mathcal{M}[1])[[\hbar]] \\ \triangle_\Phi : \mathscr{O}^+_{P,sm}(\mathcal{M}[1])[[\hbar]] &\to \mathscr{O}^+_{P,sm}(\mathcal{M}[1])[[\hbar]] \end{aligned}$$

associated to parametrices Φ and Ψ. There is also a BV bracket $\{-,-\}_\Phi$ which satisfies the usual relation with the BV Laplacian $\triangle_\Phi$. The space $\mathscr{O}^+_{P,sm}(\mathcal{M}[1])[[\hbar]]$ is the space of functionals with proper support and smooth first derivative which are at least cubic modulo $\hbar$.

The homological interpretation of these objects are as follows.

(1) For every parametrix Φ, we have the structure of 1-shifted differential graded Lie algebra on $\mathscr{O}(\mathcal{M}[1])[[\hbar]]$. The Lie bracket is $\{-,-\}_\Phi$, and the differential is
$$Q + \{I[\Phi], -\}_\Phi + \hbar \triangle_\Phi.$$
The subspace $\mathscr{O}^+_{sm,P}(\mathcal{M}[1])[[\hbar]]$ is a nilpotent sub-dgla. The Maurer-Cartan equation in this space is called the *quantum master equation*.

(2) The map $W(P(\Phi) - P(\Psi), -)$ takes solutions to the QME with parametrix Ψ to solutions with parametrix Φ. Equivalently, the Taylor terms of this map define an L_∞ isomorphism between the dglas associated to the parametrices Ψ and Φ.

If $\mathcal{L}$ is a local L_∞ algebra, then $\mathscr{O}(\mathcal{L}[1])$, with its Chevalley-Eilenberg differential, is a commutative dg algebra. We can identify the space $\mathscr{O}(\mathcal{L}[1] \oplus \mathcal{M}[1])$ of functionals on $\mathcal{L}[1] \oplus \mathcal{M}[1]$ with the completed tensor product
$$\mathscr{O}(\mathcal{L}[1] \oplus \mathcal{M}[1]) = \mathscr{O}(\mathcal{L}[1]) \widehat{\otimes}_\pi \mathscr{O}(\mathcal{M}[1]).$$
The operations $\triangle_\Phi$, $\{-,-\}_\Phi$ and $\partial_{P(\Phi)}$ associated to a parametrix on $\mathcal{M}$ extend, by $\mathscr{O}(\mathcal{L}[1])$-linearity, to operations on the space $\mathscr{O}(\mathcal{L}[1] \oplus \mathcal{M}[1])$. For instance, the operator $\partial_{P(\Phi)}$ is associated to the kernel
$$P(\Phi) \in (\mathcal{M}[1])^{\otimes 2} \subset (\mathcal{M}[1] \oplus \mathcal{L}[1])^{\otimes 2}.$$
If $\mathrm{d}_\mathcal{L}$ denotes the Chevalley-Eilenberg differential on $\mathscr{O}(\mathcal{L}[1])$, then we can form an operator $\mathrm{d}_\mathcal{L} \otimes 1$ on $\mathscr{O}(\mathcal{L}[1] \oplus \mathcal{M}[1])$. Similarly, the linear differential Q on $\mathcal{M}$ induces a derivation of $\mathscr{O}(\mathcal{M}[1])$ which we also denote by Q; we can for a derivation $1 \otimes Q$ of $\mathscr{O}(\mathcal{L}[1] \oplus \mathcal{M}[1])$.

The operators $\triangle_\Phi$ and ∂_Φ both commute with $\mathrm{d}_\mathcal{L} \otimes 1$ and satisfy the same relation described above with the operator $1 \otimes Q$.

We will let
$$\mathscr{O}^+_{sm,P}(\mathcal{L}[1] \oplus \mathcal{M}[1])[[\hbar]] \subset \mathscr{O}(\mathcal{L}[1] \oplus \mathcal{M}[1])$$
denote the space of those functionals which satisfy the following conditions.

(1) They are at least cubic modulo $\hbar$ when restricted to be functions just on $\mathcal{M}[1]$. That is, we allow functionals which are quadratic as long as they are either quadratic in $\mathcal{L}[1]$ or linear in both $\mathcal{L}[1]$ and in $\mathcal{M}[1]$, and we allow linear functionals as long as they are independent of $\mathcal{M}[1]$. Further, we work modulo the constants $\mathbb{C}[[\hbar]]$. (This clause is related to the superscript $+$ in the notation).
(2) We require our functionals to have proper support, in the usual sense (as functionals on $\mathcal{L}[1] \oplus \mathcal{M}[1]$).
(3) We require our functionals to have smooth first derivative, again in the sense we discussed before. Note that this condition involves differentiation by elements of both $\mathcal{L}[1]$ and $\mathcal{M}[1]$.

The renormalization group flow operator $W(P(\Phi) - P(\Psi), -)$ on the space $\mathscr{O}^+_{sm,P}(\mathcal{M}[1])[[\hbar]]$ extends to an $\mathscr{O}(\mathcal{L})$-linear operator on the space

$$\mathscr{O}^+_{sm,P}(\mathcal{L}[1] \oplus \mathcal{M}[1])[[\hbar]].$$

It is defined by the equation, as usual,

$$W(P(\Phi) - P(\Psi), I) = \hbar \log \exp(\hbar\partial_{P(\Phi)} - \hbar\partial_{P(\Psi)}) \exp(I/\hbar).$$

We say that an element

$$I \in \mathscr{O}^+_{sm,P}(\mathcal{L}[1] \oplus \mathcal{M}[1])[[\hbar]]$$

satisfies the quantum master equation for the parametrix Φ if it satisfies the equation

$$\mathrm{d}_{\mathcal{L}} I + QI + \{I, I\}_\Phi + \hbar\triangle_\Phi I = 0.$$

Here $\mathrm{d}_{\mathcal{L}}$ indicates the Chevalley differential on $\mathscr{O}(\mathcal{L}[1])$, extended by tensoring with 1 to an operator on $\mathscr{O}(\mathcal{L}[1] \oplus \mathcal{M}[1])$, and Q is the extension of the linear differential on $\mathcal{M}[1]$.

The renormalization group equation takes solutions to the quantum master equation for the parametrix Φ to those for the parametrix Ψ.

There are two different versions of quantum field theory with an action of a Lie algebra that we consider: an actionn and an inner action. For theories with just an action, the functionals we consider are in the quotient

$$\mathscr{O}^+_{P,sm}(\mathcal{L}[1] \oplus \mathcal{M}[1] \mid \mathcal{L}[1])[[\hbar]] = \mathscr{O}^+_{P,sm}(\mathcal{L}[1] \oplus \mathcal{M}[1]) / \mathscr{O}_{P,sm}(\mathcal{L}[1])[[\hbar]]$$

of our space of functionals by those which only depend on $\mathcal{L}$.

Now we can define our notion of a quantum field theory acted on by the local L_∞ algebra $\mathcal{L}$.

12.2.0.1 Definition. *Suppose we have a quantum field theory on M, with space of fields $\mathcal{M}[1]$. Thus, we have a collection of effective interactions*

$$I[\Phi] \in \mathscr{O}^+_{P,sm}(\mathcal{M}[1])[[\hbar]]$$

satisfying the renormalization group equation, BV master equation, and locality axiom, as detailed in subsection 8.2.9.1.

An action of $\mathcal{L}$ on this field theory is a collection of functionals

$$I^{\mathcal{L}}[\Phi] \in \mathscr{O}_{P,sm}(\mathcal{L}[1] \oplus \mathcal{M}[1] \mid \mathcal{L}[1])[[\hbar]]$$

satisfying the following properties.

(1) The renormalization group equation

$$W(P(\Phi) - P(\Psi), I^{\mathcal{L}}[\Psi]) = I^{\mathcal{L}}[\Phi].$$

(2) *Each $I[\Phi]$ must satisfy the quantum master equation (or Maurer-Cartan equation) for the dgla structure associated to the parametrix Φ. We can explicitly write out the various terms in the quantum master equation as follows:*

$$\mathrm{d}_{\mathcal{L}} I^{\mathcal{L}}[\Phi] + Q I^{\mathcal{L}}[\Phi] + \tfrac{1}{2}\{I^{\mathcal{L}}[\Phi], I^{\mathcal{L}}[\Phi]\}_{\Phi} + \hbar \triangle_{\Phi} I^{\mathcal{L}}[\Phi] = 0.$$

Here $\mathrm{d}_{\mathcal{L}}$ refers to the Chevalley-Eilenberg differential on $\mathscr{O}(\mathcal{L}[1])$, and Q to the linear differential on $\mathcal{M}[1]$. As above, $\{-,-\}_{\Phi}$ is the Lie bracket on $\mathscr{O}(\mathcal{M}[1])$ which is extended in the natural way to a Lie bracket on $\mathscr{O}(\mathcal{L}[1] \oplus \mathcal{M}[1])$.

(3) *The locality axiom, as explained in subsection 8.2.9.1, holds: saying that the support of $I^{\mathcal{L}}[\Phi]$ converges to the diagonal as the support of Φ tends to zero, with the same bounds explained in section 8.2.9.1.*

(4) *The image of $I^{\mathcal{L}}[\Phi]$ under the natural map*

$$\mathscr{O}^{+}_{sm,P}(\mathcal{L}[1] \oplus \mathcal{M}[1] \mid \mathcal{L}[1])[[\hbar]] \to \mathscr{O}^{+}_{sm,P}(\mathcal{M}[1])[[\hbar]]$$

(given by restricting to functions just of $\mathcal{M}[1]$) must be the original action functional $I[\Phi]$ defining the original theory.

An inner action is defined in exactly the same way, except that the functionals $I^{\mathcal{L}}[\Phi]$ are elements

$$I^{\mathcal{L}}[\Phi] \in \mathscr{O}_{P,sm}(\mathcal{L}[1] \oplus \mathcal{M}[1])[[\hbar]].$$

That is, we don't quotient our space of functionals by functionals just of $\mathcal{L}[1]$. We require that axioms $1-5$ hold in this context as well.

Remark: One should interpret this definition as a variant of the definition of a family of theories over a pro-nilpotent base ring $\mathcal{A}$. Indeed, if we have an $\mathcal{L}$-action on a theory on M, then the functionals $I^{\mathcal{L}}[\Phi]$ define a family of theories over the dg base ring $C^*(\mathcal{L}(M))$ of cochains on the L_∞ algebra $\mathcal{L}(M)$ of global sections of $\mathcal{L}$. In the case that M is compact, the L_∞ algebra $\mathcal{L}(M)$ often has finite-dimensional cohomology, so that we have a family of theories over a finitely-generated pro-nilpotent dg algebra.

Standard yoga from homotopy theory tells us that a $\mathfrak{g}$-action on any mathematical object (if $\mathfrak{g}$ is a homotopy Lie algebra) is the same as a family of such objects over the base ring $C^*(\mathfrak{g})$ which restrict to the given object at the central fibre. Thus, our definition of an action of the sheaf $\mathcal{L}$ of L_∞ algebras on a field theory on M gives rise to an action (in this homotopical sense) of the L_∞ algebra $\mathcal{L}(M)$ on the field theory.

However, our definition of action is stronger than this. The locality axiom we impose on the action functionals $I^{\mathcal{L}}[\Phi]$ involves both fields in $\mathcal{L}$ and in $\mathcal{M}$. As we will see later, this means that we have a homotopy action of $\mathcal{L}(U)$ on observables of our theory on U, for every open subset $U \subset M$, in a compatible way. $\Diamond$

12.3. Obstruction theory for quantizing equivariant theories

The main result of [Cos11b] as explained in section ?? states that we can construct quantum field theories from classical ones by obstruction theory. If we start with a classical field theory described by an elliptic L_∞ algebra $\mathcal{M}$, the obstruction-deformation complex is the reduced local Chevalley-Eilenberg cochain complex $C^*_{red,loc}(\mathcal{M})$, which by definition is the complex of local functionals on $\mathcal{M}[1]$ equipped with the Chevalley-Eilenberg differential.

A similar result holds in the equivariant context. Suppose we have a classical field theory with an action of a local L_∞ algebra $\mathcal{L}$. In particular, the elliptic L_∞ algebra $\mathcal{M}$ is acted on by $\mathcal{L}$, so we can form the semi-direct product $\mathcal{L} \ltimes \mathcal{M}$. Thus, we can form the local Chevalley-Eilenberg cochain complex

$$\mathscr{O}_{loc}((\mathcal{L} \ltimes \mathcal{M})[1]) = C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M}).$$

The obstruction-deformation complex for quantizing a classical field theory with an action of $\mathcal{L}$ into a quantum field theory with an action of $\mathcal{L}$ is the same as the deformation complex of the original classical field theory with an action of $\mathcal{L}$. This is the complex $C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M} \mid \mathcal{L})$, the quotient of $C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M})$ by $C^*_{red,loc}(\mathcal{L})$.

One can also study the complex controlling deformations of the action of $\mathcal{L}$ on $\mathcal{M}$, while fixing the classical theory. This is the complex we denoted by $\mathrm{Act}(\mathcal{L}, \mathcal{M})$ earlier: it fits into an exact sequence of cochain complexes

$$0 \to \mathrm{Act}(\mathcal{L}, \mathcal{M}) \to C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M} \mid \mathcal{L}) \to C^*_{red,loc}(\mathcal{M}) \to 0.$$

There is a similar remark at the quantum level. Suppose we fix a non-equivariant quantization of our original $\mathcal{L}$-equivariant classical theory $\mathcal{M}$. Then, one can ask to lift this quantization to an $\mathcal{L}$-equivariant quantization. The obstruction/deformation complex for this problem is the group $\mathrm{Act}(\mathcal{L}, \mathcal{M})$.

We can analyze, in a similar way, the problem of quantizing a classical field theory with an inner $\mathcal{L}$-action into a quantum field theory with an inner $\mathcal{L}$-acion. The relevant obstruction/deformation complex for this problem is $C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M})$. If, instead, we fix a non-equivariant quantization of the original classical theory $\mathcal{M}$, we can ask for the obstruction/deformation complex for lifting this to a quantization with an inner $\mathcal{L}$-action. The relevant obstruction-deformation complex is the complex denoted $\mathrm{InnerAct}(\mathcal{L}, \mathcal{M})$ in section 11.4. Recall that $\mathrm{InnerAct}(\mathcal{L}, \mathcal{M})$ fits into a short exact sequence of cochain complexes (of sheaves on X)

$$0 \to \mathrm{InnerAct}(\mathcal{L}, \mathcal{M}) \to C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M}) \to C^*_{red,loc}(\mathcal{M}) \to 0.$$

A more formal statement of these results about the obstruction-deformation complexes is the following.

Fix a classical field theory $\mathcal{M}$ with an action of a local L_∞ algebra $\mathcal{L}$. Let $\mathcal{T}_{\mathcal{L}}^{(n)}$ denote the simplicial set of $\mathcal{L}$-equivariant quantizations of this field theory defined modulo $\hbar^{n+1}$. The simplicial structure is defined exactly as in chapter 8.2: an n-simplex is a family of theories over the base ring $\Omega^*(\triangle^n)$ of forms on the n-simplex. Let $\mathcal{T}^{(n)}$ denote the simplicial set of quantizations without any $\mathcal{L}$-equivariance condition.

Theorem. *The simplicial sets $\mathcal{T}_{\mathcal{L}}^{(n)}$ are Kan complexes. Further, the main results of obstruction theory hold. That is, there is an obstruction map of simplicial sets*

$$\mathcal{T}_{\mathcal{L}}^{(n)} \to DK\left(C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M} \mid \mathcal{L})[1]\right).$$

(Here DK denotes the Dold-Kan functor). Further, there is a homotopy fibre diagram

$$\begin{tikzcd}
\mathscr{T}_{\mathcal{L}}^{(n+1)} \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathscr{T}_{\mathcal{L}}^{(n)} \arrow[r] & DK\left(C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M} \mid \mathcal{L})[1]\right).
\end{tikzcd}$$

Further, the natural map

$$\mathscr{T}_{\mathcal{L}}^{(n)} \to \mathscr{T}^{(n)}$$

(obtained by forgetting the $\mathcal{L}$-equivariance data in the quantization) is a fibration of simplicial sets.

Finally, there is a homotopy fibre diagram

$$\begin{tikzcd}
\mathscr{T}_{\mathcal{L}}^{(n+1)} \arrow[r] \arrow[d] & \mathscr{T}^{(n+1)} \times_{\mathscr{T}^{(n)}} \mathscr{T}_{\mathcal{L}}^{(n)} \arrow[d] \\
\mathscr{T}_{\mathcal{L}}^{(n)} \arrow[r] & DK\left(\mathrm{Act}(\mathcal{L}, \mathcal{M})[1]\right).
\end{tikzcd}$$

We should interpret the second fibre diagram as follows. The simplicial set $\mathscr{T}^{(n+1)} \times_{\mathscr{T}^{(n)}} \mathscr{T}_{\mathcal{L}}^{(n)}$ describes pairs consisting of an $\mathcal{L}$-equivariant quantization modulo $\hbar^{n+1}$ and a non-equivariant quantization modulo $\hbar^{n+2}$, which agree as non-equivariant quantizations modulo $\hbar^{n+1}$. The deformation-obstruction group to lifting such a pair to an equivariant quantization modulo $\hbar^{n+2}$ is the group $\mathrm{Act}(\mathcal{L}, \mathcal{M})$. That is, a lift exists if the obstruction class in $H^1(\mathrm{Act}(\mathcal{L}, \mathcal{M}))$ is zero, and the simplicial set of such lifts is a torsor for the simplicial Abelian group associated to the cochain complex $\mathrm{Act}(\mathcal{L}, \mathcal{M})$). At the level of zero-simplices, the set of lifts is a torsor for $H^0(\mathrm{Act}(\mathcal{L}, \mathcal{M}))$.

This implies, for instance, that if we fix a non-equivariant quantization to all orders, then the obstruction-deformation complex for making this into an equivariant quantization is $\mathrm{Act}(\mathcal{L}, \mathcal{M}))$.

Further elaborations, as detailed in chapter 8.2, continue to hold in this context. For example, we can work with families of theories over a dg base ring, and everything is fibred over the (typically contractible) simplicial set of gauge fixing conditions. In addition, all of these results hold when we work with translation-invariant objects on $\mathbb{R}^n$ and impose "renormalizability" conditions, as discussed in section **??**.

The proof of this theorem in this generality is contained in [**Cos11b**], and is essentially the same as the proof of the corresponding non-equivariant theorem. In [**Cos11b**], the term "field theory with background fields" is used instead of talking about a field theory with an action of a local L_∞ algebra.

For theories with an inner action, the same result continues to hold, except that the obstruction-deformation complex for the first statement is $C^*_{red,loc}(\mathcal{L} \ltimes \mathcal{M})$, and in the second case is $\mathrm{InnerAct}(\mathcal{L}, \mathcal{M})$.

12.3.1. Lifting actions to inner actions. Given a field theory with an action of $\mathcal{L}$, we can try to lift it to one with an inner action. For classical field theories, we have seen that the obstruction to doing this is a class in $H^1(\mathscr{O}_{loc}(\mathcal{L}[1]))$ (with, of course, the Chevalley-Eilenberg differential).

A similar result holds in the quantum setting.

12.3.1.1 Proposition. *Suppose we have a quantum field theory with an action of $\mathcal{L}$. Then there is a cochain*

$$\alpha \in \mathscr{O}_{loc}(\mathcal{L}[1])[[\hbar]] = C^*_{red,loc}(\mathcal{L})$$

of cohomological degree 1 which is closed under the Chevalley-Eilenberg differential, such that trivializing α is the same as lifting $\mathcal{L}$ to an inner action.

PROOF. This follows immediately from the obstruction-deformation complexes for constructing the two kinds of $\mathcal{L}$-equivariant field theories. However, let us explain explicitly how to calculate this obstruction class (because this will be useful later). Indeed, let us fix a theory with an action of $\mathcal{L}$, defined by functionals

$$I^{\mathcal{L}}[\Phi] \in \mathscr{O}^+_{P,sm}(\mathcal{L}[1] \oplus \mathcal{M}[1] \mid \mathcal{L}[1])[[\hbar]].$$

It is always possible to lift $I[\Phi]$ to a collection of functionals

$$\widetilde{I}^{\mathcal{L}}[\Phi] \in \mathscr{O}^+_{P,sm}(\mathcal{L}[1] \oplus \mathcal{M}[1])[[\hbar]]$$

which satisfy the RG flow and locality axioms, but may not satisfy the quantum master equation. The space of ways of lifting is a torsor for the graded abelian group $\mathscr{O}_{loc}(\mathcal{L}[1])[[\hbar]]$

of local functionals on $\mathcal{L}$. The failure of the lift $\widetilde{I}^{\mathcal{L}}[\Phi]$ to satisfy the quantum master equation is, as explained in [Cos11b], independent of Φ, and therefore is a local functional $\alpha \in \mathscr{O}_{loc}(\mathcal{L}[1])$. That is, we have

$$\alpha = \mathrm{d}_{\mathcal{L}} \widetilde{I}^{\mathcal{L}}[\Phi] + Q \widetilde{I}^{\mathcal{L}}[\Phi] + \tfrac{1}{2}\{\widetilde{I}^{\mathcal{L}}[\Phi], \widetilde{I}^{\mathcal{L}}[\Phi]\}_{\Phi} + \hbar \triangle_{\Phi} \widetilde{I}^{\mathcal{L}}[\Phi].$$

Note that functionals just of $\mathcal{L}$ are in the centre of the Poisson bracket $\{-,-\}_{\Phi}$, and are also acted on trivially by the BV operator $\triangle_{\Phi}$.

We automatically have $\mathrm{d}_{\mathcal{L}}\alpha = 0$. It is clear that to lift $I^{\mathcal{L}}[\Phi]$ to a functional $\widetilde{I}^{\mathcal{L}}[\Phi]$ which satisfies the quantum master equation is equivalent to making α exact in $C^*_{red,loc}(\mathcal{L})[[\hbar]]$. □

12.4. The factorization algebra associated to an equivariant quantum field theory

In this section, we will explain what structure one has on observables of an equivariant quantum field theory. As above, let $\mathcal{M}$ denote the elliptic L_∞ algebra on a manifold M describing a classical field theory, which is acted on by a local L_∞-algebra $\mathcal{L}$. Let us define a factorization algebra $C^*_{fact}(\mathcal{L})$ by saying that to an open subset $U \subset M$ it assigns $C^*(\mathcal{L}(U))$. (As usual, we use the appropriate completion of cochains). Note that $C^*_{fact}(\mathcal{L})$ is a factorization algebra valued in (complete filtered differentiable) commutative dg algebras on $\mathcal{M}$.

In this section we will give a brief sketch of the following result.

12.4.0.1 Proposition. *Suppose we have a quantum field theory equipped with an action of a local Lie algebra $\mathcal{L}$; let $\mathcal{M}$ denote the elliptic L_∞ algebra associated to the corresponding classical field theory. Then there is a factorization algebra of equivariant quantum observables which is a factorization algebra in modules for the factorization algebra $C^*(\mathcal{L})$ of cochains on $\mathcal{L}$. This quantizes the classical factorization algebra of equivariant observables constructed in proposition 11.3.0.1.*

PROOF. The construction is exactly parallel to the non-equivariant version which was explained in chapter 8.2, so we will only sketch the details. We define an element of $\mathrm{Obs}^q_{\mathcal{L}}(U)$ of cohomological degree k to be a family of functionals $O[\Phi]$, of cohomological degree k one for every parametrix, on the space $\mathcal{L}(M)[1] \oplus \mathcal{M}(M)[1]$ of fields of the theory. We require that, if ε is a parameter of cohomological degree $-k$ and square zero, that $I^{\mathcal{L}}[\Phi] + \varepsilon O[\Phi]$ satisfies the renormalization group equation

$$W(P(\Phi) - P(\Psi), I^{\mathcal{L}}[\Psi] + \varepsilon O[\Psi]) = I^{\mathcal{L}}[\Phi] + \varepsilon O[\Phi].$$

Further, we require the same locality axiom that was detailed in section 9.4, saying roughly that $O[\Phi]$ is supported on U for sufficiently small parametrices U.

The differential on the complex $\mathrm{Obs}^q_{\mathcal{L}}(U)$ is defined by

$$(\mathrm{d}O)[\Phi] = \mathrm{d}_{\mathcal{L}}O[\Phi] + QO[\Phi] + \{I^{\mathcal{L}}[\Phi], O[\Phi]\}_\Phi + \hbar\triangle_\Phi O[\Phi],$$

where Q is the linear differential on $\mathcal{M}[1]$, and $\mathrm{d}_{\mathcal{L}}$ corresponds to the Chevalley-Eilenberg differential on $C^*(\mathcal{L})$.

We can make $\mathrm{Obs}^q_{\mathcal{L}}(U)$ into a module over $C^*(\mathcal{L}(U))$ as follows. If $O \in \mathrm{Obs}^q_{\mathcal{L}}(U)$ and $\alpha \in C^*(\mathcal{L}(U))$, we can define a new observable $\alpha \cdot O$ defined by

$$(\alpha \cdot O)[\Phi] = \alpha \cdot (O[\Phi]).$$

This makes sense, because α is a functional on $\mathcal{L}(U)[1]$ and so can be made a functional on $\mathcal{M}(U)[1] \oplus \mathcal{L}(U)[1]$. The multiplication on the right hand side is simply multiplication of functionals on $\mathcal{M}(U)[1] \oplus \mathcal{L}(U)[1]$.

It is easy to verify that $\alpha \cdot O$ satisfies the renormalization group equation; indeed, the infinitesimal renormalization group operator is given by differentiating with respect to a kernel in $\mathcal{M}[1]^{\otimes 2}$, and so commutes with multiplication by functionals of $\mathcal{L}[1]$. Similarly, we have

$$\mathrm{d}(\alpha \cdot O) = (\mathrm{d}\alpha) \cdot O + \alpha \cdot \mathrm{d}O$$

where $\mathrm{d}O$ is the differential discussed above, and $\mathrm{d}\alpha$ is the Chevalley-Eilenberg differential applied to $\alpha \in C^*(\mathcal{L}(U)[1])$.

As is usual, at the classical level we can discuss observables at scale 0. The differential at the classical level is $\mathrm{d}_{\mathcal{L}} + Q + \{I^{\mathcal{L}}, -\}$ where $I^{\mathcal{L}} \in \mathscr{O}_{loc}(\mathcal{L}[1] \oplus \mathcal{M}[1])$ is the classical equivariant Lagrangian. This differential is the same as the differential on the Chevalley-Eilenberg differential on the cochains of the semi-direct product L_∞ algebra $\mathcal{L} \ltimes \mathcal{M}$. Thus, it is quasi-isomorphic, at the classical level, to the one discussed in proposition 11.3.0.1. $\square$

12.5. Quantum Noether's theorem

Finally, we can explain Noether's theorem at the quantum level. As above, suppose we have a quantum field theory on a manifold M with space of fields $\mathcal{M}[1]$. Let $\mathcal{L}$ be a local L_∞ algebra which acts on this field theory. Let $\alpha \in H^1(C^*_{red,loc}(\mathcal{L}))[[\hbar]]$ denote the obstruction to lifting this action to an inner action.

Recall that the factorization envelope of the local L_∞ algebra $\mathcal{L}$ is the factorization algebra whose value on an open subset $U \subset M$ is the Chevalley chain complex $C_*(\mathcal{L}_c(U))$. Given a cocycle $\beta \in H^1(C^*_{red,loc}(\mathcal{L}))$, we can form a shifted central extension

$$0 \to \mathbb{C}[-1] \to \widetilde{\mathcal{L}}_c \to \mathcal{L}_c \to 0$$

of the precosheaf $\mathcal{L}_c$ of L_∞ algebras on M. Central extensions of this form have already been discussed in section **??**.

We can then form the *twisted* factorization envelope $U^{\beta}(\mathcal{L})$, which is a factorization algebra on M. The twisted factorization envelope is defined by saying that it's value on an open subset $V \subset M$ is

$$U^{\beta}(\mathcal{L})(V) = \mathbb{C}_{c=1} \otimes_{\mathbb{C}[c]} C_*(\widetilde{\mathcal{L}}_c(V)).$$

Here the complex $C_*(\widetilde{\mathcal{L}}_c(V))$ is made into a $\mathbb{C}[c]$-module by multiplying by the central element.

We have already seen **??** that the Kac-Moody vertex algebra arises as an example of a twisted factorization envelope.

There is a $\mathbb{C}[[\hbar]]$-linear version of the twisted factorization envelope construction too: if our cocycle α is in $H^1(\mathbb{C}^*_{red,loc}(\mathcal{L}))[[\hbar]]$, then we can form a central extension of the form

$$0 \to \mathbb{C}[[\hbar]][-1] \to \widetilde{\mathcal{L}}_c[[\hbar]] \to \mathcal{L}_c[[\hbar]] \to 0.$$

This is an exact sequence of precosheaves of L_∞ algebras on M in the category of $\mathbb{C}[[\hbar]]$-modules. By performing the $\mathbb{C}[[\hbar]]$-linear version of the construction above, one finds the twisted factorization envelope $U^{\alpha}(\mathcal{L})$. This is a factorization algebra on M in the category of $\mathbb{C}[[\hbar]]$-modules, whose value on an open subset $V \subset M$ is

$$U^{\alpha}(\mathcal{L})(V) = \mathbb{C}[[\hbar]]_{c=1} \otimes_{\mathbb{C}[[\hbar]][c]} C_*(\widetilde{\mathcal{L}}_c[[\hbar]]).$$

Here Chevalley chains are taken in the $\mathbb{C}[[\hbar]]$-linear sense.

Our version of Noether's theorem will relate the factorization envelope of $\mathcal{L}_c$, twisted by the cocycle α, to the factorization algebra of quantum observables of the field theory on M. The main theorem is the following.

12.5.0.1 Theorem. *Suppose that the local $\mathcal{L}_\infty$-algebra $\mathcal{L}$ acts on a field theory on M, and that the obstruction to lifting this to an inner action is a cocycle $\alpha \in H^1(C^*_{red,loc}(\mathcal{L}))[[\hbar]]$. Then, there is a $\mathbb{C}((\hbar))$-linear homomorphism of factorization algebras*

$$U^{\alpha}(\mathcal{L})[\hbar^{-1}] \to \mathrm{Obs}^q[\hbar^{-1}].$$

(Note that on both sides we have inverted $\hbar$).

One can ask how this relates to Noether's theorem for classical field theories. In order to provide such a relationship, we need to state a version of quantum Noether's theorem which holds without inverting $\hbar$. For every open subset $V \subset M$, we define the Rees module

$$\operatorname{Rees} U^{\alpha}(\mathcal{L})(V) \subset U^{\alpha}(\mathcal{L})(V)$$

to be the submodule spanned by elements of the form $\hbar^k \gamma$ where $\gamma \in \mathrm{Sym}^{\leq k}(\mathcal{L}_c(V))$. This is a sub-$\mathbb{C}[[\hbar]]$-module, and also forms a sub-factorization algebra. The reason for the terminology is that in the case $\alpha = 0$, or more generally α is independent of $\hbar$, $\operatorname{Rees} U^{\alpha}(\mathcal{L})(V)$ is the Rees module for the filtered chain complex $C^{\alpha}_*(\mathcal{L}_c(V))$.

One can check that $\operatorname{Rees} U^\alpha(V)$ is a free $\mathbb{C}[[\hbar]]$-module and that, upon inverting $\hbar$, we find

$$(\operatorname{Rees} U^\alpha(V))\,[\hbar^{-1}] = U^\alpha(V).$$

12.5.0.2 Theorem. *The Noether map of factorization algebras*

$$U^\alpha(\mathcal{L})[\hbar^{-1}] \to \mathrm{Obs}^q[\hbar^{-1}]$$

over $\mathbb{C}((\hbar))$ *refines to a map*

$$\operatorname{Rees} U^\alpha(\mathcal{L}) \to \mathrm{Obs}^q$$

of factorization algebras over $\mathbb{C}[[\hbar]]$*.*

We would like to compare this statement to the classical version of Noether's theorem. Let α^0 denote the reduction of α modulo $\hbar$. Let $\widetilde{\mathcal{L}}_c$ denote the central extension of $\mathcal{L}_c$ arising from α^0. We have seen that the classical Noether's theorem states that there is a map of precosheaves L_∞ algebras

$$\widetilde{\mathcal{L}}_c \to \widetilde{\mathrm{Obs}}^{cl}[-1]$$

where on the right hand side, $\widetilde{\mathrm{Obs}}^{cl}[-1]$ is endowed with the structure of dg Lie algebra coming from the shifted Poisson bracket on $\widetilde{\mathrm{Obs}}^{cl}$. Further, this map sends the central element in $\widetilde{\mathcal{L}}_c$ to the unit element in $\widetilde{\mathrm{Obs}}^{cl}[-1]$.

In particular, the classical Noether map gives rise to a map of precosheaves of cochain complexes

$$\widetilde{\mathcal{L}}_c[1] \to \mathrm{Obs}^{cl}.$$

We will not use the fact that this arises from an L_∞ map in what follows. Because Obs^{cl} is a commutative factorization algebra, we automatically get a map of commutative prefactorization algebras

$$\operatorname{Sym}^* \widetilde{\mathcal{L}}_c[1] \to \mathrm{Obs}^{cl}.$$

Further, because the Noether map sends the central element to the unit observable, we get a map of commutative factorization algebras

$$\mathbb{C}_{c=1} \otimes_{\mathbb{C}[c]} \operatorname{Sym}^* \widetilde{\mathcal{L}}_c[1] \to \mathrm{Obs}^{cl}. \tag{†}$$

Now we have set up the classical Noether map in a way which is similar to the quantum Noether map. Recall that the quantum Noether map with $\hbar$ not inverted is expressed in terms of the Rees module $\operatorname{Rees} U^\alpha(\mathcal{L})$. When we set $\hbar = 0$, we can identify

$$\operatorname{Rees} U^\alpha(\mathcal{L})(V) \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}_{\hbar=0} = \operatorname{Sym}^*(\widetilde{\mathcal{L}}_c(V)) \otimes_{\mathbb{C}[c]} \mathbb{C}_{c=1}.$$

12.5.0.3 Lemma. *The quantum Noether map*

$$\operatorname{Rees} U^\alpha(\mathcal{L}) \to \mathrm{Obs}^q$$

of factorization algebras becomes, upon setting $\hbar = 0$*, the map in equation (†).*

12.5.1. Proof of the quantum Noether theorem. Before we being our proof of quantum Noether's theorem, it will be helpful to discuss the meaning (in geometric terms) of the chains and cochains of an L_∞ algebra twisted by a cocycle.

If $\mathfrak{g}$ is an L_∞ algebra, then $C^*(\mathfrak{g})$ should be thought of as functions on the formal moduli problem $B\mathfrak{g}$ associated to $\mathfrak{g}$. Similarly, $C_*(\mathfrak{g})$ is the space of distributions on $B\mathfrak{g}$. If $\alpha \in H^1(C^*(\mathfrak{g}))$, then α defines a line bundle on $B\mathfrak{g}$, or equivalently, a rank 1 homotopy representation of $\mathfrak{g}$. Sections of this line bundle are $C^*(\mathfrak{g})$ with the differential $\mathrm{d}_{\mathfrak{g}} - \alpha$, i.e. we change the differential by adding a term given by multiplication by $-\alpha$. Since α is closed and of odd degree, it is automatic that this differential squares to zero. We will sometimes refer to this complex as $C^*_\alpha(\mathfrak{g})$.

Similarly, we can define $C_{*,\alpha}(\mathfrak{g})$ to be $C_*(\mathfrak{g})$ with a differential given by adding the operator of contracting with $-\alpha$ to the usual differential. We should think of $C_{*,\alpha}(\mathfrak{g})$ as the distributions on $B\mathfrak{g}$ twisted by the line bundle associated to α; i.e. distributions which pair with sections of this line bundle.

Let $\widetilde{\mathfrak{g}}$ be the shifted central extension of $\mathfrak{g}$ associated to α. Then $C_*(\widetilde{\mathfrak{g}})$ is a module over $\mathbb{C}[c]$, where c is the central parameter. Then we can identify

$$C_*(\widetilde{\mathfrak{g}}) \otimes_{\mathbb{C}[c]} \mathbb{C}_{c=1} = C_{*,\alpha}(\mathfrak{g}).$$

A similar remark holds for cochains.

In particular, if $\mathcal{L}$ is a local L_∞ algebra on a manifold M and $\alpha \in H^1(C^*_{red,loc}(\mathcal{L}))$ is a local cochain, then

$$U^\alpha(\mathcal{L})(V) = C_{*,\alpha}(\mathcal{L}_c(V))$$

for an open subset $V \subset M$.

Now we will turn to the proof of theorems 12.5.0.1 and 12.5.0.2 and lemma 12.5.0.3, all stated in the previous section.

The first thing we need to do is to produce, for every open subset $V \subset M$, a chain map

$$C^\alpha_*(\mathcal{L}_c(V)) \to \mathrm{Obs}^q(V)[\hbar^{-1}].$$

A linear map

$$f : \mathrm{Sym}^*(\mathcal{L}_c(V)[1]) \to \mathrm{Obs}^q(V)[\hbar^{-1}]$$

is the same as a collection of linear maps

$$f[\Phi] : \mathrm{Sym}^*(\mathcal{L}_c(V)[1]) \to \mathscr{O}(\mathcal{M}(M)[1])((\hbar))$$

one for every parametrix Φ, which satisfy the renormalization group equation and the locality axiom. This, in turn, is the same as a collection of functionals

$$O[\Phi] \in \mathscr{O}(\mathcal{L}_c(V)[1] \oplus \mathcal{M}(M)[1])((\hbar))$$

satisfying the renormalization group equation and the locality axiom. We are using the natural pairing between the symmetric algebra of $\mathcal{L}_c(V)[1]$ and the space of functionals on $\mathcal{L}_c(V)[1]$ to identify a linear map $f[\Phi]$ with a functional $O[\Phi]$.

We will write down such a collection of functionals. Recall that, because we have an action of the local L_∞ algebra $\mathcal{L}$ on our theory, we have a collection of functionals

$$I^{\mathcal{L}}[\Phi] \in \mathscr{O}(\mathcal{L}_c(M)[1] \oplus \mathcal{M}_c(M)[1])[[\hbar]]$$

which satisfy the renormalization group equation and the following quantum master equation:

$$(\mathrm{d}_{\mathcal{L}} + Q) I^{\mathcal{L}}[\Phi] + \tfrac{1}{2}\{I^{\mathcal{L}}[\Phi], I^{\mathcal{L}}[\Phi]\}_\Phi + \hbar \triangle_\Phi \left(I^{\mathcal{L}}[\Phi] \right) = \alpha.$$

We will use the projections from $\mathcal{L}_c(M)[1] \oplus \mathcal{M}_c(M)[1]$ to $\mathcal{L}_c(M)[1]$ and $\mathcal{M}_c(M)[1]$ to lift functionals on these smaller spaces to functionals on $\mathcal{L}_c(M)[1] \oplus \mathcal{M}_c(M)[1]$. In particular, if as usual $I[\Phi]$ denotes the effective action of our quantum field theory, which is a function of the fields in $\mathcal{M}_c(M)[1]$, we will use the same notation to denote the lift of $I[\Phi]$ to a function of the fields in $\mathcal{L}_c(M)[1] \oplus \mathcal{M}_c(M)[1]$.

Let

$$\widehat{I}^{\mathcal{L}}[\Phi] = I^{\mathcal{L}}[\Phi] - I[\Phi] \in \mathscr{O}(\mathcal{L}_c(M)[1] \oplus \mathcal{M}_c(M)[1])[[\hbar]].$$

This functional satisfies the following master equation:

$$(\mathrm{d}_{\mathcal{L}} + Q) \widehat{I}^{\mathcal{L}}[\Phi] + \tfrac{1}{2}\{\widehat{I}^{\mathcal{L}}[\Phi], \widehat{I}^{\mathcal{L}}[\Phi]\}_\Phi + \{I[\Phi], \widehat{I}^{\mathcal{L}}[\Phi]\}_\Phi + \hbar \triangle_\Phi \left(\widehat{I}^{\mathcal{L}}[\Phi] \right) = \alpha.$$

The renormalization group equation for the functionals $\widehat{I}^{\mathcal{L}}[\Phi]$ states that

$$\exp\left(\hbar \partial_{P(\Phi)} - \hbar \partial_{P(\Psi)}\right) \exp\left(I[\Psi]/\hbar\right) \exp\left(\widehat{I}^{\mathcal{L}}[\Psi]/\hbar\right) = \exp\left(I[\Phi]/\hbar\right) \exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right).$$

This should be compared with the renormalization group equation that an observable $\{O[\Phi]\}$ in $\mathrm{Obs}^q(M)$ satisfies:

$$\exp\left(\hbar \partial_{P(\Phi)} - \hbar \partial_{P(\Psi)}\right) \exp\left(I[\Psi]/\hbar\right) O[\Psi] = \exp\left(I[\Phi]/\hbar\right) O[\Phi].$$

Note also that

$$\widehat{I}^{\mathcal{L}}[\Phi] \in \mathscr{O}(\mathcal{L}_c(V)[1] \oplus \mathcal{M}(M))[[\hbar]].$$

The point is the following. Let

$$\widehat{I}^{\mathcal{L}}_{i,k,m}[\Phi] : \mathcal{L}_c(M)^{\otimes k} \times \mathcal{M}_c(M)^{\otimes m} \to \mathbb{C}$$

denote the coefficients of $\hbar^i$ in the Taylor terms of this functional. This Taylor term is zero unless $k > 0$, and further it has proper support (which can be made as close as we like to the diagonal by making Φ small). The proper support condition implies that this Taylor term extends to a functional

$$\mathcal{L}_c(M)^{\otimes k} \times \mathcal{M}(M)^{\otimes m} \to \mathbb{C},$$

that is, only one of the inputs has to have compact support and we can choose this to be an $\mathcal{L}$-input.

From this, it follows that

$$\exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right) \in \mathscr{O}(\mathcal{L}_c(V)[1] \oplus \mathcal{M}(M)[1])((\hbar)).$$

Although there is a $\hbar^{-1}$ in the exponent on the left hand side, each Taylor term of this functional only involves finitely many negative powers of $\hbar$, which is what is required to be in the space on the right hand side of this equation.

Further, the renormalization group equation satisfied by $\exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)$ is precisely the one necessary to define (as Φ varies) an element which we denote

$$\exp\left(\widehat{I}^{\mathcal{L}}/\hbar\right) \in C^*_\alpha(\mathcal{L}_c(V), \mathrm{Obs}^q(M))[\hbar^{-1}].$$

The locality property for the functionals $\widehat{I}^{\mathcal{L}}[\Phi]$ tells us that for Φ small these functionals are supported arbitrarily close to the diagonal. This locality axiom immediately implies that

$$\exp\left(\widehat{I}^{\mathcal{L}}/\hbar\right) \in C^*_\alpha(\mathcal{L}_c(V), \mathrm{Obs}^q(V))[\hbar^{-1}].$$

Thus, we have produced the desired linear map

$$F : C_*^\alpha(\mathcal{L}_c(V)) \to \mathrm{Obs}^q(V)[\hbar^{-1}].$$

Explicitly, this linear map is given by the formula

$$F(l)[\Phi] = \left\langle l, \exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)\right\rangle$$

where $\langle -, - \rangle$ indicates the duality pairing between $C_*^\alpha(\mathcal{L}_c(V))$ and $C^*_\alpha(\mathcal{L}_c(V))$.

Next, we need to verify that F is a cochain map. Since the duality pairing between changes and cochains of $\mathcal{L}_c(V)$ (twisted by α) is a cochain map, it suffices to check that the element $\exp\left(\widehat{I}^{\mathcal{L}}/\hbar\right)$ is closed. This is equivalent to saying that, for each parametrix Φ, the following equation holds:

$$(\mathrm{d}_{\mathcal{L}} - \alpha + \hbar\triangle_\Phi + \{I[\Phi], -\}_\Phi) \exp\left(\widehat{I}^{\mathcal{L}}/\hbar\right) = 0.$$

Here, $\mathrm{d}_{\mathcal{L}}$ indicates the Chevalley-Eilenberg differential on $C^*(\mathcal{L}_c(V))$ and α indicates the operation of multiplying by the cochain α in $C^1(\mathcal{L}_c(V))$.

This equation is equivalent to the statement that

$$(\mathrm{d}_{\mathcal{L}} - \alpha + \hbar\triangle_\Phi) \exp\left(I[\Phi]/\hbar + \widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right) = 0$$

which is equivalent to the quantum master equation satisfied by $I^{\mathcal{L}}[\Phi]$.

Thus, we have produced a cochain map from $C_*^{\alpha}(\mathcal{L}_c(V))$ to $\mathrm{Obs}^q(V)[\hbar^{-1}]$. It remains to show that this cochain map defines a map of factorization algebras.

It is clear from the construction that the map we have constructed is a map of precosheaves, that is, it is compatible with the maps coming from inclusions of open sets $V \subset W$. It remains to check that it is compatible with products.

Let V_1, V_2 be two disjoint subsets of M, but contained in W. We need to verify that the following diagram commutes:

$$\begin{tikzcd} C_*^{\alpha}(\mathcal{L}_c(V_1)) \times C_*^{\alpha}(\mathcal{L}_c(V_2)) \arrow[r] \arrow[d] & \mathrm{Obs}^q(V_1)[\hbar^{-1}] \times \mathrm{Obs}^q(V_2)[\hbar^{-1}] \arrow[d] \\ C_*^{\alpha}(\mathcal{L}_c(W)) \arrow[r] & \mathrm{Obs}^q(W)[\hbar^{-1}]. \end{tikzcd}$$

Let $l_i \in C_*^{\alpha}(\mathcal{L}_c(V_i))$ for $i = 1, 2$. Let $\cdot$ denote the factorization product on the factorization algebra $C_*^{\alpha}(\mathcal{L}_c)$. This is simply the product in the symmetric algebra on each open set, coupled with the maps coming from the inclusions of open sets.

Recall that if O_i are observables in the open sets V_i, then the factorization product$O_1 O_2 \in \mathrm{Obs}^q(W)$ of these observables is defined by

$$(O_1 O_2)[\Phi] = O_1[\Phi] \cdot O_2[\Phi]$$

for Φ sufficiently small, where $\cdot$ indicates the obvious product on the space of functions on $\mathcal{M}(M)[1]$. (Strictly speaking, we need to check that for each Taylor term this identity holds for sufficiently small parametrices, but we have discussed this technicality many times before and will not belabour it now).

We need to verify that, for Φ sufficiently small,

$$F(l_1)[\Phi] \cdot F(l_2)[\Phi] = F(l_1 \cdot l_2)[\Phi] \in \mathscr{O}(\mathcal{M}(M)[1])((\hbar)).$$

By choosing a sufficiently small parametrix, we can assume that $\widehat{I}^{\mathcal{L}}[\Phi]$ is supported as close to the diagonal as we like. We can further assume, without loss of generality, that each l_i is a product of elements in $\mathcal{L}_c(V_i)$. Let us write $l_i = m_{1i} \dots m_{k_i i}$ for $i = 1, 2$ and each $m_{ji} \in \mathcal{L}_c(V_i)$. (To extend from this special case to the case of general l_i requires a small functional analysis argument using the fact that F is a smooth map, which it is. Since we restrict attention to this special case only for notation convenience, we won't give more details on this point).

Then, we can explicitly write the map F applied to the elements l_i by the formula

$$F(l_i)[\Phi] = \left\{ \frac{\partial}{\partial m_{1i}} \dots \frac{\partial}{\partial m_{k_i i}} \exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar \right) \right\} |_{0 \times \mathcal{M}(M)[1]} .$$

In other words, we apply the product of all partial derivatives by the elements $m_{ji} \in \mathcal{L}_c(V_i)$ to the function $\exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)$ (which is a function on $\mathcal{L}_c(M)[1] \oplus \mathcal{M}(M)[1]$) and then restrict all the $\mathcal{L}_c(V_i)$ variables to zero.

To show that
$$F(l_1 \cdot l_2)[\Phi] = F(l_1)[\Phi] \cdot F(l_2)[\Phi]$$
for sufficiently small Φ, it suffices to verify that
$$\left\{\frac{\partial}{\partial m_{11}} \cdots \frac{\partial}{\partial m_{k_1 1}} \exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)\right\}\left\{\frac{\partial}{\partial m_{12}} \cdots \frac{\partial}{\partial m_{k_2 2}} \exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)\right\}$$
$$= \frac{\partial}{\partial m_{11}} \cdots \frac{\partial}{\partial m_{k_1 1}} \frac{\partial}{\partial m_{12}} \cdots \frac{\partial}{\partial m_{k_2 2}} \exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right).$$
Each side can be expanded, in an obvious way, as a sum of terms each of which is a product of factors of the form

$$(\dagger) \qquad \frac{\partial}{\partial m_{j_1 i_1}} \cdots \frac{\partial}{\partial m_{j_r i_r}} \widehat{I}^{\mathcal{L}}[\Phi]$$

together with an overall factor of $\exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)$. In the difference between the two sides, all terms cancel except for those which contain a factor of the form expressed in equation ($\dagger$) where $i_1 = 1$ and $i_2 = 2$. Now, for sufficiently small parametrices,
$$\frac{\partial}{\partial m_{j_1 1}} \frac{\partial}{\partial m_{j_2 2}} \widehat{I}^{L}[\Phi] = 0$$
because $\widehat{I}^{\mathcal{L}}[\Phi]$ is supported as close as we like to the diagonal and $m_{j_1 1} \in \mathcal{L}_c(V_1)$ and $m_{j_2 2} \in \mathcal{L}_c(V_2)$ have disjoint support.

Thus, we have constructed a map of factorization algebras
$$F : U^{\alpha}(\mathcal{L}) \to \mathrm{Obs}^q[\hbar^{-1}].$$
It remains to check the content of theorem 12.5.0.2 and of lemma 12.5.0.3. For theorem 12.5.0.2, we need to verify that if
$$l \in \mathrm{Sym}^k(\mathcal{L}_c(V))$$
(for some open subset $V \subset M$) then
$$F(l) \in \hbar^{-k} \mathrm{Obs}^q(V).$$
That is, we need to check that for each Φ, we have
$$F(l)[\Phi] \in \hbar^{-k} \mathscr{O}(\mathcal{M}(M)[1])[[\hbar]].$$
Let us assume, for simplicty, that $l = m_1 \dots m_k$ where $m_i \in \mathcal{L}_c(V)$. Then the explicit formula
$$F(l)[\Phi] = \left\{\partial_{m_1} \dots \partial_{m_k} \exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)\right\} |_{\mathcal{M}(M)[1]}$$

makes it clear that the largest negative power of $\hbar$ that appears is $\hbar^{-k}$. (Note that $\widehat{I}^{\mathcal{L}}[\Phi]$ is zero when restricted to a function of just $\mathcal{M}(M)[1]$.)

Finally, we need to check lemma 12.5.0.3. This states that the classical limit of our quantum Noether map is the classical Noether map we constructed earlier. Let $l \in \mathcal{L}_c(V)$. Then the classical limit of our quantum Noether map sends l to the classical observable

$$\begin{aligned}\lim_{\hbar\to 0} \hbar F(l) &= \lim_{\Phi\to 0}\lim_{\hbar\to 0}\left\{\hbar\partial_l \exp\left(\widehat{I}^{\mathcal{L}}[\Phi]/\hbar\right)\right\}|_{\mathcal{M}(M)[1]}\\ &= \lim_{\Phi\to 0}\left\{\partial_l I^{\mathcal{L}}_{classical}[\Phi]\right\}|_{\mathcal{M}(M)[1]}\\ &= \left\{\partial_l I^{\mathcal{L}}_{classical}\right\}|_{\mathcal{M}(M)[1]}\,.\end{aligned}$$

Note that by $I^{\mathcal{L}}_{classical}[\Phi]$ we mean the scale Φ version of the functional on $\mathcal{L}[1]\oplus\mathcal{M}[1]$ which defines the inner action (at the classical level) of $\mathcal{L}$ on our classical theory, and by $I^{\mathcal{L}}_{classical}$ we mean the scale zero version.

Now, our classical Noether map is the map appearing in the last line of the above displayed equation.

This completes the proof of theorems 12.5.0.1 and 12.5.0.2 and lemma 12.5.0.3.

12.6. The quantum Noether theorem and equivariant observables

So far in this chapter, we have explained that if we have a quantum theory with an action of the local L_∞ algebra $\mathcal{L}$, then one finds a homotopical action of $\mathcal{L}$ on the quantum observables of the theory. We have also stated and proved our quantum Noether theorem: in the same situation, there is a homomorphism from the twisted factorization envelope of $\mathcal{L}$ to the quantum observables. It is natural to expect that these two constructions are closely related. In this section, we will explain the precise relationship. Along the way, we will prove a somewhat stronger version of the quantum Noether theorem. The theorems we prove in this section will allow us to formulate later a definition of the *local index* of an elliptic complex in the language of factorization algebras.

Let us now give an informal statement of the main theorem in this section. Quantum observables on U have a homotopy action of the sheaf $\mathcal{L}(U)$ of L_∞ algebras. By restricting to compactly-supported sections, we find that $\mathrm{Obs}^q(U)$ has a homotopy action of $\mathcal{L}_c(U)$. This action is compatible with the factorization structure, in the sense that the product map

$$\mathrm{Obs}^q(U)\otimes\mathrm{Obs}^q(V)\to\mathrm{Obs}^q(W)$$

(defined when $U \amalg V \subset W$) is a map of $\mathcal{L}_c(U)\oplus\mathcal{L}_c(V)$-modules, where $\mathrm{Obs}^q(W)$ is made into an $\mathcal{L}_c(U)\oplus\mathcal{L}_c(V)$ module via the natural inclusion map from this L_∞ algebra to $\mathcal{L}_c(W)$.

We can say that the factorization algebra Obs^q is an $\mathcal{L}_c$-equivariant factorization algebra.

It can be a little tricky formulating correctly all the homotopy coherences that go into such an action. When we state our theorem precisely, we will formulate this kind of action slightly differently in a way which captures all the coherences we need.

In general, for an L_∞ algebra $\mathfrak{g}$, an element $\alpha \in H^1(C^*(\mathfrak{g}))$ defines an L_∞-homomorphism $\mathfrak{g} \to \mathbb{C}$ (where $\mathbb{C}$ is given the trivial L_∞ structure). In other words, we can view such a cohomology class as a character of $\mathfrak{g}$. (More precisely, the choice of a cochain representative of α leads to such an L_∞ homomorphism, and different cochain representatives give L_∞-equivalent L_∞-homomorphisms).

Suppose that $\mathcal{L}$ is a local L_∞ algebra on M and that we have an element $\alpha \in H^1(C^*_{red,loc}(\mathcal{L}))$. This means, in particular, that for every open subset $V \subset M$ we have a character of $\mathcal{L}_c(V)$, and so an action of $\mathcal{L}_c(V)$ on $\mathbb{C}$. Let $\underline{\mathbb{C}}$ denote the trivial factorization algebra, which assigns the vector space $\mathbb{C}$ to each open set. The fact that α is local guarantees that the action of each $\mathcal{L}_c(V)$ on $\mathbb{C}$ makes $\underline{\mathbb{C}}$ into an $\mathcal{L}_c$-equivariant factorization algebra. Let us denote this $\mathcal{L}_c$-equivariant factorization algebra by $\underline{\mathbb{C}}_\alpha$.

More generally, given any $\mathcal{L}_c$-equivariant factorization algebra $\mathcal{F}$ on M, we can form a new $\mathcal{L}_c$-equivariant factorization algebra $\mathcal{F}_\alpha$, defined to be the tensor product of $\mathcal{F}$ and $\underline{\mathbb{C}}_\alpha$ in the category of factorization algebras with multiplicative $\mathcal{L}_c$-actions.

Suppose that we have a field theory on M with an action of a local L_∞ algebra $\mathcal{L}$, and with factorization algebra of quantum observables Obs^q. Let $\alpha \in H^1(C^*_{red,loc}(\mathcal{L}))[[\hbar]]$ be the obstruction to lifting this to an inner action. Let $\underline{\mathbb{C}}_\alpha[[\hbar]]$ denote the trivial factorization algebra $\underline{\mathbb{C}}[[\hbar]]$ viewed as an $\mathcal{L}_c$-equivariant factorization algebra using the character α. As we have seen above, the factorization algebra Obs^q is an $\mathcal{L}_c$-equivariant factorization algebra. We can tensor Obs^q with $\underline{\mathbb{C}}_\alpha[[\hbar]]$ to form a new $\mathcal{L}_c$-equivariant factorization algebra Obs^q_α (the tensor product is of course taken over the base ring $\mathbb{C}[[\hbar]]$). As a factorization algebra, Obs^q_α is the same as Obs^q. Only the $\mathcal{L}_c$-action has changed.

The main theorem is the following.

12.6.0.1 Theorem. *The $\mathcal{L}_c$-action on $\mathrm{Obs}^q_\alpha[\hbar^{-1}]$ is homotopically trivial.*

In other words, after twisting the action of $\mathcal{L}_c$ by the character α, and inverting $\hbar$, the action of $\mathcal{L}_c$ on observables is homotopically trivial. The trivialization of the action respects the fact that its multiplicative.

An alternative way to state the theorem is that the action of $\mathcal{L}_c$ on $\mathrm{Obs}^q[\hbar^{-1}]$ is homotopically equivalent to the α-twist of the trivial action. That is, on each open set $V \subset M$ the action of $\mathcal{L}_c(V)$ on $\mathrm{Obs}^q(V)[\hbar^{-1}]$ is by the identity times the character α.

Let us now explain how this relates to Noether's theorem. If we have an $\mathcal{L}_c$-equivariant factorization algebra $\mathcal{F}$ (valued in convenient or pro-convenient vector spaces) then on every open subset $V \subset M$ we can form the Chevalley chain complex $C_*(\mathcal{L}_c(V), \mathcal{F}(V))$ of $\mathcal{L}_c(V)$ with coefficients in $\mathcal{F}(V)$. This is defined by taking the (bornological) tensor product of $C_*(\mathcal{L}_c(V))$ with $\mathcal{F}(V)$ on every open set, with a differential which incorporates the usual Chevalley differential as well as the action of $\mathcal{L}_c(V)$ on $\mathcal{F}(V)$. The cochain complexes $C_*(\mathcal{L}_c(V), \mathcal{F}(V))$ form a new factorization algebra which we call $C_*(\mathcal{L}_c, \mathcal{F})$. (As we will see shortly in our more technical statement of the theorem, the factorization algebra $C_*(\mathcal{L}_c, \mathcal{F})$, with a certain structure of $C_*(\mathcal{L}_c)$-comodule, encodes the $\mathcal{F}$ as an $\mathcal{L}_c$-equivariant factorization algebra).

In particular, when we have an action of $\mathcal{L}$ on a quantum field theory on a manifold M, we can form the factorization algebra $C_*(\mathcal{L}_c, \mathrm{Obs}^q)$, and also the version of this twisted by α, namely $C_*(\mathcal{L}_c, \mathrm{Obs}^q_\alpha)$. We can also consider the chains of $\mathcal{L}_c$ with coefficients on Obs^q with the trivial action: this is simply $U(\mathcal{L}_c) \otimes \mathrm{Obs}^q$ (where we complete the tensor product to the bornological tensor product).

Then, the theorem above implies that we have an isomorphism of factorization algebras

$$\Phi : C_*(\mathcal{L}_c, \mathrm{Obs}^q_\alpha)[\hbar^{-1}] \cong U(\mathcal{L}_c) \otimes \mathrm{Obs}^q[\hbar^{-1}].$$

(Recall that $U(\mathcal{L}_c)$ is another name for $C_*(\mathcal{L}_c)$ with trivial coefficients, and that the tensor product on the right hand side is the completed bornological one).

Now, the action of $\mathcal{L}_c$ on Obs^q preserves the unit observable. This means that the unit map

$$\mathbb{1} : \underline{\mathbb{C}}[[\hbar]] \to \mathrm{Obs}^q$$

of factorization algebras is $\mathcal{L}_c$-equivariant. Taking Chevalley chains and twisting by α, we get a map of fatorization algebras

$$\mathbb{1} : C_*(\mathcal{L}_c, \underline{\mathbb{C}}_\alpha[[\hbar]]) = U^\alpha(\mathcal{L}_c) \to C_*(\mathcal{L}_c, \mathrm{Obs}^q_\alpha).$$

Further, there is a natural map of factorization algebras

$$\varepsilon : U(\mathcal{L}_c) \to \underline{\mathbb{C}},$$

which on every open subset is the map $C_*(\mathcal{L}_c(V)) \to \mathbb{C}$ which projects onto $\mathrm{Sym}^0 \mathcal{L}_c(V)$. (This map is the counit for a natural cocommutative coalgebra structure on $C_*(\mathcal{L}_c(V))$. Tensoring this with the identity map gives a map

$$\varepsilon \otimes \mathrm{Id} : U(\mathcal{L}_c) \otimes \mathrm{Obs}^q \to \mathrm{Obs}^q .$$

The compatibility of the theorem stated in this section with the Noether map is the following.

Theorem. *The composed map of factorization algebras*

$$U^{\alpha}(\mathcal{L}_c) \xrightarrow{\mathbb{1}} C_*(\mathcal{L}_c, \mathrm{Obs}^q_{\alpha})[\hbar^{-1}] \xrightarrow{\cong} U(\mathcal{L}_c) \otimes \mathrm{Obs}^q[\hbar^{-1}] \xrightarrow{\varepsilon \otimes \mathrm{Id}} \mathrm{Obs}^q[\hbar^{-1}]$$

is the Noether map from theorem 12.5.0.1.

This theorem therefore gives a compatibility between the action of $\mathcal{L}_c$ on observables and the Noether map.

12.6.1. Let us now turn to a more precise statement, and proof, of theorem 12.6.0.1. We first need to give a more careful statement of what it means to have a factorization algebra with a multiplicative action of $\mathcal{L}_c$, where, as usual, $\mathcal{L}$ indicates a local L_∞ algebra on a manifold M.

The first fact we need is that the factorization algebra $U(\mathcal{L})$, which assigns to every open subset $V \subset M$ the complex $C_*(\mathcal{L}_c(V))$, is a factorization algebra valued in commutative coalgebras (in the symmetric monoidal category of convenient cochain complexes).

To see this, first observe that for any L_∞ algebra $\mathfrak{g}$, the cochain complex $C_*(\mathfrak{g})$ is a cocommutative dg coalgebra. The coproduct

$$C_*(\mathfrak{g}) \to C_*(\mathfrak{g}) \otimes C_*(\mathfrak{g})$$

is the map induced from the diagonal map of Lie algebras $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$, combined with the Chevalley-Eilenberg chain complex functor. Here we are using the fact that there is a natural isomorphism

$$C_*(\mathfrak{g} \oplus \mathfrak{h}) \cong C_*(\mathfrak{g}) \otimes C_*(\mathfrak{h}).$$

We want, in the same way, to show that $C_*(\mathcal{L}_c(V))$ is a cocommutative coalgebra, for every open subset $V \subset M$. The diagonal map $\mathcal{L}_c(V) \to \mathcal{L}_c(V) \oplus \mathcal{L}_c(V)$ gives us, as above, a putative coproduct map

$$C_*(\mathcal{L}_c(V)) \to C_*(\mathcal{L}_c(V) \oplus \mathcal{L}_c(V)).$$

The only point which is non-trivial is to verify that the natural map

$$C_*(\mathcal{L}_c(V)) \widehat{\otimes}_\beta C_*(\mathcal{L}_c(V)) \to C_*(\mathcal{L}_c(V) \oplus \mathcal{L}_c(V))$$

is an isomorphism, where on the right hand side we use the completed bornological tensor product of convenient vector spaces.

The fact that this map is an isomorphism follows from the fact that for any two manifolds X and Y, we have a natural isomorphism

$$C^\infty_c(X) \widehat{\otimes}_\beta C^\infty_c(Y) \cong C^\infty_c(X \times Y).$$

Thus, for every open subset $V \subset M$, $U(\mathcal{L})(V)$ is a cocommutative coalgebra. The fact that the assignment of the cocommutative coalgebra $C_*(\mathfrak{g})$ to an L_∞ algebra $\mathfrak{g}$ is functorial immediately implies that $U(\mathcal{L})(V)$ is a prefactorization algebra in the category of cocommutative coalgebras. (It is a factorization algebra, and not just a pre-algebra, because the forgetful functor from cocommutative coalgebras to cochain complexes preserves colimits).

Now we can give a formal definition of a factorization algebra with a multiplicative $\mathcal{L}_c$-action.

12.6.1.1 Definition. *Let $\mathcal{F}$ be a factorization algebra on a manifold M valued in convenient vector spaces, and let $\mathcal{L}$ be a local L_∞ algebra. Then, a multiplicative $\mathcal{L}_c$-action on $\mathcal{F}$ is is the following:*

1. *A factorization algebra $\mathcal{F}^{\mathcal{L}}$ in the category of (convenient) comodules for $U(\mathcal{L})$.*
2. *Let us give $\mathcal{F}$ the trivial coaction of $U(\mathcal{L})$. Then, we have a map of dg $U(\mathcal{L})$-comodule factorization algebras $\mathcal{F} \to \mathcal{F}^{\mathcal{L}}$.*
3. *For every open subset $V \subset M$, $\mathcal{F}^{\mathcal{L}}$ is quasi-cofree: this means that there is an isomorphism*
$$\mathcal{F}^{\mathcal{L}} \cong U(\mathcal{L}_c)(V)\widehat{\otimes}_\beta \mathcal{F}(V).$$
of graded, but not dg, $U(\mathcal{L}_c)(V)$ comodules, such that the given map from $\mathcal{F}(V)$ is obtained by tensoring the identity on $\mathcal{F}(V)$ with the coaugmentation map $\mathbb{C} \to U(\mathcal{L}_c)(V)$. (The coaugmentation map is simply the natural inclusion of $\mathbb{C}$ into $C_*(\mathcal{L}_c(V)) = \operatorname{Sym}^* \mathcal{L}_c(V)[1]$).

More generally, suppose that $\mathcal{F}$ is a convenient factorization algebra with a complete decreasing filtration. We give $U(\mathcal{L})$ a complete decreasing filtration by saying that $F^i(U(\mathcal{L}))(V) = 0$ for $i > 0$. In this situation, a multiplicative $\mathcal{L}_c$-action on $\mathcal{F}$ is a complete filtered convenient $U(\mathcal{L})$ comodule $\mathcal{F}^{\mathcal{L}_c}$ with the same extra data and properties as above, except that the tensor product is the one in the category of complete filtered convenient vector spaces.

One reason that this is a good definition is the following.

12.6.1.2 Lemma. *Suppose that $\mathcal{F}$ is a (complete filtered) convenient factorization algebra with a multiplicative $\mathcal{L}_c$ action in the sense above. Then, for every open subset $V \subset M$, there is an L_∞ action of $\mathcal{L}_c(V)$ on $\mathcal{F}(V)$ and an isomorphism of dg $C_*(\mathcal{L}_c(V))$-comodules*
$$C_*(\mathcal{L}_c(V), \mathcal{F}(V)) \cong \mathcal{F}^{\mathcal{L}_c}(V),$$
where on the left hand side we take chains with coefficients in the L_∞-module $\mathcal{F}(V)$.

PROOF. Let $\mathfrak{g}$ be any L_∞ algebra. There is a standard way to translate between L_∞ $\mathfrak{g}$-modules and $C_*(\mathfrak{g})$-comodules: if W is an L_∞ $\mathfrak{g}$-module, then $C_*(\mathfrak{g}, W)$ is a $C_*(\mathfrak{g})$-comodule. Conversely, to give a differential on $C_*(\mathfrak{g}) \otimes W$ making it into a $C_*(\mathfrak{g})$-comodule with the property that the map $W \to C_*(\mathfrak{g}) \otimes W$ is a cochain map, is the same as to give an L_∞ action of $\mathfrak{g}$ on W. Isomorphisms of comodules (which are the identity on the copy of W contained in $C_*(\mathfrak{g}, W)$) are the same as L_∞ equivalences.

This lemma just applies these standard statements to the symmetric monoidal category of convenient cochain complexes. □

To justify the usefulness of our definition to the situation of field theories, we need to show that if we have a field theory with an action of $\mathcal{L}$ then we get a factorization algebra with a multiplicative $\mathcal{L}_c$ action in the sense we have explained.

12.6.1.3 Proposition. *Suppose we have a field theory on a manifold M with an action of a local L_∞ algebra $\mathcal{L}$. Then quantum observables Obs^q of the field theory have a multiplicative $\mathcal{L}_c$-action in the sense we described above.*

PROOF. We need to define the space of elements of $\mathrm{Obs}^{q,\mathcal{L}_c}$. □

Now we can give the precise statement, and proof, of the main theorem of this section. Suppose that we have a quantum field theory on a manifold M, with an action of a local L_∞ algebra $\mathcal{L}$. Let $\alpha \in H^1(C^*_{red,loc}(\mathcal{L}))[[\hbar]]$ be the obstruction to lifting this to an inner action.

12.7. Noether's theorem and the local index

In this section we will explain how Noether's theorem – in the stronger form formulated in the previous section – gives rise to a definition of the *local index* of an elliptic complex with an action of a local L_∞ algebra.

Let us explain what we mean by the local index. Suppose we have an elliptic complex on a compact manifold M. We will let $\mathscr{E}(U)$ denote the cochain complex of sections of this elliptic complex on an open subset $U \subset M$.

Then, the cohomology of $\mathscr{E}(M)$ is finite dimensional, and the index of our elliptic complex is defined to be the Euler characteristic of the cohomology. We can write this as

$$\mathrm{Ind}(\mathscr{E}(M)) = \mathrm{STr}_{H^*(\mathscr{E}(M))} \mathrm{Id}\,.$$

That is, the index is the super-trace (or graded trace) of the identity operator on cohomology.

More generally, if $\mathfrak{g}$ is a Lie algebra acting on global sections of our elliptic complex $\mathscr{E}(M)$, then we can consider the character of $\mathfrak{g}$ on $H^*(\mathscr{E}(M))$. If $X \in \mathfrak{g}$ is any element, the character can be written as

$$\mathrm{Ind}(X, \mathscr{E}(M)) = \mathrm{STr}_{H^*(\mathscr{E}(M))} X.$$

Obviously, the usual index is the special case when $\mathfrak{g}$ is the one-dimensional Lie algebra acting on $\mathscr{E}(M)$ by scaling.

We can rewrite the index as follows. For any endomorphism X of $H * (\mathscr{E}(M))$, the trace of X is the same as the trace of X acting on the determinant of $H^*(\mathscr{E}(M))$. Note that for this to work, we need $H^*(\mathscr{E}(M))$ to be treated as a super-line: it is even or odd depending on whether the Euler characteristic of $H^*(\mathscr{E}(M))$ is even or odd.

It follows that the character of the action of a Lie algebra $\mathfrak{g}$ on $\mathscr{E}(M))$ can be encoded entirely in the natural action of $\mathfrak{g}$ on the determinant of $H^*(\mathscr{E}(M))$. In other words, the character of the $\mathfrak{g}$ action is the same data as the one-dimensional $\mathfrak{g}$-representation $\det H^*(\mathscr{E}(M))$.

Now suppose that $\mathfrak{g}$ is global sections of a sheaf $\mathcal{L}$ of dg Lie algebras (or L_∞ algebras) on M. We will further assume that $\mathcal{L}$ is a local L_∞ algebra. Let us also assume that the action of $\mathfrak{g} = \mathcal{L}(M)$ on $\mathscr{E}(M)$ arises from a local action of the sheaf $\mathcal{L}$ of L_∞ algebras on the sheaf $\mathscr{E}$ of cochain complexes.

Then, one can ask the following question: is there some way in which the character of the $\mathcal{L}(M)$ action on $\mathscr{E}(M)$ can be expressed in a local way on the manifold? Since, as we have seen, the character of the $\mathcal{L}(M)$ action is entirely expressed in the homotopy $\mathcal{L}(M)$ action on the determinant of the cohomology of $\mathscr{E}(M)$, this question is equivalent to the following one: is it possible to express the determinant of the cohomology of $\mathscr{E}(M)$ in a way local on the manifold M, in an $\mathcal{L}$-equivariant way?

Now, $\mathscr{E}(M)$ is a sheaf, so that we can certainly describe $\mathscr{E}(M)$ in a way local on M. Informally, we can imagine $\mathscr{E}(M)$ as being a direct sum of its fibres at various points in M. More formally if we choose a cover $\mathfrak{U}$ of M, then the Čech double complex for $\mathfrak{U}$ with coefficients in the sheaf $\mathscr{E}$ produces for us a complex quasi-isomorphic to $\mathscr{E}(M)$. This double complex is an additive expression describing $\mathscr{E}(M)$ in terms of sections of $\mathscr{E}$ in the open cover $\mathfrak{U}$ of M.

Heuristically, the Čech double complex gives a formula of the form

$$\mathscr{E}(M) \sim \sum_i \mathscr{E}(U_i) - \sum_{i,j} \mathscr{E}(U_i \cap U_j) + \sum_{i,j,k} \mathscr{E}(U_i \cap U_j \cap U_k) - \dots$$

which we should imagine as the analog of the inclusion-exclusion formula from combinatorics. If $\mathfrak{U}$ is a finite cover and each $\mathscr{E}(U)$ has finite-dimensional cohomology, this formula becomes an identity upon taking Euler characteristics.

Since M is compact, one can also view $\mathscr{E}(M)$ as the global sections of the cosheaf of compactly supported sections of $\mathscr{E}$, and then Čech homology gives us a similar expression.

The determinant functor from vector spaces to itself takes sums to tensor products. We thus could imagine that the determinant of the cohomology of $\mathscr{E}(M)$ can be expressed in a local way on the manifold M, but where the direct sums that appear in sheaf theory are replaced by tensor products.

Factorization algebras have the feature that the value on a disjoint union is a tensor product (rather than a direct sum as appears in sheaf theory). That is, factorization algebras are multiplicative versions of cosheaves.

It is therefore natural to express that the determinant of the cohomology of $\mathscr{E}(M)$ can be realized as global sections of a factorization algebra, just as $\mathscr{E}(M)$ is global sections of a cosheaf.

It turns out that this is the case.

12.7.0.1 Lemma. *Let $\mathscr{E}$ be any elliptic complex on a compact manifold M. Let us form the free cotangent theory to the Abelian elliptic Lie algebra $\mathscr{E}[-1]$. This cotangent theory has elliptic complex of fields $\mathscr{E} \oplus \mathscr{E}^![-1]$.*

Let $\mathrm{Obs}^q_{\mathscr{E}}$ denote the factorization algebra of observables of this theory. Then, there is a quasi-isomorphism

$$H^*(\mathrm{Obs}^q_{\mathscr{E}}(M)) = \det H^*(\mathscr{E}(M))[d]$$

where d is equal to the Euler characteristic of $H^*(\mathscr{E}(M))$ modulo 2.

Recall that by $\det H^*(\mathscr{E}(M))$ we mean

$$\det H^*(\mathscr{E}(M)) = \otimes_i \left\{ \det H^i(\mathscr{E}(M)) \right\}^{(-1)^i}.$$

This lemma therefore states that the cohomology of global observables of the theory is the determinant of the cohomology of $\mathscr{E}(M)$, with its natural $\mathbb{Z}/2$ grading. The proof of this lemma, although easy, will be given at the end of this section.

This lemma shows that the factorization algebra $\mathrm{Obs}^q_{\mathscr{E}}$ is a local version of the determinant of the cohomology of $\mathscr{E}(M)$. One can then ask for a local version of the index. Suppose that $\mathcal{L}$ is a local L_∞ algebra on M which acts linearly on $\mathscr{E}$. Then, as we have seen, the precosheaf of L_∞-algebras given by compactly supported sections $\mathcal{L}_c$ of $\mathcal{L}$ acts on the factorization algebra Obs^q. We have also seen that, up to coherent homotopies which respect the factorization algebra structure, the action of $\mathcal{L}_c(U)$ on $\mathrm{Obs}^q(U)$ is by a character α times the identity matrix.

12.7.0.2 Definition. *In this situation, the* local index *is the multiplicative $\mathcal{L}_c$-equivariant factorization algebra $\mathrm{Obs}^q_{\mathscr{E}}$.*

This makes sense, because as we have seen, the action of $\mathcal{L}(M)$ on $\mathrm{Obs}^q_{\mathscr{E}}(M)$ is the same data as the character of the $\mathcal{L}(M)$ action on $\mathscr{E}(M)$, that is, the index.

Theorem 12.6.0.1 tells us that the multiplicative action of $\mathcal{L}_c$ on $\mathrm{Obs}^q_{\mathscr{E}}$ is through the character α of $\mathcal{L}_c$, which is also the obstruction to lifting the action to an inner action.

12.7.1. Proof of lemma 12.7.0.1. Before we give the (simple) proof, we should clarify some small points. Recall that for a free theory, there are two different versions of quantum observables we can consider. We can take our observables to be polynomial functions on the space of fields, and not introduce the formal parameter $\hbar$; or we can take our observables to be formal power series on the space of fields, in which case one needs to introduce the parameter $\hbar$. These two objects encode the same information: the second construction is obtained by applying the Rees construction to the first construction. We will give the proof for the first (polynomial) version of quantum observables. A similar statement holds for the second (power series) version, but one needs to invert $\hbar$ and tensor the determinant of cohomology by $\mathbb{C}((\hbar))$.

Globally, polynomial quantum observables can be viewed as the space $P(\mathscr{E}(M))$ of polynomial functions on $\mathscr{E}(M)$, with a differential which is a sum of the linear differential Q on $\mathscr{E}(M)$ with the BV operator. Let us compute the cohomology by a spectral sequence associated to a filtration of $\mathrm{Obs}^q_{\mathscr{E}}(M)$. The filtration is the obvious increasing filtration obtained by declaring that

$$F^i \,\mathrm{Obs}^q_{\mathscr{E}}(M) = \mathrm{Sym}^{\leq i}(\mathscr{E}(M) \oplus \mathscr{E}^!(M)[-1])^\vee.$$

The first page of this spectral sequence is cohomology of the associated graded. The associated graded is simply the symmmetric algebra

$$H^* \,\mathrm{Gr}\,\mathrm{Obs}^q_{\mathscr{E}}(M) = \mathrm{Sym}^*\left(H^*(\mathscr{E}(M))^\vee \oplus H^*(\mathscr{E}^!(M)[-1])^\vee\right).$$

The differential on this page of the spectral sequence comes from the BV operator associated to the non-degenerate pairing between $H^*(\mathscr{E}(M))$ and $H^*(\mathscr{E}^!(M))[-1]$. Note that $H^*(\mathscr{E}^!(M))$ is the dual to $H^*(\mathscr{E}(M))$.

It remains to show that the cohomology of this secondary differential yields the determinant of $H^*(\mathscr{E}(M))$, with a shift.

We can examine a more general problem. Given any finite-dimensional graded vector space V, we can give the algebra $P(V \oplus V^*[-1])$ of polynomial functions on $V \oplus V^*[-1]$ a BV operator $\triangle$ arising from the pairing between V and $V^*[-1]$. Then, we need to produce an isomorphism

$$H^*(P(V \oplus V^*[-1]), \triangle) \cong \det(V)[d]$$

where the shift d is equal modulo 2 to the Euler characteristic of V.

Sending V to $H^*(P(V \oplus V^*[-1]), \triangle)$ is a functor from the groupoid of finite-dimensional graded vector spaces and isomorphisms between them, to the category of graded vector spaces. It sends direct sums to tensor products. It follows that to check whether or not it returns the determinant, one needs to check that it does in the case that V is a graded line.

Thus, let us assume that $V = \mathbb{C}[k]$ for some $k \in \mathbb{Z}$. We will check that our functor returns $V[1]$ if k is even and V^* if k is odd. Thus, viewed as a $\mathbb{Z}/2$ graded line, our functor returns $\det V$ with a shift by the Euler characteristic of V.

To check this, note that
$$P(V \oplus V^*[-1]) = \mathbb{C}[x, y]$$
where x is of cohomological degree k and y is of degree $-1-k$. The BV operator is
$$\triangle = \frac{\partial}{\partial x}\frac{\partial}{\partial y}.$$
A simple calculation shows that the cohomology of this complex is 1 dimensional, spanned by x if k is odd and by y if k is even. Since x is a basis of V^* and y is a basis of V, this completes the proof.

12.8. The partition function and the quantum Noether theorem

Our formulation of the quantum Noether theorem goes beyond a statement just about symmetries (in the classical sense of the word). It also involves deformations, which are symmetries of cohomological degree 1, as well as symmetries of other cohomological degree. Thus, it has important applications when we consider families of field theories.

The first application we will explain is that the quantum Noether theorem leads to a definition of the *partition function* of a perturbative field theory.

Suppose we have a family of field theories which depends on a formal parameter c, the coupling constant. (Everything we will say will work when the family depends on a number of formal parameters, or indeed on a pro-nilpotent dg algebra). For example, we could start with a free theory and deform it to an interacting theory. An example of such a family of scalar field theories is given by the action functional
$$S(\phi) = \int \phi(\triangle + m^2)\phi + c\phi^4.$$
We can view such a family of theories as being a *single* theory – in this case the free scalar field theory – with an action of the Abelian L_∞ algebra $\mathbb{C}[1]$. Indeed, by definition, an action of an L_∞ algebra $\mathfrak{g}$ on a theory is a family of theories over the dg ring $C^*(\mathfrak{g})$ which specializes to the original theory upon reduction by the maximal ideal $C^{>0}(\mathfrak{g})$.

We have seen (lemma **??**) that actions of $\mathfrak{g}$ on a theory are the same thing as actions of the local Lie algebra $\Omega^*_X \otimes \mathfrak{g}$. In this way, we see that a family of theories over the base ring $\mathbb{C}[[c]]$ is the same thing as a single field theory with an action of the local abelian L_∞ algebra $\Omega^*_X[-1]$.

Here is our definition of the partition function. We will give the definition in a general context, for a field theory acted on by a local L_∞ algebra; afterwards, we will analyze what it means for a family of field theories depending on a formal parameter c.

The partition function is only defined for field theories with some special properties. Suppose we have a theory on a compact manifold M, described classically by a local L_∞ algebra $\mathcal{M}$ with an invariant pairing of degree -3. Suppose that $H^*(\mathcal{M}(M)) = 0$; geometrically, this means we are perturbing around an solated solution to the equations of motion on the compact manifold M. This happens, for instance, with a massive scalar field theory.

This assumption implies that $H^*(\mathrm{Obs}^q(M)) = \mathbb{C}[[\hbar]]$. There is a preferred $\mathbb{C}[[\hbar]]$-linear isomorphism which sends the observable $1 \in H^0(\mathrm{Obs}^q(M))$ to the basis vector of $\mathbb{C}$.

Suppose that this field theory is equipped with an inner action of a local L_∞ algebra $\mathcal{L}$. Then, proposition **??** tells us that for any open subset $U \subset M$, the action of $\mathcal{L}_c(U)$ on $\mathrm{Obs}^q(U)$ is homotopically trivialized. In particular, since M is compact, the action of $\mathcal{L}(M)$ on $\mathrm{Obs}^q(M)$ is homotopically trivialized.

A theory with an $\mathcal{L}$-action is the same as a family of theories over $B\mathcal{L}$. The complex $\mathrm{Obs}^q(M)^{\mathcal{L}(M)}$ of $\mathcal{L}(M)$-equivariant observables should be interpreted as the $C^*(\mathcal{L}(M))$-module of sections of the family of observables over $B\mathcal{L}(M)$.

Since the action of $\mathcal{L}(M)$ is trivialized, we have a quasi-isomorphism of $C^*(\mathcal{L})[[\hbar]]$-modules

$$\mathrm{Obs}^q(M)^{\mathcal{L}(M)} \simeq \mathrm{Obs}^q(M) \otimes C^*(\mathcal{L}).$$

Since $\mathrm{Obs}^q(M)$ is canonically quasi-isomorphic to $\mathbb{C}[[\hbar]]$, we get a quasi-isomorphism

$$\mathrm{Obs}^q(M)^{\mathcal{L}(M)} \simeq C^*(\mathcal{L})[[\hbar]].$$

12.8.0.1 Definition. *The partition function is the element in $C^*(\mathcal{L})[[\hbar]]$ which is the image of the observable $1 \in \mathrm{Obs}^q(M)^{\mathcal{L}(M)}$.*

Another way to interpret this is as follows. The fact that $\mathcal{L}$ acts linearly on our theory implies that the family of global observables over $B\mathcal{L}(M)$ is equipped with a flat connection. Since we have also trivialized the central fibre, via the quasi-isomorphism $\mathrm{Obs}^q(M) \simeq \mathbb{C}[[\hbar]]$, this whole family is trivialized. The observable 1 then becomes a section of the trivial line bundle on $B\mathcal{L}(M)$ with fibre $\mathbb{C}[[\hbar]]$, that is, an element of $C^*(\mathcal{L}(M))[[\hbar]]$.

Let us now analyze some examples of this definition.

Example: Let us first see what this definition amounts to when we work with one-dimensional topological field theories, which (in our formalism) are encoded by associative algebras.

Thus, suppose that we have a one-dimensional topological field theory, whose factorization algebra is described by an associative algebra A. We will view A as a field theory on S^1. For any interval $I \subset S^1$, observables of theory are A, but observables on S^1 are the Hochschild homology $HH_*(A)$.

Suppose that $\mathfrak{g}$ is a Lie algebra with an inner action on A, given by a Lie algebra homomorphism $\mathfrak{g} \to A$. The analog of proposition **??** in this situation is the statement that that the action of $\mathfrak{g}$ on the Hochschild homology of A is trivialized. At the level of HH_0, this is clear, as $HH_0(A) = A/[A, A]$ and bracketing with any element of A clearly acts by zero on $A/[A, A]$.

The obstruction to extending an action of $\Omega^*_X[-1]$ to an inner action is an element in $H^1(C^*_{red,loc}(\Omega^*_X[-1]))$. The D-module formulation **??** of the complex of local cochains of a local Lie algebra gives us a quasi-isomorphism

$$C^*_{red,loc}(\Omega^*_X[-1]) \cong c\Omega^*(X)[[c]][d]$$

where $d = \dim X$ and c is a formal parameter. Therefore, there are no obstructions to lifting an action of $\Omega^*_X[-1]$ to an inner action. However, there is an ambiguity in giving such a lift, coming from the space $cH^d(X)[[c]]$.

Let us suppose that we have a field theory with an inner action of $\Omega^*_X[-1]$. Let Obs^q denote observables of this theory. Then, Noether's theorem gives us a map of factorization algebras

$$\widehat{U}^{BD}(\Omega^*_{X,c}) \to \mathrm{Obs}^q$$

where Obs^q denotes the factorization algebra of observables of our theory, and $\widehat{U}^{BD}(\Omega^*_X[-1])$ is the completed BD envelope factorization algebra of Ω^*_X. Since $\Omega^*_X[-1]$ is abelian, then the BD envelope is simply a completed symmetric algebra:

$$\widehat{U}^{BD}(\Omega^*_c(U)[-1]) = \widehat{\mathrm{Sym}}^*(\Omega^*_c(U))[[\hbar]].$$

In particular, if M is a compact manifold, then we have a natural isomorphism

$$\widehat{\mathrm{Sym}}^*(H^*(M))[[\hbar]] = H^*\left(\widehat{U}^{BD}(\Omega^*_c(M)[-1])\right).$$

Applying Noether's theorem, we get a map

$$\widehat{\mathrm{Sym}}^*(H^0(M))[[\hbar]] \to H^0(\mathrm{Obs}^q(M)).$$

APPENDIX A

Background

A.1. Lie algebras and L_∞ algebras

Lie algebras, and their homotopical generalization L_∞ algebras, appear throughout this book in a variety of contexts. It might surprise the reader that we never use their representation theory or almost any aspects emphasized in textbooks on Lie theory. Instead, we primarily use dg Lie algebras as a convenient language for formal derived geometry. In this section, we overview homological constructions with dg Lie algebras, and in the following section, we overview the relationship with derived geometry.

We use these ideas in the following settings.

- We use the Chevalley-Eilenberg complex to construct a large class of factorization algebras, via the *factorization envelope* of a sheaf of dg Lie algebras. This class includes the observables of free field theories and the Kac-Moody vertex algebras.
- We use the Lie-theoretic approach to deformation functors to motivate our approach to classical field theory.
- We introduce the notion of a *local* Lie algebra to capture the symmetries of a field theory and prove generalizations of Noether's theorem.

We also use Lie algebras in the construction of gauge theories in the usual way.

A.1.1. Dg Lie algebras and L_∞ algebras. We now quickly extend and generalize homologically the notion of a Lie algebra. Our base ring will now be a commutative algebra R over a characteristic zero field $\mathbb{K}$, and we encourage the reader to keep in mind the simplest case: where $R = \mathbb{R}$ or $\mathbb{C}$. Of course, one can generalize the setting considerably, with a little care, by working in a symmetric monoidal category (with a linear flavor); the cleanest approach is to use operads.

Before introducing L_∞ algebras, we treat the simplest homological generalization.

A.1.1.1 Definition. *A* dg Lie algebra *over R is a $\mathbb{Z}$-graded R-module $\mathfrak{g}$ such that*

(1) *there is a differential*

$$\cdots \xrightarrow{d} \mathfrak{g}^{-1} \xrightarrow{d} \mathfrak{g}^0 \xrightarrow{d} \mathfrak{g}^1 \to \cdots$$

making $(\mathfrak{g}, d)$ *into a dg R-module;*

(2) *there is a bilinear bracket* $[-,-] : \mathfrak{g} \otimes_R \mathfrak{g} \to \mathfrak{g}$ *such that*
- $[x,y] = -(-1)^{|x||y|}[y,x]$ *(graded antisymmetry),*
- $d[x,y] = [dx,y] + (-1)^{|x|}[x,dy]$ *(graded Leibniz rule),*
- $[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[y,[x,z]]$ *(graded Jacobi rule),*

where $|x|$ *denotes the cohomological degree of* $x \in \mathfrak{g}$.

In other words, a dg Lie algebra is an algebra over the operad *Lie* in the category of dg R-modules. In practice — and for the rest of the section — we require the graded pieces $\mathfrak{g}^k$ to be projective R-modules so that we do not need to worry about the tensor product or taking duals.

Here are several examples.

(a) We construct the dg analog of $\mathfrak{gl}_n$. Let (V, d_V) be a cochain complex over $\mathbb{K}$. Let $\mathrm{End}(V) = \oplus_n \mathrm{Hom}^n(V,V)$ denote the graded vector space where Hom^n consists of the linear maps that shift degree by n, equipped with the differential

$$d_{\mathrm{End}\, V} = [d_V, -] : f \mapsto d_V \circ f - (-1)^{|f|} f \circ d_V.$$

The commutator bracket makes $\mathrm{End}(V)$ a dg Lie algebra over $\mathbb{K}$.

(b) For M a smooth manifold and $\mathfrak{g}$ an ordinary Lie algebra (such as $su(2)$), the tensor product $\Omega^*(M) \otimes_{\mathbb{R}} \mathfrak{g}$ is a dg Lie algebra where the differential is simply the exterior derivative and the bracket is

$$[\alpha \otimes x, \beta \otimes y] = \alpha \wedge \beta \otimes [x,y].$$

We can view this dg Lie algebra as living over $\mathbb{K}$ or over the commutative dg algebra $\Omega^*(M)$. This example appears naturally in the context of gauge theory.

(c) For X a simply-connected topological space, let $\mathfrak{g}_X^{-n} = \pi_{1+n}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and use the Whitehead product to provide the bracket. Then $\mathfrak{g}_X$ is a dg Lie algebra with zero differential. This example appears naturally in rational homotopy theory.

We now introduce a generalization where we weaken the Jacobi rule on the brackets in a systematic way. After providing the (rather convoluted) definition, we sketch some motivations.

A.1.1.2 Definition. *An* L_∞ algebra over R *is a* $\mathbb{Z}$*-graded, projective R-module* $\mathfrak{g}$ *equipped with a sequence of multilinear maps of cohomological degree* $2-n$

$$\ell_n : \underbrace{\mathfrak{g} \otimes_R \cdots \otimes_R \mathfrak{g}}_{n \ \ times} \to \mathfrak{g},$$

with $n = 1, 2, \ldots$, satisfying the following properties.

(1) *Each bracket ℓ_n is graded-antisymmetric, so that*
$$\ell_n(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = -(-1)^{|x_i||x_{i+1}|}\ell_n(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)$$
for every n-tuple of elements and for every i between 1 and $n-1$.

(2) *Each bracket ℓ_n satisfies the n-Jacobi rule, so that*
$$0 = \sum_{k=1}^{n}(-1)^k \sum_{\substack{i_1<\cdots<i_k \\ j_{k+1}<\cdots<j_n \\ \{i_1,\ldots,j_n\}=\{1,\ldots,n\}}} (-1)^\varepsilon \ell_{n-k+1}(\ell_k(x_{i_1}, \ldots, x_{i_k}), x_{j_{k+1}}, \ldots, x_{j_n}).$$
Here $(-1)^\varepsilon$ denotes the sign for the permutation
$$\begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ i_1 & \cdots & i_k & j_{k+1} & \cdots & j_n \end{pmatrix}$$
acting on the element $x_1 \otimes \cdots \otimes x_n$ given by the alternating-Koszul sign rule, where the transposition $ab \mapsto ba$ acquires sign $-(-1)^{|a||b|}$.

For small values of n, we recover familiar relations. For example, the 1-Jacobi rule says that $\ell_1 \circ \ell_1 = 0$. In other words, ℓ_1 is a differential! Momentarily, let's denote ℓ_1 by d and ℓ_2 by the bracket $[-,-]$. The 2-Jacobi rule then says that
$$-[dx_1, x_2] + [dx_2, x_1] + d[x_1, x_2] = 0,$$
which encodes the graded Leibniz rule. Finally, the 3-Jacobi rule rearranges to
$$\begin{aligned}&[[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2] \\ &\qquad = d\ell_3(x_1, x_2, x_3) + \ell_3(dx_1, x_2, x_3) + \ell_3(dx_2, x_3, x_1) + \ell_3(dx_3, x_1, x_2).\end{aligned}$$
In short, $\mathfrak{g}$ does not satisfy the usual Jacobi rule *on the nose* but the failure is described by the other brackets. In particular, at the level of cohomology, the usual Jacobi rule *is* satisfied.

Example: There are numerous examples of L_∞ algebras throughout the book, but many are simply dg Lie algebras spiced with analysis. We describe here a small, algebraic example of interest in topology and elsewhere (see, for instance, [**Hen08**], [**BC04**], [**BR10**]). The *String Lie 2-algebra* $string(n)$ is the graded vector space $so(n) \oplus \mathbb{R}\beta$, where β has degree 1 — thus $string(n)$ is concentrated in degrees 0 and 1 — equipped with two nontrivial brackets:
$$\ell_2(x, y) = \begin{cases} [x, y], & x, y \in so(n) \\ 0, & x = \beta \end{cases}$$
$$\ell_3(x, y, z) = \mu(x, y, z)\beta \quad x, y, z \in so(n),$$
where μ denotes $\langle -, [-,-]\rangle$, the canonical (up to scale) 3-cocycle on $so(n)$ arising from the Killing form. This L_∞ algebra arises as a model for the "Lie algebra" of $String(n)$,

which itself appears in various guises (as a topological group, as a smooth 2-group, or as a more sophisticated object in derived geometry). ◊

There are two important cochain complexes associated to an L_∞ algebra, which generalize the two Chevalley-Eilenberg complexes we defined earlier.

A.1.1.3 Definition. *For* $\mathfrak{g}$ *an* L_∞ *algebra, the* Chevalley-Eilenberg complex for homology $C_*\mathfrak{g}$ *is the dg cocommutative coalgebra*

$$\mathrm{Sym}_R(\mathfrak{g}[1]) = \bigoplus_{n=0}^{\infty} \left((\mathfrak{g}[1])^{\otimes n}\right)_{S_n}$$

equipped with the coderivation d *whose restriction to cogenerators* $d_n : \mathrm{Sym}^n(\mathfrak{g}[1]) \to \mathfrak{g}[1]$ *are precisely the higher brackets* ℓ_n*.*

Remark: The coproduct $\Delta : C_*\mathfrak{g} \to C_*\mathfrak{g} \otimes_R C_*\mathfrak{g}$ is given by running over the natural ways that one can "break a monomial into two smaller monomials." Namely,

$$\Delta(x_1 \cdots x_n) = \sum_{\sigma \in S_n} \sum_{1 \leq k \leq n-1} (x_{\sigma(1)} \cdots x_{\sigma(k)}) \otimes (x_{\sigma(k+1)} \cdots x_{\sigma(n)}).$$

A coderivation respects the coalgebra analog of the Leibniz property, and so it is determined by its behavior on cogenerators. ◊

This coalgebra $C_*\mathfrak{g}$ conveniently encodes all the data of the L_∞ algebra $\mathfrak{g}$. The coderivation d puts all the brackets together into one operator, and the equation $d^2 = 0$ encodes all the higher Jacobi relations. It also allows for a concise definition of a map between L_∞ algebras.

A.1.1.4 Definition. *A* map of L_∞ algebras $F : \mathfrak{g} \rightsquigarrow \mathfrak{h}$ *is given by a map of dg cocommutative coalgebras* $F : C_*\mathfrak{g} \to C_*\mathfrak{h}$*.*

Note that a map of L_∞ algebras is *not* determined just by its behavior on $\mathfrak{g}$, which is why we use $\rightsquigarrow$ to denote such a morphism. Unwinding the definition above, one discovers that such a morphism consists of a linear map $\mathrm{Sym}^n(\mathfrak{g}[1]) \to \mathfrak{h}$ for each n, satisfying compatibility conditions ensuring that we get a map of coalgebras.

To define the other Chevalley-Eilenberg complex $C^*\mathfrak{g}$, we use the graded linear dual of $\mathfrak{g}$,

$$\mathfrak{g}^\vee = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_R(\mathfrak{g}^n, R)[n],$$

which is the natural notion of dual in this context.

A.1.1.5 Definition. *For $\mathfrak{g}$ an L_∞ algebra, the* Chevalley-Eilenberg complex for cohomology *$C^*\mathfrak{g}$ is the dg commutative algebra*

$$\widehat{\mathrm{Sym}}_R(\mathfrak{g}[1]^\vee) = \prod_{n=0}^{\infty} \left((\mathfrak{g}[1]^\vee)^{\otimes n} \right)_{S_n}$$

equipped with the derivation d whose Taylor coefficients $d_n : \mathfrak{g}[1]^\vee \to \mathrm{Sym}^n(\mathfrak{g}[1]^\vee)$ are dual to the higher brackets ℓ_n.

We emphasize that this dg algebra is *completed* with respect to the filtration by powers of the ideal generated by $\mathfrak{g}[1]^\vee$. This filtration will play a crucial role in the setting of deformation theory.

A.1.2. References. We highly recommend [Get09] for an elegant and efficient treatment of L_∞ algebras (over $\mathbb{K}$) (and simplicial sets and also how these constructions fit together with deformation theory). The book of Kontsevich and Soibelman [KS] provides a wealth of examples, motivation, and context.

A.2. Derived deformation theory

In physics, one often studies very small perturbations of a well-understood system, wiggling an input infinitesimally or deforming an operator by a small amount. Asking questions about how a system behaves under small changes is ubiquitous in mathematics, too, and there is an elegant formalism for such problems in the setting of algebraic geometry, known as *deformation theory*. Here we will give a very brief sketch of *derived* deformation theory, where homological ideas are mixed with classical deformation theory.

A major theme of this book is that perturbative aspects of field theory — both classical and quantum — are expressed cleanly and naturally in the language of derived deformation theory. In particular, many constructions from physics, like the the Batalin-Vilkovisky formalism, obtain straightforward interpretations. Moreover, derived deformation theory suggests how to rephrase standard results in concise, algebraic terms and also suggests how to generalize these results substantially (see, for instance, the discussion on Noether's theorem).

In this section, we begin with a quick overview of formal deformation theory in algebraic geometry. We then discuss its generalization in derived algebraic geometry. Finally, we explain the powerful relationship between deformation theory and L_∞ algebras, which we exploit throughout the book.

A.2.1. The formal neighborhood of a point. Let $\mathcal{S}$ denote some category of spaces, such as smooth manifolds or complex manifolds or schemes. The Yoneda lemma implies we can understand any particular space $X \in \mathcal{S}$ by understanding how other spaces $Y \in \mathcal{S}$ map into X. That is, the functor represented by X,

$$\begin{array}{rccc} h_X : & \mathcal{S}^{op} & \to & Sets \\ & Y & \mapsto & \mathcal{S}(Y,X) \end{array},$$

knows everything about X as a space of type $\mathcal{S}$. We call h_X the *functor of points of* X, and this functorial perspective on geometry will guide our work below. Although abstract at first acquaintance, this perspective is especially useful for thinking about general features of geometry.

Suppose we want to describe what X looks like near some point $p \in X$. Motivated by the perspective of functor of points, we might imagine describing "X near p" by some kind of functor. The input category ought to capture all possible "small neighborhoods of a point" permitted in $\mathcal{S}$, so that we can see how such models map into X near p. We now make this idea precise in the setting of algebraic geometry.

Let $\mathcal{S} = Sch_{\mathbb{C}}$ denote the category of schemes over $\mathbb{C}$. Every such scheme X consists of a topological space X_{top} equipped with a sheaf of commutative $\mathbb{C}$-algebras $\mathscr{O}_X$ (satisfying various conditions we will not specify). We interpret the algebra $\mathscr{O}_X(U)$ on the open set U as the "algebra of functions on U." Every commutative $\mathbb{C}$-algebra R determines a scheme $\operatorname{Spec} R$ where the prime ideals of R provide the set of points of the topological space $(\operatorname{Spec} R)_{top}$ and where the stalk of $\mathscr{O}$ at a prime ideal $\mathfrak{P}$ is precisely the localization of R with respect to $R - \mathfrak{P}$. We call such a scheme $\operatorname{Spec} R$ an *affine scheme*. By definition, every scheme admits an open cover by affine schemes.

It is a useful fact that the functor of points h_X of a scheme X is determined by its behavior on the subcategory $Aff_{\mathbb{C}}$ of affine schemes. By construction, $Aff_{\mathbb{C}}$ is the opposite category to $CAlg_{\mathbb{C}}$, the category of commutative $\mathbb{C}$-algebras. Putting these facts together, we know that every scheme X provides a functor from $CAlg_{\mathbb{C}}$ to $Sets$. Here are two examples.

Example: Consider the polynomial $q(x,y) = x^2 + y^2 - 1$. The functor

$$\begin{array}{rccc} h_X : & CAlg_{\mathbb{C}} & \to & Sets \\ & R & \mapsto & \{(a,b) \in R^2 \mid 0 = q(a,b) = a^2 + b^2 - 1\} \end{array}$$

corresponds to the affine scheme $\operatorname{Spec} S$ for the algebra $S = \mathbb{C}[x,y]/(q)$. This functor simply picks out solutions to the equation $q(x,y) = 0$ in the algebra R, which we might call the "unit circle" in R^2. Generalizing, we see that any system of polynomials (or ideal in an algebra) defines a similar functor of "solutions to the system of equations." $\lozenge$

Example: Consider the scheme SL_2, viewed as the functor

$$\begin{array}{rccc} SL_2 : & CAlg_{\mathbb{C}} & \to & Sets \\ & R & \mapsto & \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in R \text{ such that } 1 = ad - bc \right\}. \end{array}$$

Note that $SL_2(\mathbb{C})$ is precisely the set that we usually mean. One can check as well that this functor factors through the category of groups. $\Diamond$

The notion of "point" in this category is given by $\operatorname{Spec}\mathbb{C}$, which is the locally ringed space given by a one-point space $\{*\}$ equipped with $\mathbb{C}$ as its algebra of functions. A *point in the scheme X* is then a map $p : \operatorname{Spec}\mathbb{C} \to X$. Every point is contained in some affine patch $U \cong \operatorname{Spec} R \subset X$, so it suffices to understand points in affine schemes. It is now possible to provide an answer to the question, "What are the affine schemes that look like small thickenings of a point?"

A.2.1.1 Definition. *A commutative* $\mathbb{C}$*-algebra* A *is* artinian *if* A *is finite-dimensional as a* $\mathbb{C}$*-vector space. A* local algebra A *with unique maximal ideal* $\mathfrak{m}$ *is artinian if and only if there is some integer* n *such that* $\mathfrak{m}^n = 0$*.*

Any local artinian algebra $(A, \mathfrak{m})$ provides a scheme $\operatorname{Spec} A$ whose underlying topological space is a point but whose scheme structure has "infinitesimal directions" in the sense that every function $f \in \mathfrak{m}$ is "small" because $f^n = 0$ for some n. Let $Art_{\mathbb{C}}$ denote the category of local artinian algebras, which we will view as the category encoding "small neighborhoods of a point."

Remark: Hopefully it seems reasonable to choose $Art_{\mathbb{C}}$ as a model for "small neighborhoods of a point." There are other approaches imaginable but this choice is quite useful. In particular, the most obvious topology for schemes — the Zariski topology — is quite coarse, so that open sets are large and hence do not reflect the idea of "zooming in near the point." Instead, we use schemes whose space is just a point but have interesting but tractable algebra. $\Diamond$

A point $p : \operatorname{Spec}\mathbb{C} \to \operatorname{Spec} R$ corresponds to a map of algebras $P : R \to \mathbb{C}$. Every local artinian algebra $(A, \mathfrak{m})$ has a distinguished map $Q : A \to A/\mathfrak{m} \cong \mathbb{C}$. Given a point p in $\operatorname{Spec} R$, we obtain a functor

$$\begin{array}{cccc} h_p : & Art_{\mathbb{C}} & \to & Sets \\ & (A, \mathfrak{m}) & \mapsto & \{F : R \to A \mid P = Q \circ F\} \end{array} .$$

Geometrically, this condition on ϕ means p is the composition $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec} A \overset{\operatorname{Spec} F}{\to} \operatorname{Spec} R$. The map F thus describes some way to "extend infinitesimally" away from the point p in X. A concrete example is in order.

Example: Our favorite point in SL_2 is given by the identity element $\mathbb{1}$. Let $h_{\mathbb{1}}$ denote the associated functor of artinian algebras. We can describe the tangent space $T_{\mathbb{1}} SL_2$ using it, as follows. Consider the artinian algebra $\mathbb{D} = \mathbb{C}[\varepsilon]/(\varepsilon^2)$, often called the *dual numbers*.

Then

$$\begin{aligned} h_{\mathbb{1}}(\mathbb{D}) &= \left\{ M = \begin{pmatrix} 1+s\varepsilon & t\varepsilon \\ u\varepsilon & 1+v\varepsilon \end{pmatrix} \;\middle|\; \begin{matrix} s,t,u,v \in \mathbb{C} \text{ and} \\ 1 = (1+s\varepsilon)(1+v\varepsilon) - tu\varepsilon^2 = 1 + (s+v)\varepsilon \end{matrix} \right\} \\ &\cong \{N \in M_2(\mathbb{C}) \mid \operatorname{Tr} N = 0\} \\ &= \mathfrak{sl}_2(\mathbb{C}), \end{aligned}$$

where the isomorphism is given by $M = \mathbb{1} + \varepsilon N$. Thus, we have recovered the underlying set of the Lie algebra. $\diamondsuit$

For any point p in a scheme X, the set $h_p(\mathbb{D})$ is the tangent space to p in X. By considering more complicated artinian algebras, one can study the "higher order jets" at p. We say that h_p describes the *formal neighborhood* of p in X. The following proposition motivates this terminology.

A.2.1.2 Proposition. *Let $P : R \to \mathbb{C}$ be a map of algebras (i.e., we have a point $p : \operatorname{Spec}\mathbb{C} \to \operatorname{Spec} R$). Then*

$$h_p(A) = CAlg_{\mathbb{C}}(\widehat{R}_p, A),$$

where

$$\widehat{R}_p = \varprojlim R/\mathfrak{m}_p^n$$

is the completed local ring given by the inverse limit over powers of $\mathfrak{m}_p = \ker P$, the maximal ideal given by the functions vanishing at p.

In other words, the functor h_p is not represented by a local artinian algebra (unless R is artinian), but it is represented inside the larger category $CAlg_{\mathbb{C}}$. When R is noetherian, the ring $\widehat{R}_p$ is given by an inverse system of local artinian algebras, so we say h_p is *pro*-represented. When R is a regular ring (such as a polynomial ring over $\mathbb{C}$), $\widehat{R}_p$ is isomorphic to formal power series. This important example motivates the terminology of *formal neighborhood*.

There are several properties of such a functor h_p that we want to emphasize, as they guide our generalization in the next section. First, by definition, $h_p(\mathbb{C})$ is simply a point, namely the point p. Second, we can study h_p in stages, by a process we call *artinian induction*. Observe that every local artinian algebra $(A, \mathfrak{m})$ is equipped with a natural filtration

$$A \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \cdots \supset \mathfrak{m}^n = 0.$$

Thus, every local artinian algebra can be constructed iteratively by a sequence of *small* extensions, namely a short exact sequence of vector spaces

$$0 \to I \hookrightarrow B \xrightarrow{f} A \to 0$$

where $f : B \to A$ is a surjective map of algebras and I is an ideal in B such that $\mathfrak{m}_B I = 0$. We can thus focus on understanding the maps $h_p(f) : h_p(B) \to h_p(A)$, which are simpler to analyze. In summary, h_p is completely determined by how it behaves with respect to small extensions.

A third property is categorical in nature. Consider a pullback of local artinian algebras

$$\begin{tikzcd} B \times_A C \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & A \end{tikzcd}$$

and note that $B \times_A C$ is local artinian as well. Then the natural map

$$h_p(B \times_A C) \to h_p(B) \times_{h_p(A)} h_p(C)$$

is surjective — in fact, it is an isomorphism. (This property will guide us in the next subsection.)

As an example, we describe how to study small extensions for the model case. Let $(R, \mathfrak{m}_R)$ be a complete local ring with residue field $R/\mathfrak{m}_R \cong \mathbb{C}$ and with finite-dimensional *tangent space* $T_R = (\mathfrak{m}_R/\mathfrak{m}_R^2)^\vee$. Consider the functor $h_R : A \mapsto CAlg(R, A)$, which describes the formal neighborhood of the closed point in Spec R. The following proposition provides a tool for understanding the behavior of h_R on small extensions.

A.2.1.3 Proposition. *For every small extension*

$$0 \to I \hookrightarrow B \xrightarrow{f} A \to 0,$$

there is a natural exact sequence of sets

$$0 \to T_R \otimes_{\mathbb{C}} I \to h_R(B) \xrightarrow{f \circ -} h_R(A) \xrightarrow{ob} O_R \otimes I,$$

where exact *means that a map $\phi \in h_R(A)$ lifts to a map $\tilde{\phi} \in h_R(B)$ if and only if $ob(\phi) = 0$ and the space of liftings is an affine space for the vector space $T_R \otimes_{\mathbb{C}} I$.*

Here *ob* denotes the *obstruction* to lifting maps, and O_R is a set where an obstruction lives. An obstruction space O_R only depends on the algebra R, not on the small extension. One can construct an obstruction space as follows. If $d = \dim_{\mathbb{C}} T_R$, there is a surjection of algebras

$$r : S = \mathbb{C}[[x_1, \dots, x_d]] \to R$$

such that $J = \ker r$ satisfies $J \subset \mathfrak{m}_S^2$, where $\mathfrak{m}_S = (x_1, \dots, x_d)$ is the maximal ideal of S. In other words, Spec R can be embedded into the formal neighborhood of the origin in $\mathbb{A}^d$, and minimally, in some sense. Then O_R is $(J/\mathfrak{m}_S J)^\vee$. For a proof of the proposition, see Theorem 6.1.19 of [FGI+05].

This proposition hints that something homotopical lurks behind the scenes, and that the exact sequence of sets is the truncation of a longer sequence. For a discussion of these ideas and the modern approach to deformation theory, we highly recommend the 2010 ICM talk of Lurie [**Lur10**].

A.2.1.1. *References.* The textbook of Eisenbud and Harris [**EH00**] is a lovely introduction to the theory of schemes, full of examples and motivation. There is an extensive discussion of the functor of points approach to geometry, carefully compared to the locally ringed space approach. For an introduction to deformation theory, we recommend the article of Fantechi and Göttsche in [**FGI$^+$05**]. Both texts provide extensive references to the literature.

A.2.2. Formal moduli spaces. The functorial perspective on algebraic geometry suggests natural generalizations of the notion of a scheme by changing the source and target categories. For instance, stacks arise as functors from $CAlg_{\mathbb{C}}$ to the category of groupoids, allowing one to capture the idea of a space "with internal symmetries." It is fruitful to generalize even further, by enhancing the source category from commutative algebras to dg commutative algebras (or simplicial commutative algebras) and by enhancing the target category from sets to simplicial sets. (Of course, one needs to simultaneously adopt a more sophisticated version of category theory, namely ∞-category theory.) This generalization is the subject of derived algebraic geometry, and much of its power arises from the fact that it conceptually integrates geometry, commutative algebra, and homotopical algebra. As we try to show in this book, the viewpoint of derived geometry provides conceptual interpretations of constructions like Batalin-Vilkovisky quantization.

We now outline the derived geometry version of studying the formal neighborhood of a point. Our aim to pick out a class of functors that capture our notion of a formal derived neighborhood.

A.2.2.1 Definition. *An* artinian dg algebra *A is a dg commutative algebra over $\mathbb{C}$ such that*

(1) *each component A^k is finite-dimensional, $\dim_{\mathbb{C}} A^k = 0$ for $k << 0$ and for $k > 0$, and*
(2) *A has a unique maximal ideal $\mathfrak{m}$, closed under the differential, and $A/\mathfrak{m} = \mathbb{C}$.*

Let $dgArt_{\mathbb{C}}$ denote the category of artinian algebras, where morphisms are simply maps of dg commutative algebras.

Note that, as we only want to work with local rings, we simply included it as part of the definition. Note as well that we require A to be concentrated in nonpositive degrees. (This second condition is related to the Dold-Kan correspondence: we want A to correspond to a simplicial commutative algebra.)

We now provide an abstract characterization of a functor that behaves like the formal neighborhood of a point, motivated by our earlier discussion of functors h_p.

A.2.2.2 Definition. *A* formal moduli problem *is a functor*

$$F : dgArt_{\mathbb{C}} \to \text{sSet}$$

such that

(1) *$F(\mathbb{C})$ is a contractible Kan complex,*
(2) *F sends a surjection of dg artinian algebras to a fibration of simplicial sets, and*
(3) *for every pullback diagram in $dgArt$*

$$\begin{tikzcd} B \times_A C \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & A \end{tikzcd}$$

the map $F(B \times_A C) \to F(B) \times_{F(A)} F(C)$ is a weak homotopy equivalence.

Note that since surjections go to fibrations, the strict pullback $F(B) \times_{F(A)} F(C)$ agrees with the homotopy pullback $F(B) \times^h_{F(A)} F(C)$.

We now describe a large class of examples. Let R be a commutative dg algebra over $\mathbb{C}$ whose underlying graded algebra is $\widehat{\mathrm{Sym}} V$, where V is a $\mathbb{Z}$-graded vector space, and whose differential d_R is a degree 1 derivation. It has a unique maximal ideal generated by V. Let h_R denote the functor into simplicial sets whose n-simplices are

$$h_R(A)_n = \{f : R \to A \otimes \Omega^*(\triangle^n) \mid f \text{ a map of unital dg commutative algebras}\}$$

and whose structure maps arise from those between the de Rham complexes of simplices. Then h_R is a formal moduli problem.

A.2.2.1. *References.* We are modeling our approach on Lurie's, as explained in his ICM talk [**Lur10**] and his paper on deformation theory [**Lur**]. For a discussion of these ideas in our context of field theory, see [**?**].

A.2.3. The role of L_∞ algebras in deformation theory. There is another algebraic source of formal moduli functors — L_∞ algebras — and, perhaps surprisingly, formal moduli functors arising in geometry often manifest themselves in this form. We begin by introducing the Maurer-Cartan equation for an L_∞ algebra $\mathfrak{g}$ and explaining how it provides a formal moduli functor. *This construction is at the heart of our approach to classical field theory.* We then describe several examples from geometry and algebra.

A.2.3.1 Definition. *Let $\mathfrak{g}$ be an L_∞ algebra. The* Maurer-Cartan equation *(or MC equation) is*

$$\sum_{n=1}^{\infty} \frac{1}{n!} \ell_n(\alpha^{\otimes n}) = 0,$$

where α denotes a degree 1 element of $\mathfrak{g}$.

Note that when we consider the dg Lie algebra $\Omega^*(M) \otimes \mathfrak{g}$, with M a smooth manifold and $\mathfrak{g}$ an ordinary Lie algebra, the MC equation becomes the equation

$$d\alpha + \frac{1}{2}[\alpha, \alpha] = 0.$$

A $\mathfrak{g}$-connection $\alpha \in \Omega^1 \otimes \mathfrak{g}$ on the trivial principal G-bundle on M is *flat* if and only if it satisfies the MC equation. (This is the source of the name Maurer-Cartan.)

There are two other perspectives on the MC equation. First, observe that a map of commutative dg algebras $\underline{\alpha} : C^*\mathfrak{g} \to \mathbb{C}$ is determined by its behavior on the generators $\mathfrak{g}^\vee[-1]$ of the algebra $C^*\mathfrak{g}$. Hence $\underline{\alpha}$ is a linear functional of degree 0 on $\mathfrak{g}^\vee[-1]$ — or, equivalently, a degree 1 element α of $\mathfrak{g}$ — that commutes with differentials. This condition $\underline{\alpha} \circ d = 0$ is precisely the MC equation for α. The second perspective uses the coalgebra $C_*\mathfrak{g}$, rather than the algebra $C^*\mathfrak{g}$. A solution to the MC equation α is equivalent to giving a map of cocommutative dg coalgebras $\tilde{\alpha} : \mathbb{C} \to C_*\mathfrak{g}$.

Now observe that L_∞ algebras behave nicely under base change: if $\mathfrak{g}$ is an L_∞ algebra over $\mathbb{C}$ and A is a commutative dg algebra over $\mathbb{C}$, then $\mathfrak{g} \otimes A$ is an L_∞ algebra (over A and, of course, $\mathbb{C}$). Solutions to the MC equation go along for the ride as well. For instance, a solution α to the MC equation of $\mathfrak{g} \otimes A$ is equivalent to both a map of commutative dg algebras $\underline{\alpha} : C^*\mathfrak{g} \to A$ and a map of cocommutative dg coalgebras $\tilde{\alpha} : A^\vee \to C_*\mathfrak{g}$. Again, we simply unravel the conditions of such a map restricted to (co)generators. As maps of algebras compose, solutions play nicely with base change. Thus, we can construct a functor out of the MC solutions.

A.2.3.2 Definition. *For an L_∞ algebra $\mathfrak{g}$, its* Maurer-Cartan functor

$$\mathrm{MC}_\mathfrak{g} : dgArt_\mathbb{C} \to \mathrm{sSet}$$

sends $(A, \mathfrak{m})$ to the simplicial set whose n-simplices are solutions to the MC equation in $\mathfrak{g} \otimes \mathfrak{m} \otimes \Omega^*(\triangle^n)$.

We remark that tensoring with the nilpotent ideal $\mathfrak{m}$ makes $\mathfrak{g} \otimes \mathfrak{m}$ is nilpotent. This condition then ensures that the simplicial set $\mathrm{MC}_\mathfrak{g}(A)$ is a Kan complex [**Hin01**] [**Get09**]. In fact, their work shows the following.

A.2.3.3 Theorem. *The Maurer-Cartan functor* $\mathrm{MC}_\mathfrak{g}$ *is a formal moduli problem.*

In fact, every formal moduli problem is represented — up to a natural notion of weak equivalence — by the MC functor of an L_∞ algebra [**Lur**].

A.2.3.1. *References.* For a clear, systematic introduction with an expository emphasis, we highly recommend Manetti's lecture [**Man09**], which carefully explains how dg Lie algebras relate to deformation theory and how to use them in algebraic geometry. The unpublished book [**KS**] contains a wealth of ideas and examples; it also connects these ideas to many other facets of mathematics. The article of Hinich [**Hin01**] is the original published treatment of derived deformation theory, and it provides one approach to necessary higher category theory. For the relation with L_∞ algebras, we recommend [**Get09**], which contains elegant arguments for many of the ingredients, too. Finally, see Lurie's [**Lur**] for a proof that every formal moduli functor is described by a dg Lie algebra.

APPENDIX B

Functions on spaces of sections

Our focus throughout the book is on the "observables of a field theory," where for us the fields of a field theory are sections of a vector bundle and the observables are polynomial (or power series) functions on these fields. In this appendix, we will use the setting introduced in appendix **??** to give a precise meaning to this notion of observable.

B.1. Classes of functions on the space of sections of a vector bundle

Let M be a manifold and E a graded vector bundle on M. Let $U \subset M$ be an open subset. In this section we will introduce some notation for various classes of functionals on sections $\mathscr{E}(U)$ of E on U. These spaces of functionals will all be differentiable cochain complexes (or pro-cochain complexes) as described in appendix **??**. (In this appendix, however, the differential will always be trivial, so that it is natural to think of these spaces of functionals as differentiable pro-graded vector spaces.)

Recall the following notations:

- $\mathscr{E}(M)$ denotes the vector space of smooth sections of E over M,
- $\mathscr{E}_c(M)$ denotes the vector space of compactly supported smooth sections of E over M,
- $\overline{\mathscr{E}}(M)$ denotes the vector space of distributional sections of E over M, and
- $\overline{\mathscr{E}}_c(M)$ denotes the vector space of compactly supported distributional sections of E over M.

We can view these spaces as living in LCTVS, BVS, CVS, or DVS, as suits us, thanks to the discussion in appendix **??**. In LCTVS, there is a standard isomorphism between the continuous linear dual $\mathscr{E}(M)^*$, equipped with the strong topology, and $\overline{\mathscr{E}}^!_c(M)$, the compactly supported distributional sections of the bundle $E^! = E^\vee \otimes \mathrm{Dens}_M$. Likewise, there is an isomorphism between $\mathscr{E}_c(M)^*$ and $\overline{\mathscr{E}}^!(M)$.

B.1.1. Functions. Given an ordinary vector space V, the symmetric algebra $\mathrm{Sym}\, V^*$ on the dual space V^* provides a natural class of functions on V. Similarly, the completed symmetric algebra $\widehat{\mathrm{Sym}}\, V^*$ describes the formal power series centered at the origin, which is interpreted as functions on the formal neighborhood of the origin in V. We wish to describe the analogs of these constructions when the vector space is $\mathscr{E}(U)$, and hence we need to be careful in our choice of tensor products and ambient category. In the end, we will show that two natural approaches coincide and thus provide our definition.

From the point of view of topological vector spaces, a natural approach is use the completed projective tensor product $\widehat{\otimes}_\pi$ and follow the general recipe for constructing symmetric algebras. Because we will consider other approaches as well, we will call this construction the π-symmetric powers and define it as

$$\mathrm{Sym}^n_\pi\, \mathscr{E}^!_c(U) = \left(\mathscr{E}^!_c(U)^{\widehat{\otimes}_\pi n}\right)_{S_n},$$
$$\mathrm{Sym}^n_\pi\, \overline{\mathscr{E}}^!_c(U) = \left(\overline{\mathscr{E}}^!_c(U)^{\widehat{\otimes}_\pi n}\right)_{S_n},$$

where the subscript S_n denotes the coinvariants with respect to the action of this symmetric group. Then we define the uncompleted π-symmetric algebra as

$$\mathrm{Sym}_\pi\, \mathscr{E}^!_c(U) = \bigoplus_{n=0}^{\infty} \mathrm{Sym}^n_\pi\, \mathscr{E}^!_c(U)$$

and the completed π-symmetric algebra

$$\widehat{\mathrm{Sym}}_\pi \mathscr{E}^!_c(U) = \prod_{n=0}^{\infty} \mathrm{Sym}^n_\pi\, \mathscr{E}^!_c(U).$$

Using the same formulas, one defines $\mathrm{Sym}_\pi\, \overline{\mathscr{E}}^!_c(U)$ and $\widehat{\mathrm{Sym}}_\pi \overline{\mathscr{E}}^!_c(U)$.

If one views $\mathscr{E}^!_c(U)$ and $\overline{\mathscr{E}}^!_c(U)$ as convenient vector spaces, the natural choice is to work with the tensor product $\widehat{\otimes}_\beta$ and then to follow the standard procedure for constructing symmetric algebras. In short, we define the uncompleted β-symmetric algebra as

$$\mathrm{Sym}_\beta\, \mathscr{E}^!_c(U) = \bigoplus_{n=0}^{\infty} \left(\mathscr{E}^!_c(U)^{\widehat{\otimes}_\beta n}\right)_{S_n}$$

and the completed β-symmetric algebra

$$\widehat{\mathrm{Sym}}_\beta \mathscr{E}^!_c(U) = \prod_{n=0}^{\infty} \left(\overline{\mathscr{E}}^!_c(U)^{\widehat{\otimes}_\beta n}\right)_{S_n}.$$

Using the same formulas, one defines $\mathrm{Sym}_\beta\, \overline{\mathscr{E}}^!_c(U)$ and $\widehat{\mathrm{Sym}}_\beta \overline{\mathscr{E}}^!_c(U)$.

Thankfully, these two constructions provide the same differentiable vector spaces, via proposition **??**.

B.1.1.1 Lemma. *As graded differentiable vector spaces, there are isomorphisms*

$$\begin{aligned}
\mathrm{Sym}_\pi \mathscr{E}_c^!(U) &\cong \mathrm{Sym}_\beta \mathscr{E}_c^!(U),\\
\mathrm{Sym}_\pi \overline{\mathscr{E}}^!_c(U) &\cong \mathrm{Sym}_\beta \overline{\mathscr{E}}^!_c(U),\\
\widehat{\mathrm{Sym}}_\pi \mathscr{E}_c^!(U) &\cong \widehat{\mathrm{Sym}}_\beta \mathscr{E}_c^!(U),\\
\widehat{\mathrm{Sym}}_\pi \overline{\mathscr{E}}^!_c(U) &\cong \widehat{\mathrm{Sym}}_\beta \overline{\mathscr{E}}^!_c(U).
\end{aligned}$$

In light of this lemma, we often write $\mathscr{O}(\mathscr{E}(U))$ for $\widehat{\mathrm{Sym}}_\pi \overline{\mathscr{E}}^!_c(U)$, as it is naturally interpreted as the algebra of formal power series on $\mathscr{E}(U)$. (This notation emphasizes the role of the construction rather than its inner workings.) Similarly, we write $\mathscr{O}(\overline{\mathscr{E}}(U))$ for $\widehat{\mathrm{Sym}}_\pi \mathscr{E}_c^!(U)$ and so on for $\mathscr{O}(\mathscr{E}_c(U))$ and $\mathscr{O}(\overline{\mathscr{E}}_c(U))$.

These completed spaces of functionals are all products of the differentiable vector spaces of symmetric powers, and so they are themselves differentiable vector spaces. We will equip all of these spaces of functionals with the structure of a differentiable pro-vector space, induced by the filtration

$$F^i \mathscr{O}(\mathscr{E}(U)) = \prod_{n \geq i} \mathrm{Sym}^i \overline{\mathscr{E}}^!_c(U)$$

(and similarly for $\mathscr{O}(\mathscr{E}_c(U))$, $\mathscr{O}(\overline{\mathscr{E}}(U))$ and $\mathscr{O}(\overline{\mathscr{E}}_c(U))$).

The natural product $\mathscr{O}(\mathscr{E}(U))$ is compatible with the differentiable structure, making $\mathscr{O}(\mathscr{E}(U))$ into a commutative algebra in the multicategory of differentiable graded pro-vector spaces. The same holds for the spaces of functionals $\mathscr{O}(\mathscr{E}_c(U))$, $\mathscr{O}(\overline{\mathscr{E}}(U))$ and $\mathscr{O}(\overline{\mathscr{E}}_c(U))$.

B.1.2. One-forms. Recall that for V is a vector space, we view the formal neighborhood of the origin as having the ring of functions $\mathscr{O}(V) = \widehat{\mathrm{Sym}}(V^\vee)$. Then we likewise define the space of one-forms on this formal scheme as

$$\Omega^1(V) = \mathscr{O}(V) \otimes V^\vee.$$

There is a universal derivation, called the exterior derivative map,

$$\mathrm{d} : \mathscr{O}(V) \to \Omega^1(V).$$

In components the exterior derivative is just the composition

$$\mathrm{Sym}^{n+1} V^\vee \to (V^\vee)^{\otimes n+1} \to \mathrm{Sym}^n(V^\vee) \otimes V^\vee,$$

where the maps are the inclusion followed by the natural projection, up to an overall combinatorial constant. (As a concrete example, note that $\mathrm{d}(xy) = y\mathrm{d}x + x\mathrm{d}y$ can be computed by taking the tensor representative $(x \otimes y + y \otimes x)/2$ for xy and then projecting off the last factor.)

This construction extends naturally to our context. We define

$$\Omega^1(\mathscr{E}(U)) = \mathscr{O}(\mathscr{E}(U)) \widehat{\otimes}_\beta \overline{\mathscr{E}}^!_c(U),$$

where we take the associated differentiable vector space. In concrete terms,

$$\Omega^1(\mathscr{E}(U)) = \prod_n \mathrm{Sym}^n(\overline{\mathscr{E}}^!_c(U)) \widehat{\otimes}_\beta \overline{\mathscr{E}}^!_c(U).$$

In this way, $\Omega^1(\mathscr{E}(U))$ becomes a differentiable pro-cochain complex, where the filtration is defined by

$$F^i\Omega^1(\mathscr{E}(U)) = \prod_{n \geq i-1} \mathrm{Sym}^n(\overline{\mathscr{E}}^!_c(U)) \widehat{\otimes}_\beta \overline{\mathscr{E}}^!_c(U).$$

Further, $\Omega^1(\mathscr{E}(U))$ is a module for the commutative algebra $\mathscr{O}(\mathscr{E}(U))$, where the module structure is defined in the multicategory of differentiable pro-vector spaces.

In a similar way, define the exterior derivative

$$\mathrm{d} : \mathscr{O}(\mathscr{E}(U)) \to \Omega^1(\mathscr{E}(U))$$

by saying that on components it is given by the same formula as in the finite-dimensional case.

B.1.3. Other classes of sections of a vector bundle. Before we introduce our next class of functionals — those with proper support — we need to introduce some further notation concerning classes of sections of a vector bundle.

Let $f : M \to N$ be a smooth fibration between two manifolds. Let E be a vector bundle on M. We say a section $s \in \Gamma(M, E)$ has *compact support over* f if the map

$$f : \mathrm{Supp}(s) \to N$$

is proper. We let $\Gamma_{c/f}(M, E)$ denote the space of sections with compact support over f. It is a differentiable vector space: if X is an auxiliary manifold, a smooth map $X \to \Gamma_{c/f}(M, E)$ is a section of the bundle $\pi_M^* E$ on $X \times M$ that has compact support relative to the map

$$M \times X \to N \times X.$$

(It is straightforward to write down a flat connection on $C^\infty(X, \Gamma_{c/f}(M, E))$, using arguments of the type described in section **??** of appendix **??**.)

Next, we need to consider spaces of the form $\overline{\mathscr{E}}(M) \widehat{\otimes}_\beta \mathscr{F}(N)$, where M and N are manifolds and E, F are vector bundles on the manifolds M, N, respectively. We want a more geometric interpretation of this tensor product.

We will view $\overline{\mathscr{E}}(M)\widehat{\otimes}_\beta\mathscr{F}(N)$ as a subspace

$$\overline{\mathscr{E}}(M)\widehat{\otimes}_\beta\mathscr{F}(N) \subset \overline{\mathscr{E}}(M)\widehat{\otimes}_\beta\overline{\mathscr{F}}(N).$$

It consists of those elements D with the property that, if $\phi \in \mathscr{E}_c^!(M)$, then map

$$\begin{array}{rccc} D(\phi): & \mathscr{F}_c^!(N) & \to & \mathbb{R} \\ & \psi & \mapsto & D(\phi \otimes \psi) \end{array}$$

comes from an element of $\mathscr{F}(N)$. Alternatively, $\overline{\mathscr{E}}(M)\widehat{\otimes}_\beta\mathscr{F}(N)$ is the space of continuous linear maps from $\mathscr{E}_c^!(M)$ to $\mathscr{F}(N)$.

We can similarly define $\overline{\mathscr{E}}_c(M)\widehat{\otimes}_\beta\mathscr{F}(N)$ as the subspace of those elements of $\overline{\mathscr{E}}(M)\widehat{\otimes}_\beta\mathscr{F}(N)$ that have compact support relative to the projection $M \times N \to N$.

These spaces form differentiable vector spaces in a natural way: a smooth map from an auxiliary manifold X to $\overline{\mathscr{E}}(M)\widehat{\otimes}_\beta\mathscr{F}(N)$ is an element of $\overline{\mathscr{E}}(N)\widehat{\otimes}_\beta\mathscr{F}(N)\widehat{\otimes}_\beta C^\infty(X)$. Similarly, a smooth map to $\overline{\mathscr{E}}_c(M)\widehat{\otimes}_\beta\mathscr{F}(N)$ is an element of $\overline{\mathscr{E}}(M)\widehat{\otimes}_\beta\mathscr{F}(N)\widehat{\otimes}_\beta C^\infty(X)$ whose support is compact relative to the map $M \times N \times X \to N \times X$.

B.1.4. Functions with proper support. Recall that

$$\Omega^1(\mathscr{E}_c(U)) = \mathscr{O}(\mathscr{E}_c(U))\widehat{\otimes}_\beta\overline{\mathscr{E}}^!(U).$$

We can thus define a subspace

$$\mathscr{O}(\mathscr{E}(U))\widehat{\otimes}_\beta\overline{\mathscr{E}}^!(U) \subset \Omega^1(\mathscr{E}_c(U)).$$

The Taylor components of elements of this subspace are in the space

$$\mathrm{Sym}^n(\overline{\mathscr{E}}_c^!(U))\widehat{\otimes}_\beta\overline{\mathscr{E}}^!(U),$$

which in concrete terms is the S_n coinvariants of

$$\overline{\mathscr{E}}_c^!(U)^{\widehat{\otimes}_\beta n}\widehat{\otimes}_\beta\overline{\mathscr{E}}^!(U).$$

B.1.4.1 Definition. *A function* $\Phi \in \mathscr{O}(\mathscr{E}_c(U))$ *has* proper support *if*

$$\mathrm{d}\Phi \in \mathscr{O}(\mathscr{E}(U))\widehat{\otimes}_\beta\overline{\mathscr{E}}^!(U) \subset \mathscr{O}(\mathscr{E}_c(U))\widehat{\otimes}_\beta\overline{\mathscr{E}}^!(U).$$

The reason for the terminology is as follows. Let $\Phi \in \mathscr{O}(\mathscr{E}_c(U))$ and let

$$\Phi_n \in \mathrm{Hom}(\mathscr{E}_c(U)^{\widehat{\otimes}_\beta n}, \mathbb{R})$$

be the nth term in the Taylor expansion of Φ. Then Φ has proper support if and only if, for all n, the composition with any projection map

$$\mathrm{Supp}(\Phi_n) \subset U^n \to U^{n-1}$$

is proper.

We will let

$$\mathscr{O}^P(\mathscr{E}_c(U)) \subset \mathscr{O}(\mathscr{E}_c(U))$$

be the subspace of functions with proper support. Note that functions with proper support are *not* a subalgebra.

Because $\mathscr{O}^P(\mathscr{E}_c(U))$ fits into a fiber square

$$\begin{array}{ccc} \mathscr{O}^P(\mathscr{E}_c(U)) & \to & \mathscr{O}(\mathscr{E}(U))\widehat{\otimes}_\beta\mathscr{E}_c(U)^\vee \\ \downarrow & & \downarrow \\ \mathscr{O}(\mathscr{E}_c(U)) & \to & \mathscr{O}(\mathscr{E}_c(U))\widehat{\otimes}_\beta\mathscr{E}_c(U)^\vee \end{array}$$

it has a natural structure of a differentiable pro-vector space.

B.1.5. Functions with smooth first derivative.

B.1.5.1 Definition. *A function $\Phi \in \mathscr{O}(\mathscr{E}_c(U))$ has* smooth first derivative *if* $\mathrm{d}\Phi$*, which is* a priori *an element of*

$$\Omega^1(\mathscr{E}_c(U)) = \mathscr{O}(\mathscr{E}_c(U))\widehat{\otimes}_\beta\overline{\mathscr{E}}^!(U),$$

is an element of the subspace

$$\mathscr{O}(\mathscr{E}_c(U))\widehat{\otimes}_\beta\mathscr{E}^!(U).$$

In other words, the 1-form $\mathrm{d}\Phi$ can be evaluated on a distributional tangent vectors from $\overline{\mathscr{E}}$, and not just smooth tangent vectors.

Note that we can identify, concretely, $\mathscr{O}(\mathscr{E}_c(U))\widehat{\otimes}_\beta\mathscr{E}^!(U)$ with the space

$$\prod_n \mathrm{Sym}^n\,\overline{\mathscr{E}}^!(U)\widehat{\otimes}_\beta\mathscr{E}^!(U)$$

and

$$\mathrm{Sym}^n\,\overline{\mathscr{E}}^!(U)\widehat{\otimes}_\beta\mathscr{E}^!(U) \subset \overline{\mathscr{E}}^!(U)^{\widehat{\otimes}_\beta n}\widehat{\otimes}_\beta\mathscr{E}^!(U).$$

(Spaces of the form $\mathscr{E}(U)\widehat{\otimes}_\beta\overline{\mathscr{E}}(U)$ were described concretely above.)

Thus $\mathscr{O}(\mathscr{E}_c(U))\widehat{\otimes}_\beta\mathscr{E}^!(U)$ is a differentiable pro-vector space. It follows that the space of functionals with smooth first derivative is a differentiable pro-vector space, since it is defined by a fiber diagram of such objects.

An even more concrete description of the space $\mathscr{O}^{sm}(\mathscr{E}_c(U))$ of functionals with smooth first derivative is as follows.

B.1.5.2 Lemma. *A functional $\Phi \in \mathscr{O}(\mathscr{E}_c(U))$ has smooth first derivative if each of its Taylor components*

$$D_n\Phi \in \mathrm{Sym}^n\,\overline{\mathscr{E}}^!(U) \subset \overline{\mathscr{E}}^!(U)^{\widehat{\otimes}_\beta n}$$

lies in the intersection of all the subspaces

$$\overline{\mathscr{E}}^!(U)^{\widehat{\otimes}_\beta k}\widehat{\otimes}_\beta\mathscr{E}^!(U)\widehat{\otimes}_\beta\overline{\mathscr{E}}^!(U)^{\widehat{\otimes}_\beta n-k-1}$$

for $0 \le k \le n-1$.

PROOF. The proof is a simple calculation. □

Note that the space of functions with smooth first derivative is a subalgebra of $\mathscr{O}(\mathscr{E}_c(U))$. We will denote this subalgebra by $\mathscr{O}^{sm}(\mathscr{E}_c(U))$. Again, the space of functions with smooth first derivative is a differentiable pro-vector space, as it is defined as a fiber product.

Similarly, we can define the space of functions on $\mathscr{E}(U)$ with smooth first derivative, $\mathscr{O}^{sm}(\mathscr{E}(U))$ as those functions whose exterior derivative lies in $\mathscr{O}(\mathscr{E}(U))\widehat{\otimes}_\beta\mathscr{E}_c^!(U) \subset \Omega^1(\mathscr{E}(U))$.

B.1.6. Functions with smooth first derivative and proper support. We are particularly interested in those functions which have both smooth first derivative and proper support. We will refer to this subspace as $\mathscr{O}^{P,sm}(\mathscr{E}_c(U))$. The differentiable structure on $\mathscr{O}^{P,sm}(\mathscr{E}_c(U))$ is, again, given by viewing it as defined by the fiber diagram

$$\begin{array}{ccc} \mathscr{O}^{P,sm}(\mathscr{E}_c(U)) & \to & \mathscr{O}(\mathscr{E}(U))\widehat{\otimes}_\beta\mathscr{E}^!(U) \\ \downarrow & & \downarrow \\ \mathscr{O}(\mathscr{E}_c(U)) & \to & \mathscr{O}(\mathscr{E}_c(U))\widehat{\otimes}_\beta\overline{\mathscr{E}}^!(U). \end{array}$$

We have inclusions

$$\mathscr{O}^{sm}(\mathscr{E}(U)) \subset \mathscr{O}^{P,sm}(\mathscr{E}_c(U)) \subset \mathscr{O}^{sm}(\mathscr{E}_c(U)),$$

where each inclusion has dense image.

B.2. Derivations

As before, let M be a manifold, E a graded vector bundle on M, and U an open subset of M. In this section we will define derivations of algebras of functions on $\mathscr{E}(U)$.

To start with, recall that for V a finite dimensional vector space, which we treat as a formal scheme, the algebra of function is $\mathscr{O}(V) = \prod \mathrm{Sym}^n V^\vee$, the formal power series on V. We then identify the space of continuous derivations of $\mathscr{O}(V)$ with $\mathscr{O}(V) \otimes V$. We view these derivations as the space of vector fields on V and use the notation $\mathrm{Vect}(V)$.

In a similar way, we define the space of vector fields $\mathrm{Vect}(\mathscr{E}(U))$ of vector fields on $\mathscr{E}(U)$ as

$$\mathrm{Vect}(\mathscr{E}(U)) = \mathscr{O}(\mathscr{E}(U))\widehat{\otimes}_\beta\mathscr{E}(U) = \prod_n \left(\mathrm{Sym}^n(\overline{\mathscr{E}}^!_c(U))\widehat{\otimes}_\beta\mathscr{E}(U)\right).$$

We have already seen (section B.1) how to define the structure of differentiable pro-vector space on spaces of this nature.

In this section we will show the following.

B.2.0.1 Proposition. $\operatorname{Vect}(\mathscr{E}(U))$ *has a natural structure of Lie algebra in the multicategory of differentiable pro-vector spaces. Further,* $\mathscr{O}(\mathscr{E}(U))$ *has an action of the Lie algebra* $\operatorname{Vect}(\mathscr{E}(U))$ *by derivations, where the structure map* $\operatorname{Vect}(\mathscr{E}(U)) \times \mathscr{O}(\mathscr{E}(U)) \to \mathscr{O}(\mathscr{E}(U))$ *is smooth.*

PROOF. To start with, let's look at the case of a finite-dimensional vector space V, to get an explicit formula for the Lie bracket on $\operatorname{Vect}(V)$, and the action of $\operatorname{Vect}(V)$ on $\mathscr{O}(V)$. Then, we will see that these formulae make sense when $V = \mathscr{E}(U)$.

Let $X \in \operatorname{Vect}(V)$, and let us consider the Taylor components $D_n X$, which are multilinear maps

$$V \times \cdots \times V \to V.$$

Our conventions are such that

$$D_n(X)(v_1, \dots, v_n) = \left(\frac{\partial}{\partial v_1} \cdots \frac{\partial}{\partial v_n} X \right)(0) \in V$$

Here, we are differentiating vector fields on V using the trivialization of the tangent bundle to this formal scheme arising from the linear structure.

Thus, we can view $D_n X$ as in the endomorphism operad of the vector space V.

If $A : V^{\times n} \to V$ and $B : V^{\times m} \to V$, let us define

$$A \circ_i B(v_1, \dots, v_{n+m-1}) = A(v_1, \dots, v_{i-1}, B(v_i, \dots, v_{i+m-1}), v_{i+m}, \dots, v_{n+m-1}).$$

If A, B are symmetric (under S_n and S_m, respectively), then define

$$A \circ B = \sum_{i=1}^{n} A \circ_i B.$$

Then, if X, Y are vector fields, the Taylor components of $[X, Y]$ satisfy

$$D_n([X, Y]) = \sum_{k+l=n+1} c_{k,l} \left(D_k X \circ D_l Y - D_l Y \circ D_k X \right)$$

where $c_{k,l}$ are combinatorial constants whose values are irrelevant for our purposes.

Similarly, if $f \in \mathscr{O}(V)$, the Taylor components of f are multilinear maps

$$D_n f : V^{\times n} \to \mathbb{C}.$$

In a similar way, if X is a vector field, we have

$$D_n(Xf) = \sum_{k+l=n+1} c'_{k,l} D_k(X) \circ D_k(f).$$

Thus, we see that in order to define the Lie bracket on $\mathrm{Vect}(\mathscr{E}(U))$, we need to give maps of differentiable vector spaces

$$\circ_i : \mathrm{Hom}(\mathscr{E}(U)^{\widehat{\otimes}_\beta n}, \mathscr{E}(U)) \times \mathrm{Hom}(\mathscr{E}(U)^{\widehat{\otimes}_\beta m}, \mathscr{E}(U)) \to \mathrm{Hom}(\mathscr{E}(U)^{\widehat{\otimes}_\beta (n+m-1)}, \mathscr{E}(U))$$

where here Hom indicates the space of continuous linear maps, treated as a differentiable vector space. Similarly, to define the action of $\mathrm{Vect}(\mathscr{E}(U))$ on $\mathscr{O}(\mathscr{E}(U))$, we need to define a composition map

$$\circ_i : \mathrm{Hom}(\mathscr{E}(U)^{\widehat{\otimes}_\beta n}, \mathscr{E}(U)) \times \mathrm{Hom}(\mathscr{E}(U)^{\widehat{\otimes}_\beta m}) \to \mathrm{Hom}(\mathscr{E}(U)^{\widehat{\otimes}_\beta n+m-1}).$$

We will treat the first case; the second is similar.

Now, if X is an auxiliary manifold, a smooth map

$$X \to \mathrm{Hom}(\mathscr{E}(U)^{\widehat{\otimes}_\beta m}, \mathscr{E}(U))$$

is the same as a continuous multilinear map

$$\mathscr{E}(U)^{\times m} \to \mathscr{E}(U) \widehat{\otimes}_\beta C^\infty(X).$$

Here, "continuous" means for the product topology.

This is the same thing as a continuous $C^\infty(X)$-multilinear map

$$\Phi : (\mathscr{E}(U) \widehat{\otimes}_\beta C^\infty(X))^{\times m} \to \mathscr{E}(U) \widehat{\otimes}_\beta C^\infty(X).$$

If

$$\Psi : (\mathscr{E}(U) \widehat{\otimes}_\beta C^\infty(X))^{\times n} \to \mathscr{E}(U) \widehat{\otimes}_\beta C^\infty(X).$$

is another such map, then it is easy to define $\Phi \circ_i \Psi$ by the usual formula:

$$\Phi \circ_i \Psi(v_1, \dots, v_{n+m-1}) = \Phi(v_1, \dots v_{i-1}, \Psi_i(v_i, \dots, v_{m+i-1}), \dots, v_{n+m-1})$$

if $v_i \in \mathscr{E}(U) \widehat{\otimes}_\beta C^\infty(X)$. This map is $C^\infty(X)$ linear. $\square$

Bibliography

[AHS78] M. F. Atiyah, N. J. Hitchin, and I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A **362** (1978), no. 1711, 425–461. MR 506229 (80d:53023)

[AKSZ97] M. Alexandrov, M. Kontsevich, A. Schwarz, and O. Zaboronsky, *The geometry of the master equation and topological field theory*, Internat. J. Modern Phys. **12** (1997), no. 7, 1405–1429.

[BC04] John C. Baez and Alissa S. Crans, *Higher-dimensional algebra. VI. Lie 2-algebras*, Theory Appl. Categ. **12** (2004), 492–538. MR 2068522 (2005m:17039)

[BD04] Alexander Beilinson and Vladimir Drinfeld, *Chiral algebras*, American Mathematical Society Colloquium Publications, vol. 51, American Mathematical Society, Providence, RI, 2004. MR MR2058353 (2005d:17007)

[BGV92] Nicole Berline, Ezra Getzler, and Michèle Vergne, *Heat kernels and Dirac operators*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 298, Springer-Verlag, Berlin, 1992. MR MR1215720 (94e:58130)

[BR10] John C. Baez and Christopher L. Rogers, *Categorified symplectic geometry and the string Lie 2-algebra*, Homology, Homotopy Appl. **12** (2010), no. 1, 221–236. MR 2638872 (2011g:70065)

[CCRF$^+$98] A.S. Cattaneo, P. Cotta-Ramusino, F. Fucito, M. Martellini, M. Rinaldi, A. Tanzini, and M. Zeni, *Four-dimensional Yang–Mills theory as a deformation of topological BF theory*, Communications in Mathematical Physics **197** (1998), no. 3, 571–621.

[Cos10] Kevin Costello, *A geometric construction of the Witten genus, I*, Proceedings of the International Congress of Mathematicians, (Hyderabad, 2010), 2010.

[Cos11a] ______, *A geometric construction of the Witten genus, II (preprint)*.

[Cos11b] ______, *Renormalization and effective field theory*, Surveys and monographs, American Mathematical Society, 2011.

[Cos13] ______, *Notes on supersymmetric and holomorphic field theories in dimensions 2 and 4*, Pure Appl. Math. Q. **9** (2013), no. 1, 73–165. MR 3126501

[EH00] David Eisenbud and Joe Harris, *The geometry of schemes*, Graduate Texts in Mathematics, vol. 197, Springer-Verlag, New York, 2000. MR 1730819 (2001d:14002)

[FGI$^+$05] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli, *Fundamental algebraic geometry*, Mathematical Surveys and Monographs, vol. 123, American Mathematical Society, Providence, RI, 2005, Grothendieck's FGA explained. MR 2222646 (2007f:14001)

[Get09] Ezra Getzler, *Lie theory for nilpotent L_∞-algebras*, Ann. of Math. (2) **170** (2009), no. 1, 271–301. MR 2521116 (2010g:17026)

[GG11] Ryan Grady and Owen Gwilliam, *One-dimensional Chern-Simons theory and the $\widehat{A}$ genus*.

[Hen08] André Henriques, *Integrating L_∞-algebras*, Compos. Math. **144** (2008), no. 4, 1017–1045. MR 2441255 (2011h:18021)

[Hin01] Vladimir Hinich, *DG coalgebras as formal stacks*, J. Pure Appl. Algebra **162** (2001), no. 2-3, 209–250. MR 1843805 (2002f:14008)

[Kap05] Anton Kapustin, *Chiral de Rham complex and the half-twisted sigma-model*.

[Kon93] Maxim Kontsevich, *Formal (non)commutative symplectic geometry*, The Gel′fand Mathematical Seminars, 1990–1992, Birkhäuser Boston, Boston, MA, 1993, pp. 173–187. MR 94i:58212

[Kos85] J-L. Koszul, *Crochet de Schouten-Nijenhuis et cohomologie*, Elie Cartan et les mathematiques d'aujourd'hui, Asterisque, 1985, pp. 257–271.

[KS] Maxim Kontsevich and Yan Soibelman, *Deformation theory, volume I*.

[KS06] ______, *Notes on A-infinity algebras, A-infinity categories and non-commutative geometry, I*.

[KW06] Anton Kapustin and Edward Witten, *Electric-Magnetic Duality and the Geometric Langlands Program*.

[Lur] Jacob Lurie, *Derived Algebraic Geometry X: Formal Moduli Problems*, available at http://math.mit.edu/~lurie/papers.

[Lur09] ______, *On the classification of topological field theories*.

[Lur10] ______, *Moduli problems for ring spectra*, Proceedings of the International Congress of Mathematicians. Volume II (New Delhi), Hindustan Book Agency, 2010, pp. 1099–1125. MR 2827833 (2012i:55008)

[Lur11] ______, *Derived Algebraic Geometry X : Formal moduli problems*.

[Man09] Marco Manetti, *Differential graded Lie algebras and formal deformation theory*, Algebraic geometry—Seattle 2005. Part 2, Proc. Sympos. Pure Math., vol. 80, Amer. Math. Soc., Providence, RI, 2009, pp. 785–810. MR 2483955 (2009m:17015)

[PTVV11] T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi, *Quantization and derived moduli spaces*.

[Qui69] Daniel Quillen, *Rational homotopy theory*, Ann. of Math. (2) **90** (1969), no. 2, 205–295.

[Sch93] A. Schwarz, *Geometry of Batalin-Vilkovisky quantization*, Comm. Math. Phys. **155** (1993), no. 2, 249–260.

[Seg10] Ed Segal, *Personal communication*, 2010.

[Tar87] N.N. Tarkhanov, *On Alexander duality for elliptic complexes*, Math. USSR Sbornik **58** (1987), no. 1.

[Toë06] Bertrand Toën, *Higher and derived stacks: a global overview*.

[Vez11] Gabriele Vezzosi, *Derived critical loci I - basics*.

[VW94] Cumrun Vafa and Edward Witten, *A strong coupling test of S-duality*.

[Wit05] Edward Witten, *Two-dimensional models with (0,2) supersymmetry: perturbative aspects*.