I. The Limit Laws

Assumptions: c is a constant and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist

	Limit Law in symbols	Limit Law in words
1	$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$	The limit of a sum is equal to the sum of the limits.
2	$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$	The limit of a difference is equal to the difference of the limits.
3	$\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$	The limit of a constant times a function is equal to the constant times the limit of the function.
4	$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)]$	The limit of a product is equal to the product of the limits.
5	$\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ $\left(\text{if } \lim_{x\to a} g(x)\neq 0\right)$	The limit of a quotient is equal to the quotient of the limits.
6	$\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$	where n is a positive integer
7	$\lim_{x\to a} c = c$	The limit of a constant function is equal to the constant.
8	$\lim_{x\to a} x = a$	The limit of a linear function is equal to the number x is approaching.
9	$\lim_{x\to a} x^n = a^n$	where n is a positive integer
10	$\lim_{x\to a}\sqrt[n]{x} = \sqrt[n]{a}$	where n is a positive integer & if n is even, we assume that $a > 0$
11	$\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$	where n is a positive integer & if n is even, we assume that $\lim_{x\to a} f(x) > 0$

Direct Substitution Property: If f is a polynomial or rational function and a is in the domain of f, then $\lim_{x\to a} f(x) =$

"Simpler Function Property": If $f(x) = g(x)$ when $x \neq a$ then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, as long as the limit exists.

ex#1 Given $\lim_{x\to 3} f(x) = 2$, $\lim_{x\to 3} g(x) = -1$, $\lim_{x\to 3} h(x) = 3$ use the Limit Laws find $\lim_{x\to 3} \sqrt{f(x)h(x) - x^2 g(x)}$

ex#2 Evaluate $\lim_{x\to 2} \frac{2x^2+1}{x^2+6x-4}$, if it exists, by using the Limit Laws.

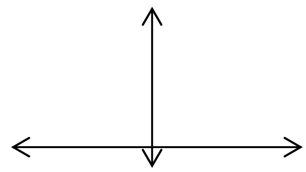
ex#3 Evaluate: $\lim_{x\to 1} 2x^2 + 3x - 5$

ex#4 Evaluate: $\lim_{x\to 0} \frac{1-(1-x)^2}{x}$

ex#5 Evaluate: $\lim_{h\to 0} \frac{\sqrt{h+4}-2}{h}$

Two Interesting Functions

1. *Absolute Value Function*



Definition: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Geometrically: The absolute value of a number indicates its distance from another number.

$|x - c| = a$ means the number x is exactly _____ units away from the number _____.

$|x - c| < a$ means: The number x is within _____ units of the number _____.

How to solve equations and inequalities involving absolute value:

Solve: |3x + 2| = 7

Solve: |x - 5| < 2

What does |x - 5| < 2 mean geometrically?

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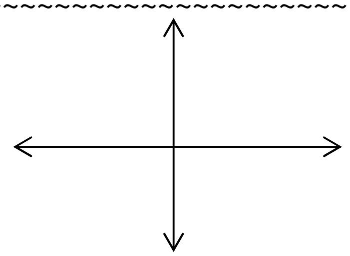
2. *The Greatest Integer Function*

Definition: $[\![x]\!] =$ the largest integer that is less than or equal to x.



ex 6 [[5]] =

ex 7 [[5.999]] =

ex 8 [[$\sqrt{3}$]] =

ex 9 [[-2.6]] =

Theorem 1: $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a+} f(x)$

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ex#10 Prove that the $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

<u>ex#11</u> What is $\lim_{x \to 3} [[x]]$?

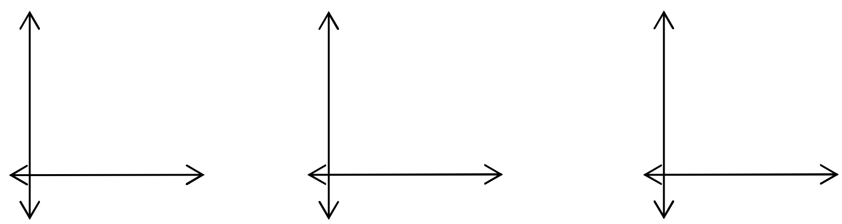
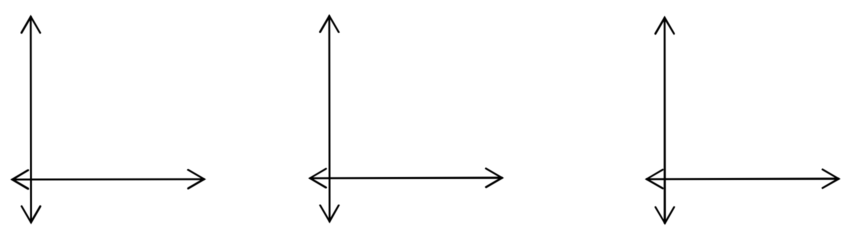
Theorem 2: If $f(x) \le g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a then $\lim_{x \to a} f(x) \le \lim_{x \to a} g(x)$.

<u>ex12</u> Find $\lim_{x \to 0} x^2 \sin\frac{1}{x}$. To find this limit, let's start by graphing it. Use your graphing calculator.

The Squeeze Theorem: If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a) and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$ then $\lim_{x \to a} g(x) = L$

Definitions of Limits at Large Numbers

	Definition in Words	Precise Mathematical Definition
Large POSITIVE numbers	Let f be a function defined on some interval (a, ∞). Then $\lim_{x\to\infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large in a positive direction.	Let f be a function defined on some interval (a, ∞). Then $\lim_{x\to\infty} f(x) = L$ if for every $\varepsilon > 0$ there is a corresponding number N such that if $x > N$ then $\lvert f(x) - L\rvert < \varepsilon$
Large NEGATIVE numbers	Let f be a function defined on some interval $(-\infty,a)$. ∞). Then $\lim_{x\to-\infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large in a negative direction.	Let f be a function defined on some interval $(-\infty,a)$. Then $\lim_{x\to-\infty} f(x) = L$ if for every $\varepsilon > 0$ there is a corresponding number N such that if $x < N$ then $\lvert f(x) - L\rvert < \varepsilon$

	Definition	What this can look like…
Horizontal Asymptote	The line $y = L$ is a **horizontal asymptote** of the curve $y = f(x)$ if either is true: 1. $\lim_{x\to\infty} f(x) = L$ *or* 2. $\lim_{x\to-\infty} f(x) = L$	
Vertical Asymptote	The line $x = a$ is a **vertical asymptote** of the curve $y = f(x)$ if *at least one* of the following is true: 1. $\lim_{x\to a} f(x) = \infty$ 2. $\lim_{x\to a^-} f(x) = \infty$ 3. $\lim_{x\to a^+} f(x) = \infty$ 4. $\lim_{x\to a} f(x) = -\infty$ 5. $\lim_{x\to a^-} f(x) = -\infty$ 6. $\lim_{x\to a^+} f(x) = -\infty$	

Theorem

- If $r > 0$ is a rational number then $\lim_{x\to\infty} \frac{1}{x^r} = 0$
- If $r > 0$ is a rational number such that x^r is defined for all x then $\lim_{x\to-\infty} \frac{1}{x^r} = 0$

ex#1 Find the limit: $\lim_{x\to\infty} \frac{3}{x^5}$

ex#2 Find the limit: $\lim_{x\to\infty} \frac{x^3 + 2x}{5x^3 - x^2 + 4}$

ex#3 Find the limit: $\lim_{x\to\infty} \sqrt{9x^2 + x} - 3x$

ex#4 Find the limit: $\lim_{x\to\infty} \cos x$

ex#5 Find the vertical and horizontal asymptotes of the graph of the function: $f(x)=\dfrac{\sqrt{2x^2+1}}{3x-5}$

Defn. *Matrix Multiplication* is a "map" (or function) that inputs two matrices, of size $n \times m$ and $m \times p$ for some $m, n, p \in \mathbb{N}$, and outputs a matrix of size $n \times p$. That is, matrix multiplication is a map

$$M_{n\times m}(\mathbb{R}) \times M_{m\times p}(\mathbb{R}) \longrightarrow M_{n\times p}(\mathbb{R})$$

How?

To multiply $A \cdot B = C$ for matrices $A \in M_{n\times m}(\mathbb{R})$, $B \in M_{m\times p}(\mathbb{R})$, think of the leftmost matrix (here, A) as a set of rows and the rightmost matrix (here, B) as a set of columns. Then the $c_{ij} \in \mathbb{R}$ entry of the new, output matrix $C \in M_{n\times p}(\mathbb{R})$ is obtained by taking the dot product of the ith row of A with the jth column of B.

Recall that c_{ij} means the entry in the ith row, jth column of the matrix C.

That is,

write $$A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} \text{---} & \vec{v}_1 & \text{---} \\ \text{---} & \vec{v}_2 & \text{---} \\ & \vdots & \\ \text{---} & \vec{v}_n & \text{---} \end{bmatrix}$$ where $\vec{v}_i$ is the ith row of the matrix A for $1 \le i \le n$

and write $$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_p \\ | & | & & | \end{bmatrix}$$ where $\vec{w}_j$ is the jth column of the matrix B for $1 \le j \le p$

and to multiply, take dot products described above:

$$C = A \cdot B = \begin{bmatrix} \text{---} & \vec{v}_1 & \text{---} \\ \text{---} & \vec{v}_2 & \text{---} \\ & \vdots & \\ \text{---} & \vec{v}_n & \text{---} \end{bmatrix} \begin{bmatrix} | & | & & | \\ \vec{w}_1 & \vec{w}_2 & \cdots & \vec{w}_p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \vec{v}_1 \cdot \vec{w}_1 & \vec{v}_1 \cdot \vec{w}_2 & \cdots & \vec{v}_1 \cdot \vec{w}_p \\ \vec{v}_2 \cdot \vec{w}_1 & \vec{v}_2 \cdot \vec{w}_2 & \cdots & \vec{v}_2 \cdot \vec{w}_p \\ \vdots & \vdots & & \vdots \\ \vec{v}_n \cdot \vec{w}_1 & \vec{v}_n \cdot \vec{w}_2 & \cdots & \vec{v}_n \cdot \vec{w}_p \end{bmatrix}$$

$$\in M_{n\times p}(\mathbb{R}) \checkmark$$

Ex As a special case,

B could be a column vector so that

$B \in M_{m\times 1}(\mathbb{R})$.

For example, take $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix} \in M_{2\times 3}(\mathbb{R})$

and $B = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \in M_{3\times 1}(\mathbb{R})$.

Then the product $A \cdot B$ is

$$AB = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 2 + 1\cdot 1 + 0\cdot 1 \\ 2\cdot(-1) + 0\cdot 1 + 3\cdot 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \in M_{2\times 1}(\mathbb{R}) \checkmark$$

Ex Multiplying matrices may not work if they do not have the correct dimension!

$$\begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$ is not well-defined

since the left matrix has 3 column and the right matrix has 2 rows.

\# columns on left $\underline{\underline{\text{MUST}}}$ # rows on right

in order for matrix multiplication to be well-defined.

Ex
$$\begin{bmatrix} 3 & 5 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 \\ 1 & 2 \\ 0 & 3 \\ 1 & 1 \end{bmatrix} =$$

3x4 matrix times 4x2 matrix gives 3x2 matrix

$$= \begin{bmatrix} 3\cdot 1 + 5\cdot 1 + 1\cdot 0 + 0\cdot 1 & 3\cdot 5 + 5\cdot 2 + 1\cdot 3 + 0\cdot 1 \\ 0\cdot 1 + 0\cdot 1 + (-1)\cdot 0 + 0\cdot 1 & 0\cdot 5 + 0\cdot 2 + (-1)\cdot 3 + 0\cdot 1 \\ 1\cdot 1 + 2\cdot 1 + 0\cdot 0 + 1\cdot 1 & 1\cdot 5 + 2\cdot 2 + 0\cdot 3 + 1\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & \boxed{28} \\ 0 & \boxed{-3} \\ 4 & \boxed{10} \end{bmatrix}$$

Ex
$$\begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\cdot(-1) + 1\cdot 0 + 2\cdot 1 \\ 0\cdot(-1) + 1\cdot 0 + 0\cdot 1 \\ 1\cdot(-1) + (-1)0 + 1\cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

nonzero matrix · nonzero matrix = zero matrix

Interesting

Notice how neither the matrix on the left nor the matrix on the right need be all equal to zero in order for their matrix product to be a fully zero matrix !

Note: The above possibility for matrices is in contrast to the familiar property of multiplication of numbers which states that if $xy=0$ either $x=0$ or $y=0$ for $x,y \in \mathbb{R}$

(NOT TRUE FOR MATRICES !)

Ex

$$\begin{bmatrix} 2 & 6 \\ 5 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & 2 \end{bmatrix}$$

Ex

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 6 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 5 & 7 \end{bmatrix}$$

Also Interesting!

Matrix multiplication is not commutative

$$A \cdot B \neq B \cdot A$$

This shows that we must be very careful when executing matrix multiplication.

(Clearly this property of matrices differs greatly from commutativity of multiplication of real numbers where $xy = yx$ for $x, y \in \mathbb{R}$)

$5 \cdot 2 = 10$ ✓
$2 \cdot 5 = 10$ ✓

Additional Properties of Matrix Multiplication (Proofs left to student)

Distributivity: $A \cdot (B + C) = A \cdot B + AC$ for $A \in M_{n \times m}(\mathbb{R})$, $B, C \in M_{m \times p}(\mathbb{R})$ for some $m, n, p \in \mathbb{N}$

Associativity: $(A \cdot B) \cdot C = A \cdot (B \cdot C)$

Commutes With Scalar Multiplication:

$$A \cdot (kB) = k \cdot (A \cdot B) \text{ for } k \in \mathbb{R}$$

LECTURE NOTES

WHEN: principal case involves square roots, where the dominant power is the second power

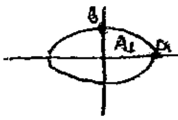
$\boxed{\text{3 cases}}$:

①	$\sqrt{a^2-x^2}$	$\xrightarrow[\text{substitute}]{}$	$x=a\sin\theta$	$\xrightarrow[\text{domain}]{}$ $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$	$\xrightarrow[\text{use}]{}$	$\sqrt{1-\sin^2\theta}=\cos\theta$
②	$\sqrt{a^2+x^2}$	$\longrightarrow$	$x=a\tan\theta$	$\longrightarrow -\frac{\pi}{2}<\theta<\frac{\pi}{2}$	$\longrightarrow$	$\sqrt{\tan^2\theta+1}=\sec\theta$
③	$\sqrt{x^2-a^2}$	$\longrightarrow$	$x=a\sec\theta$	$\longrightarrow$ $0\le\theta<\pi/2$, $\pi\le\theta<3\pi/2$	$\longrightarrow$	$\sqrt{\sec^2\theta-1}=\tan\theta$

key examples:

(A): Case 1. Find the area of an ellipse

$A=4A_1$. The equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y=\frac{b}{a}\sqrt{a^2-x^2}$ (in 1st quadrant)



$A_1=\int_0^a y\,dx=\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx=\frac{b}{a}\int_0^a\sqrt{a^2-x^2}\,dx$ Use $x=a\sin\theta \Rightarrow dx=a\cos\theta\,d\theta$

$\Rightarrow A_1=\frac{b}{a}\int_0^{\pi/2}\sqrt{a^2(1-\sin^2\theta)}\;a\cos\theta\,d\theta \underset{\sqrt{a^2}=a}{=} ba\int_0^{\pi/2}\cos^2\theta\,d\theta=ba\int_0^{\pi/2}\left[\frac{1}{2}+\frac{\cos(2\theta)}{2}\right]d\theta$

$=\dots=\frac{\pi}{4}ab$

$\Rightarrow A=4A_1=\pi ab$

Notice that for $a=b \Rightarrow A=\pi a^2 \longrightarrow$ circle's area.

(B): Case 2. Evaluate $I=\int\sqrt{a^2+x^2}\,dx \underset{\substack{x=a\tan\theta\\dx=a\sec^2\theta\,d\theta}}{=} \int\underbrace{\sqrt{a^2(1+\tan^2\theta)}}_{\sec^2\theta}\,a\sec^2\theta\,d\theta=$

$=a^2\int\sec^3\theta\,d\theta$ But this is an integral we have evaluated (see Trigonometric Integrals)

$\int\sec^3\theta\,d\theta=\frac{1}{2}\left[\ln|\sec\theta+\tan\theta|+\tan\theta\sec\theta\right]$

But we still need to express everything in terms of x.

$\tan\theta=x/a \Rightarrow \sec\theta=\sqrt{1+\tan^2\theta}=\sqrt{1+\frac{x^2}{a^2}}=\frac{\sqrt{x^2+a^2}}{a}$

$$\Rightarrow I=\frac{a^2}{2}\left[\ln\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right|+\frac{x\sqrt{x^2+a^2}}{a^2}\right]+C$$

similarly: Evaluate $I=\int\frac{1}{\sqrt{x^2+a^2}}\,dx \underset{\substack{x=a\tan\theta\\dx=a\sec^2\theta\,d\theta}}{=} \int\frac{a\sec^2\theta\,d\theta}{a\sqrt{\tan^2\theta+1}}=\int\sec\theta\,d\theta=$

$=\ln|\sec\theta+\tan\theta|$

$=\ln\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right|+C$

(C): Case 3. Evaluate $I=\int\frac{x^2}{\sqrt{x^2-a^2}}\,dx \underset{\substack{x=a\sec\theta\\dx=a\sec\theta\tan\theta\,d\theta}}{=} \int\frac{a^2\sec^2\theta}{\sqrt{a^2\underbrace{(\sec^2\theta-1)}_{\tan^2\theta}}}\,a\sec\theta\tan\theta\,d\theta=a^2\int\sec^3\theta\,d\theta$

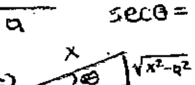
$\Rightarrow I=\frac{a^2}{2}\left[\ln|\sec\theta+\tan\theta|+\tan\theta\sec\theta\right]$ (see above)

But now $\sec\theta=\frac{x}{a}$ * $\tan\theta=\sqrt{\sec^2\theta-1}=\sqrt{\frac{x^2}{a^2}-1}=\frac{\sqrt{x^2-a^2}}{a}$

$$\Rightarrow I=\frac{a^2}{2}\left[\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+\frac{\sqrt{x^2-a^2}\,x}{a^2}\right]+C$$

* An alternative method to find $\tan\theta$ (knowing $\sec\theta$):

$\sec\theta=\frac{x}{a} \Rightarrow \cos\theta=\frac{a}{x} \Rightarrow$



$\Rightarrow \tan\theta=\frac{\sqrt{x^2-a^2}}{a}$

$\cos\theta=a/x$

$\sin\theta=\sqrt{x^2-a^2}/x$ etc.

$$= \int \frac{1}{a^2}\frac{1}{\sec\theta}\,d\theta = \frac{1}{a^2}\int \cos\theta\,d\theta = \frac{1}{a^2}\sin\theta + C \underset{\substack{\text{see previous}\\ \text{example}\\ \text{for } \sin\theta}}{=} \frac{1}{a^2}\frac{\sqrt{x^2-a^2}}{x} + C$$

→ More general examples where the leading order (power) inside the radical is a square:

$$I = \int \sqrt{x^2-2x}\,dx = \int \sqrt{(x^2-2x+1)-1}\,dx = \int \sqrt{(x-1)^2-1}\,dx \underset{u=x-1}{=} \int \sqrt{u^2-1}\,du$$

use $u = \sec\theta$, $du = \tan\theta\sec\theta\,d\theta$ $\Rightarrow I = \int \tan\theta \cdot \tan\theta \cdot \sec\theta\,d\theta = \int \tan^2\theta\sec\theta\,d\theta$

This is again an integral we have calculated (see trigonometric integrals). We found that

$$I = \frac{1}{2}\left[\tan\theta\sec\theta - \ln\left|\tan\theta + \sec\theta\right|\right] = \frac{1}{2}\left[\sqrt{(x-1)^2-1}\,(x-1) - \ln\left|(x-1)+\sqrt{(x-1)^2-1}\right|\right]$$

$\sec\theta = \frac{x-1}{a}$

$\tan\theta = \frac{\sqrt{(x-1)^2-a^2}}{a}$ (here $a=1$)

The more general case:

$\int \sqrt{x^2+ax+b}\,dx$: Consider $\sqrt{(x+\frac{a}{2})^2 + (b-\frac{a^2}{4})}$

set $u = x + \frac{a}{2}$

If $b = \frac{a^2}{4}$ → simple

If $b > \frac{a^2}{4} \Rightarrow D = \sqrt{b-\frac{a^2}{4}}$ use $\left\{\begin{array}{l} u = D\tan\theta \\ du = D\sec^2\theta\,d\theta \end{array}\right\} \Rightarrow \sqrt{x^2+ax+b} = D\sec\theta$

If $b < \frac{a^2}{4} \Rightarrow D^2 = -(b - \frac{a^2}{4})$ use $\left\{\begin{array}{l} u = D\sec\theta \\ du = D\sec\theta\tan\theta\,d\theta \end{array}\right\} \Rightarrow \sqrt{x^2+ax+b} = D\tan\theta$

$D = \sqrt{\frac{a^2}{4} - b}$

⇓

Now one can substitute in the integral → we get integrals like the ones above.

Note: For definite integrals, remember to always <u>also</u> substitute the integral limits!

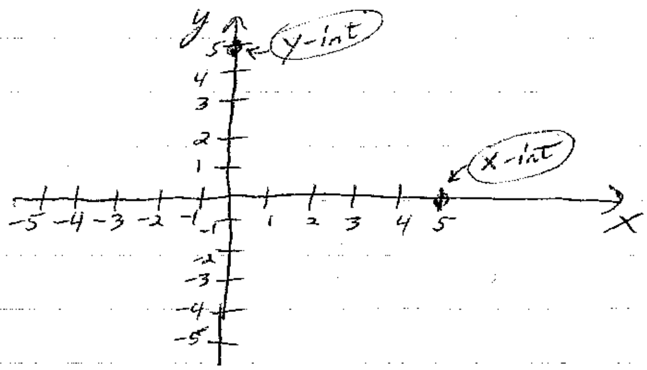
② (cont'd)

x	y
0	5
5	0

Each pair of values which make the equation true represents a solution to the equation

Each pair of values (x, y) represent a point on a graph, just like when we graphed the solution to one equation with <u>one</u> <u>variable</u>.

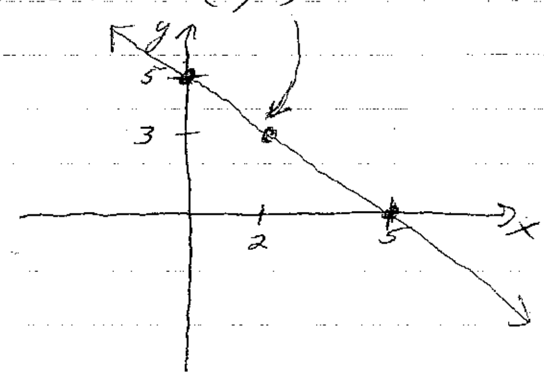
Only now, since we have an equation with <u>two</u> variables, we need a graph with <u>two</u> <u>distinct</u> number lines (one horizontal and one vertical)



The set of all possible solutions to the equation with two variables will be represented by <u>a line</u> on a graph. (Thus <u>linear</u> <u>equation</u> → first degree variables)

Typically, to graph a line we like to have <u>three</u> points (just to be safe).

So, say $x = 2$ and $y = 3$ → $(2,3)$



when it's clear all points line up then it's safe to draw the line.

⊛ But sometimes, we'd like to solve <u>two</u> equations at the same time ....

[Day 2 1/31/03]

1) Solving 2x2 systems.

(nice pt. of int.) I do. (A) Solve $3x-y=3$, $x+y=1$ — First by graphing. Then by substitution. Then by elimination.

(Class ex: TGHI (same as above)) They do. (B) $2x+3y=6$, $-x+2y=4$ — same as above.

* eliminate the y-terms here for something different

(pt. of int. not nice fractions) graph first. (C) $3x+2y=1$, $2x-y=2$ — First by graphing. Then, split class up into 2 sections. (middle section ; both side sections) by subst. / by elimin.

[Day 3 2/2/03]

(parallel lines) Class ex: no soln [false statement] so this const. (D) $x+4y=4$, $2x+8y=5$ — Divide by region; 3 sections. each does it differently. (or, divide by 2 sections for Sub & elimin. Then do graph.)

* Ask for answers from subst. + elim. first. Then ask graphers.

~~class ex:~~ [when this not in class] (coincid. lines) infinite solutions [true statement] (E) $2x+3y=6$, $4x+6y=12$ — same as for (D). *(switch sections for methods) (have them write system in notes before doing class ex:)

for D & E, graph last

* also, express general solutions either in terms of x or y.

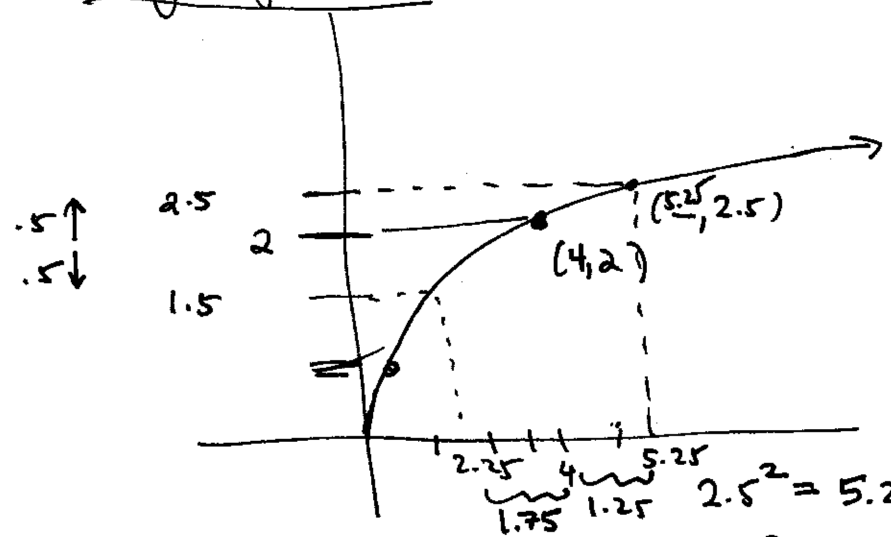
ex: $2x+3y=6 \rightarrow 2x=-3y+6$

$x=\dfrac{-3y+6}{2}$ (or $x=-\frac{3}{2}y+3$)

so the general set of solution pts. is $\left(\dfrac{-3y+6}{2}, y\right)$

[or you could solve for y and express in terms of x]

Def of limit



$y=\sqrt{x}$

$\lim_{x\to 4}\sqrt{x}=\sqrt{4}=2.$

$2.5^2 = 5.25$

$1.5^2 = 2.25$

$a = 4$

I want: $|\sqrt{x}-2| < .5$ $\quad$ $1.5 < \sqrt{x} < 2.5$

if $\underline{\underline{2.25}} < x < 5.25$ then this will be true

$0 < |x-4| < 1.25$ $\quad$ $\underline{\underline{2.75}} < x < 5.25$, $x \neq 4$.

Repeat: if $0<|x-4|<1.25$, then $|\sqrt{x}-2|<.5$

Def of limit says: if you want $|\overset{f(x)}{\sqrt{x}} - 2| < \varepsilon$ small any #

then you can find $\delta > 0$ so that

if $0 < |x-4| < \delta$, then $|\sqrt{x}-2| < \varepsilon$

E.g. for $\varepsilon = .5$, we can choose $\delta = 1.25$ (or anything smaller)

$\lim_{x \to a} f(x) = L$ means

that if you want $|f(x) - L| < \varepsilon$ $(\varepsilon > 0)$

then you can find $\delta > 0$ so that

$0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$

$\varepsilon = .001 \longrightarrow$ there's a δ

$\varepsilon = .000002 \longrightarrow$ there's a delta.

$f(x) = 2x - 3$

$\lim_{x \to 5} f(x) = 2(5) - 3 = 7$ Let's show this using def'n.

$|f(x) - L| < \varepsilon$

$$|2x - 3 - (7)| < \varepsilon$$
$$\Leftrightarrow |2x - 10| < \varepsilon$$
$$\Leftrightarrow |2(x-5)| < \varepsilon$$
$$\Leftrightarrow |x - 5| < \varepsilon/2$$

If you want $|2x - 3 - (7)| < \varepsilon$, choose $\delta = \frac{\varepsilon}{2}$

then $0 < |x - 5| < \underset{\frac{\varepsilon}{2}}{\overset{\shortparallel}{\delta}}$ implies $|2x - 3 - (7)| < \varepsilon$