Weyl group multiple Dirichlet series

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Basic problem

Let Φ be an irreducible root system of rank r.

Our goal: explain general construction of multiple Dirichlet series in r complex variables $\mathbf{s} = (s_1, \ldots, s_r)$

$$Z(\mathbf{s}) = \sum_{c_1, \ldots, c_r} \frac{a(c_1, \ldots, c_r)}{c_1^{s_1} \ldots c_r^{s_r}}$$

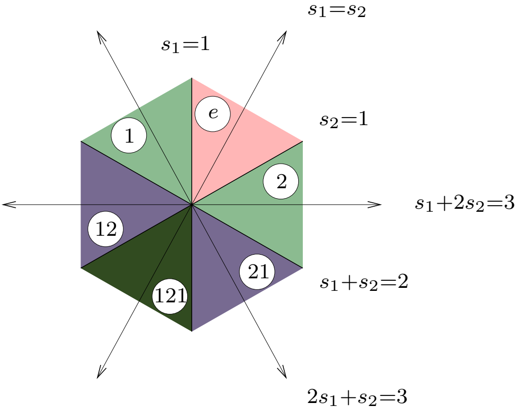
satisfying a group of functional equations isomorphic to the Weyl group W of Φ.

The functional equations intermix all the variables, and are closely related to the usual action of W on the space containing Φ.

Example

Let $\Phi = A_2$, $W = \langle \sigma_1, \sigma_2 \mid \sigma_i^2 = 1, \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$. The desired functional equations look like

$$\sigma_1\colon s_1 \to 2 - s_1, s_2 \to s_1 + s_2 - 1, \quad \sigma_2\colon s_1 \to s_1 + s_2 - 1, s_2 \to 2 - s_2$$



Why?

- Such series provide tools for certain problems in analytic number theory (moments, mean values, ...).
- Conjecturally these series arise as Fourier–Whittaker coefficients of Eisenstein series on metaplectic groups

$$1 \to \mu_n \to \tilde{G}(\mathbb{A}_F) \to G(\mathbb{A}_F) \to 1$$

  This has been proved in some cases (type A and type B (double covers)).
- The series are built out of arithmetically interesting data, such as Gauss sums, nth power residue symbols, Hilbert symbols, and (sometimes) L-functions.
- The objects that arise in the construction have interesting relationships with combinatorics, representation theory, and statistical mechanics.

Maass and the half-integral weight Eisenstein series

Let $E^*(z,s)$ be the half-integral weight Eisenstein series on $\Gamma_0(4)$:

$$E^*(z,s) = \sum_{\Gamma_\infty \backslash \Gamma_0(4)} j_{1/2}(\gamma, z)^{-1} \Im(\gamma z)^{s/2}.$$

Maass showed that its dth Fourier coefficient is essentially

$$L(s, \chi_d),$$

where χ_d is the quadratic character attached to $\mathbb{Q}(\sqrt{d}/\mathbb{Q})$.

Essentially means up to the Euler 2-factor, archimedian factors, and certain correction factors that have to be inserted when d isn't squarefree.

Siegel, Goldfeld–Hoffstein

Siegel (1956), Goldfeld–Hoffstein (1985):

$$Z(s,w) = \int_0^\infty (E^*(iy, s/2) - \text{const term})\, y^w \frac{dy}{y}.$$

The result is a Dirichlet series roughly of the form

$$Z(s,w) \approx \sum_d \frac{L(s,\chi_d)}{d^w}.$$

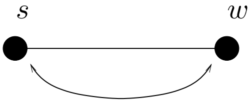
This behaves well in s since it's built from the Dirichlet L-functions, and it turns out to have nice analytic properties in w as well. Goldfeld–Hoffstein used this to get estimates for sums like

$$\sum_{\substack{|d|<X \\ d \text{ fund.}}} L(1,\chi_d), \qquad \sum_{\substack{|d|<X \\ d \text{ fund.}}} L(\tfrac{1}{2},\chi_d).$$

Siegel, Goldfeld–Hoffstein

$Z(s, w)$ satisfies a functional equation in s, again because of the Dirichlet L-functions. But it turns out that it satisfies extra functional equations.

In fact, Z satisfies a group of 12 functional equations, and is an example of a Weyl group multiple Dirichlet series of type A_2. There is a subgroup of functional equations isomorphic to $S_3 = W(A_2)$, and an extra one swapping s and w that corresponds to the outer automorphism of the Dynkin diagram:

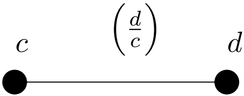


Connection to A_2

Why is this series related to root system A_2 (besides the fact that there are two variables and I drew the picture that way)?

Imagine expanding the L-functions in the rough definition:

$$Z(s,w) = \sum_d \frac{L(s,\chi_d)}{d^w} = \sum_d d^w \sum_c \left(\frac{d}{c}\right) c^{-s} = \sum_{d,c} \left(\frac{d}{c}\right) c^{-s} d^{-w}.$$



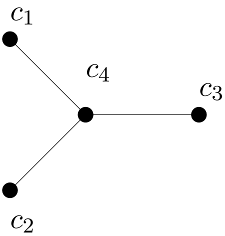
The general shape

Heuristically, the multiple Dirichlet series looks like

$$Z(\mathbf{s}) = \sum_{c_1,\dots,c_r} \frac{a(c_1,\dots,c_r)}{c_1^{s_1}\dots c_r^{s_r}}$$

where $a(c_1,\dots,c_r)$ is a product of nth power residue symbols corresponding to the edges of the Dynkin diagram.

For instance D_4, $n = 2$ leads to a series related to the third moment of quadratic Dirichlet L-functions.



Setup

- F number field with $2n$th roots of unity
- S set of places of F containing archimedian, ramified, and such that $\mathcal{O}_S$ is a PID
- Φ irreducible simply-laced root system of rank r
- $\{\alpha_1, \ldots, \alpha_r\}$ the simple roots
- $\mathbf{m} = (m_1, \ldots, m_r)$ r-tuple of integers in $\mathcal{O}_S$
- $\mathbf{s} = (s_1, \ldots, s_r)$ r-tuple of complex variables

Setup

- $F_S = \prod_{v \in S} F_v$
- $\mathcal{M}(\Phi)$ certain finite-dimensional space of complex-valued functions on $(F_S^\times)^r$ (to deal with Hilbert symbols and units)
- $\Psi \in \mathcal{M}(\Phi)$
- $H(\mathbf{c}; \mathbf{m})$ to be specified later . . . this is the most important part of the definition

The multiple Dirichlet series

Then the multiple Dirichlet series looks like

$$Z(\mathbf{s}; \mathbf{m}, \Psi; \Phi, n) = \sum_{\mathbf{c}} \frac{H(\mathbf{c}; \mathbf{m})\, \Psi(\mathbf{c})}{\prod |c_i|^{s_i}},$$

where $\mathbf{c} = (c_1, \ldots, c_r)$ and each c_i ranges over $(\mathcal{O}_S \smallsetminus \{0\})/\mathcal{O}_S^\times$.

The function H

The coefficients H have to be carefully defined to guarantee that Z satisfies the desired group of functional equations. General considerations tell us how to define H in the following cases:

- When $c_1 \cdots c_r$ and $c'_1 \cdots c'_r$ are relatively prime, one uses a "twisted multiplicativity" to construct $H(\mathbf{cc}';\mathbf{m})$ from $H(\mathbf{c};\mathbf{m})$ and $H(\mathbf{c}';\mathbf{m})$. One puts

$$H(\mathbf{cc}';\mathbf{m}) = \varepsilon(\mathbf{c},\mathbf{c}')H(\mathbf{c};\mathbf{m})H(\mathbf{c}';\mathbf{m}),$$

  where $\varepsilon(\mathbf{c},\mathbf{c}')$ is a root of unity built out of residue symbols and root data:

$$\varepsilon(\mathbf{c},\mathbf{c}') = \prod_{i=1}^{r}\left(\frac{c_i}{c'_i}\right)\left(\frac{c'_i}{c_i}\right)\prod_{i — j}\left(\frac{c_i}{c'_j}\right)\left(\frac{c'_i}{c_j}\right).$$

The function H

- When $(c_1 \cdots c_r, m'_1 \cdots m'_r) = 1$, we can define $H(\mathbf{c}; \mathbf{m}\mathbf{m}')$ in terms of $H(\mathbf{c}; \mathbf{m})$ and certain power residue symbols:

$$H(\mathbf{c}; \mathbf{m}\mathbf{m}') = \prod_{j=1}^{r} \left(\frac{m'_j}{c_j} \right) H(\mathbf{c}; \mathbf{m})$$

The function H

So we reduce the definition of H to that of

$$H(\varpi^{k_1}, \ldots, \varpi^{k_r}; \varpi^{l_1}, \ldots, \varpi^{l_r}),$$

where ϖ is a prime in $\mathcal{O}_S$.

This leads naturally to the generating function

$$\begin{aligned} N &= N(x_1, \ldots, x_r) \\ &= \sum_{k_1, \ldots, k_r \geq 0} H(\varpi^{k_1}, \ldots, \varpi^{k_r}; \varpi^{l_1}, \ldots, \varpi^{l_r}) x_1^{k_1} \cdots x_r^{k_r} \end{aligned}$$

(**m** is fixed). One can ask what properties this series has to satisfy so that one can prove Z satisfies the right group of functional equations.

The function N

$$N = N(x_1, \ldots, x_r) = \sum_{k_1, \ldots, k_r \geq 0} H(\varpi^{k_1}, \ldots, \varpi^{k_r}) x_1^{k_1} \cdots x_r^{k_r}.$$

If one puts $x_i = q^{-s_i}$, where $q = |\mathcal{O}_S/\varpi|$, then one can see that the global functional equations imply N must transform a certain way under a certain W-action.

This leads to a connection with characters of representations of $\mathfrak{g}$, the simple complex Lie algebra attached to Φ.

In this relationship the monomials correspond to certain weight spaces.

Building N

The connection with characters leads to two approaches to defining N:

- *Crystal graphs.* These are models for $\mathfrak{g}$ representations that have various combinatorial incarnations (Gelfand–Tsetlin patterns, tableaux, Proctor patterns, Littlemann path model, ...). One tries to extract a statistic from the combinatorial model to define the coefficients of N. (Brubaker–Bump–Friedberg, Beineke–Brubaker–Frechette, Chinta–PG)
- *Weyl character formula.* This is an explicit expression for a given character as a ratio of two polynomials. We take this approach and define a deformation of Weyl's formula that reflects the metaplectacity (metaplectaciousness?) of the setup. (Chinta–PG, Bucur–Diaconu)

WCF

- Λ_w weight lattice of Φ
- $\{\omega_1, \ldots, \omega_r\}$ fundamental weights
- $\rho = \sum \omega_i$ the Weyl vector
- $\mathbb{Z}[y_1^{\pm 1}, \ldots, y_r^{\pm 1}]$ group ring of the weight lattice ($y_i \leftrightarrow \omega_i$)
- θ a dominant weight

Then according to Weyl the character of the irreducible representation of highest weight θ is

$$\chi_\theta = \frac{\sum_{w \in W} \operatorname{sgn}(w) \mathbf{y}^{w(\theta+\rho)-\rho}}{\prod_{\alpha>0}(1-\mathbf{y}^{-\alpha})} = \sum_{w \in W} \operatorname{sgn}(w) \mathbf{y}^{w(\theta+\rho)-\rho} \frac{1}{\Delta(\mathbf{y})}.$$

$\Delta(\mathbf{y}) = \prod_{\alpha>0}(1-\mathbf{y}^{-\alpha})$.

Deformation of the WCF

Our goal now is to define the W-action leading to H. For the application to multiple Dirichlet series, we normalize things slightly differently. Thus we work with the root lattice, introduce some $q = |\mathcal{O}_S/\varpi|$ powers, shift the character around, ...

- Λ root lattice of Φ
- $d\colon \Lambda \to \mathbb{Z}$ height function on the roots
- $A \simeq \mathbb{C}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$ complex group ring of Λ ($x_i \leftrightarrow \alpha_i$)
- $\tilde{A} \simeq \mathbb{C}(x_1, \dots, x_r)$ fraction field of A
- $\theta = \rho + \sum l_i \omega_i$ a *strictly* dominant weight (recall that we're defining $H(\mathbf{c}; \mathbf{m})$ when $\mathbf{m} = (\varpi^{l_1}, \dots, \varpi^{l_r})$)

The action on monomials

We let the Weyl group act on monomials through a "change of variables" map. This is essentially the same as the geometric action of W on the root lattice (except for the q power).

If $f(\mathbf{x}) = \mathbf{x}^{\beta}$, we put

$$f(w\mathbf{x}^{\beta}) = q^{d(w^{-1}\beta-\beta)}\mathbf{x}^{w^{-1}\beta}.$$

Affine action of W

Given any $\lambda \in \Lambda$, we put

$$w \bullet \lambda = w(\lambda - \theta) + \theta,$$

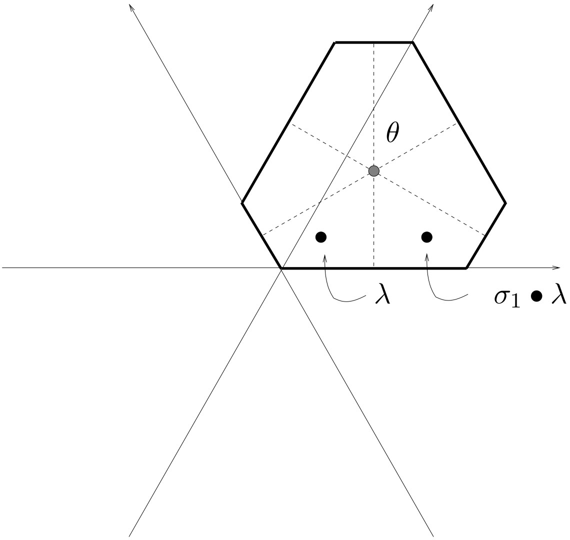
where the action on the right hand side is the usual action on the root lattice. This just performs an affine reflection of $\Lambda \otimes \mathbb{R}$ (the same as the usual w reflection but shifted to have center θ).

If σ_i is a simple reflection, we put

$$\mu_i(\lambda) = d(\sigma_i \bullet \lambda - \lambda).$$

This is just the multiple of α_i needed to go from λ to $\sigma_i \bullet \lambda$.

Affine action of W



Gauss sums

Choose some complex numbers $\gamma(i)$, $i = 1, \ldots, n-1$ such that $\gamma(i)\gamma(n-i) = 1/q$. Put $\gamma(0) = -1$.

Ultimately these numbers will be Gauss sums (the same ones appearing in the metaplectic cocycle), but actually any complex numbers satisfying these relations will work.

Extend $\gamma(i)$ to all integers by reducing i mod n.

Homogeneous decomposition

The action on a monomial $f(\mathbf{x}) = \mathbf{x}^\beta$ depends on the congruence class of the monomial mod $n\Lambda$.

To treat general rational functions, we decompose $\tilde{A}$ into homogeneous parts

$$\tilde{A} = \bigoplus_{\lambda \in \Lambda/n\Lambda} \tilde{A}_\lambda.$$

A_λ consists of those rational function f/g where all monomials in g lie in $n\Lambda$ and those in f map to λ modulo $n\Lambda$.

e.g.,

$$\frac{1 - xy}{x^2 - y^2} = \frac{1}{x^2 - y^2} - \frac{xy}{x^2 - y^2}$$

Finally

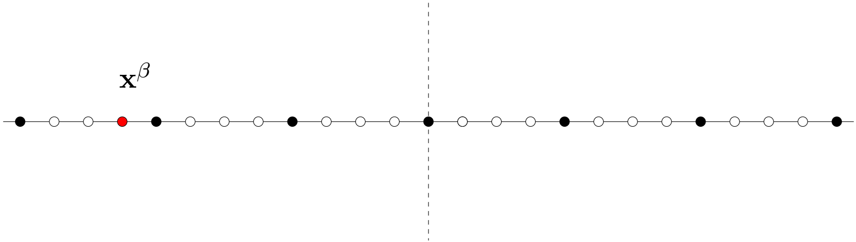
Theorem (Chinta–PG) Suppose $f \in A_\beta$. Let σ_i be a simple reflection and let $(k)_n$ be the remainder upon division of k by n. Then

$$
\begin{aligned}
(f|_\theta \sigma_i)(\mathbf{x}) = {} & (qx_i)^{l_i+1-(\mu_i(\beta))_n} \frac{1-1/q}{1-q^{n-1}x_i^n} f(\sigma_i \mathbf{x}) && (P) \\
& - \gamma(\mu_i(\beta)) \cdot (qx_i)^{l_i+1-n} \frac{1-(qx_i)^n}{(1-q^{n-1}x_i^n)} f(\sigma_i \mathbf{x}) && (Q)
\end{aligned}
$$

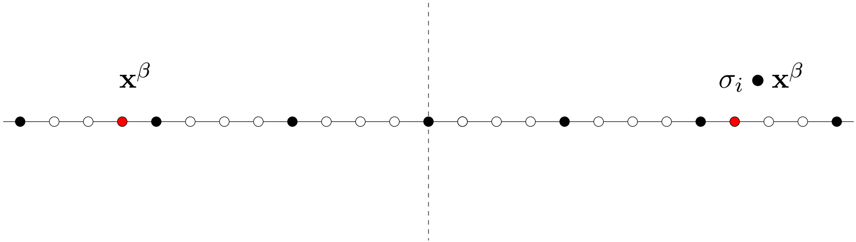
extends to a W-action on $\mathbb{C}(x_1, \ldots, x_r)$.

The W-action

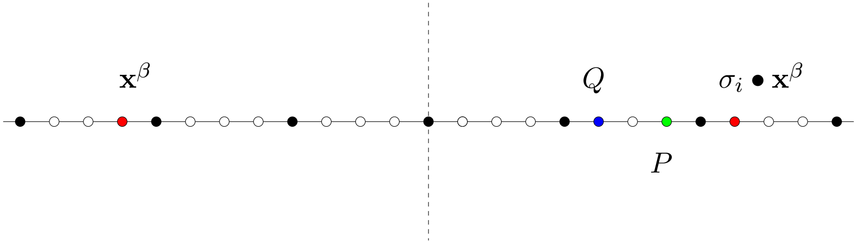
The W-action



The W-action



The W-action



Making the multiple Dirichlet series

Theorem (Chinta–PG)

- Put $\Delta(\mathbf{x}) = \prod_{\alpha>0}(1 - q^n \mathbf{x}^{n\alpha})$ and $D(\mathbf{x}) = \prod_{\alpha>0}(1 - q^{n-1}\mathbf{x}^{n\alpha})$. Then
$$h(\mathbf{x}) = \sum_{w \in W} \frac{(1|_\theta w)(\mathbf{x})}{\Delta(w\mathbf{x})}$$
is a rational function such that hD is a polynomial.
- Let $N = hD$, define H by
$$N = \sum_{k_1,\ldots,k_r \geq 0} H(\varpi^{k_1}, \ldots, \varpi^{k_r}; \varpi^{l_1}, \ldots, \varpi^{l_r}) x_1^{k_1} \cdots x_r^{k_r},$$
and specialize the $\gamma(i)$ to the appropriate Gauss sums. Then the resulting multiple Dirichlet series $Z(\mathbf{s}; \mathbf{m}, \Psi; \Phi, n)$ has analytic continuation to $\mathbb{C}^r$ and satisfies a group of functional equations isomorphic to W.

A_2 examples $(n = 2)$

Here $g_1 = q\gamma(1)$ and the notation (a, b) means

$$\theta = (a + 1)\omega_1 + (b + 1)\omega_2.$$

- $(0, 0)$: $1 + g_1x + g_1y - g_1qx^2y - g_1qxy^2 - q^2x^2y^2$
- $(1, 0)$: $1 - qx^2 + g_1y - g_1qx^2y + g_1q^2x^2y + q^3x^3y - g_1q^3x^2y^3 - q^4x^3y^3$
- $(1, 1)$: $1 - qx^2 - qy^2 + q^2x^2y^2 - q^3x^2y^2 + q^4x^4y^2 + q^4x^2y^4 - q^5x^4y^4$
- $(2, 1)$: $1 - qx^2 + q^2x^2 + g_1q^2x^3 - qy^2 + q^2x^2y^2 - 2q^3x^2y^2 + q^4x^2y^2 - g_1q^3x^3y^2 + g_1q^4x^3y^2 + q^4x^4y^2 - q^5x^4y^2 - g_1q^5x^5y^2 + q^4x^2y^4 - q^5x^2y^4 - g_1q^5x^3y^4 + g_1q^6x^3y^4 - q^5x^4y^4 + q^6x^4y^4 + g_1q^6x^5y^4 - g_1q^7x^5y^4 + q^7x^3y^5 - q^8x^5y^5$

Open questions

- The WCF method works for all Φ, whereas the crystal graph approach has only been worked out for some (classical) Φ. Can one do the latter for all Φ uniformly? (Kim–Lee, McNamara)
- Prove that Z is a Whittaker coefficient of a metaplectic Eisenstein series. (Chinta–Offen)
- Prove that the crystal graph descriptions and the WCF descriptions coincide. (Chinta–Offen + McNamara)
- Develop multiple Dirichlet series on affine Weyl groups and crystallographic Coxeter groups (Bucur–Diaconu, Lee)
- What is the geometric interpretation of Weyl group multiple Dirichlet series over function fields?

References

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